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Nonlocal Symmetry and Generating Solutions for the Inhomogeneous Burgers Equation

Valentyn TYCHYNIN † *and Olexandr RASIN* ‡

† *Prydniprovsk State Academy of Civil Engineering and Architecture, 24a Chernyshevsky Str., 49000 Dnipropetrovsk, Ukraine*
E-mail: *tychynin@pgasa.dp.ua, tychynin@ukr.net*

‡ *Dnipropetrovsk National University, 13 Naukovy Per., 49050 Dnipropetrovsk, Ukraine*
E-mail: *sasha_rasin@rambler.ru*

In the present paper we consider a class of inhomogeneous Burgers equations. Nonlocal transformations of a dependent variable that establish relations between various equations of this class were constructed. We identified the subclass of the equations, invariant under the appropriate substitution. The formula of non-local superposition for the inhomogeneous Burgers equation was constructed. We also present examples of generation of solutions.

1 Non-local invariance of the inhomogeneous Burgers equation

Let us consider an inhomogeneous Burgers equation:

$$u_t + uu_x - u_{xx} = c, \qquad u = u(x,t), \tag{1}$$

where $c = c(x,t)$ is an arbitrary smooth function, and $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$.

Let us make the first order non-local substitution of the dependent variable:

$$u = f(t, x, v, v_x), \tag{2}$$

where $v = v(x,t)$ is the new dependent variable.

We seek the transformation of the equation (1) into another inhomogeneous Burgers equation:

$$v_t + vv_x - v_{xx} = g, \tag{3}$$

with arbitrary smooth function $g = g(x,t)$. To obtain this transformation we substitute (2) and its differential prolongations into equation (1). For differential prolongations of the equation (3) we obtain the determining relation:

$$\begin{aligned} &f_t - f_v v_x v + f_v g - f_{v_x} v_{xx} v - f_{v_x} {v_x}^2 + f_{v_x} g_x + f f_x + f f_v v_x + f f_{v_x} v_{xx} - f_{xx} \\ &\qquad - 2 f_{vx} v_x - 2 f_{v_x x} v_{xx} - f_{vv} {v_x}^2 - 2 v_x f_{v v_x} v_{xx} - f_{v_x v_x} {v_{xx}}^2 - c = 0. \end{aligned}$$

Having split these expressions with respect to the derivative v_{xx}, we obtain the system of equations:

$$\begin{aligned} &f_{v_x v_x} = 0, \\ &-f_{v_x} v + f f_{v_x} - 2 f_{v v_x} v_x - 2 f_{v_x x} = 0, \\ &f_t - f_v v_x v + f_v g + f f_v v_x - f_{v_x} {v_x}^2 + f_{v_x} g_x + f f_x - c - f_{vv} {v_x}^2 - f_{xx} - 2 f_{vx} v_x = 0. \end{aligned} \tag{4}$$

The nontrivial solution of the system (1) is

$$f = \frac{-2v_x + 4F_x + v^2 - 2vF}{v - 2F}, \tag{5}$$

$$c = 2F_{xx} + 2F_t + 4F_xF, \tag{6}$$
$$g = -2F_{xx} + 2F_t + 4F_xF. \tag{7}$$

Here $F = F(x,t)$ is an arbitrary smooth function.

Let us consider a non-local invariance transformation, namely $c = g$. The equations (6), (7) give us the condition $F_{xx} = 0$. Integration yields $F = A(t)x + B(t)$, where $A(t)$ and $B(t)$ are arbitrary smooth functions. So we have found the non-local invariant transformation for the inhomogeneous Burgers equation in a general form:

$$f = \frac{2v_x - 4A - v^2 + 2vAx + 2vB}{-v + 2Ax + 2B}, \tag{8}$$
$$g = c = 2A_tx + 2B_t + 4A^2x + 4AB. \tag{9}$$

Substituting $c = g = 0$ into expressions (6), (7), we find the system:

$$2F_{xx} + 2F_t + 4F_xF = 0,$$
$$-2F_{xx} + 2F_t + 4F_xF = 0.$$

The general and trivial solutions of this system are:

$$F = \frac{x + c_1}{2t + 2c_2}, \qquad F = 0. \tag{10}$$

Here c_1, c_2 are an arbitrary constants. Having substituted (10) into (5), we get the following non-local invariant transformations of the homogeneous Burgers equation for the general and trivial solutions respectively:

$$u = \frac{2v_xt + 2v_xc_2 - 2 - v^2t - v^2c_2 + vx + vc_1}{-vt - vc_2 + x + c_1}, \qquad u = \frac{-2v_x + v^2}{v}.$$

Example 1. Using (5), (6), (7) we transform the inhomogeneous Burgers equation:

$$u_t + uu_x - u_{xx} = 8\frac{\sin x}{\cos^3 x}, \tag{11}$$

into homogeneous one. We obtain the system of the equations for F by substituting $c = 8\frac{\sin x}{\cos^3 x}$ and $g = 0$ in (6), (7):

$$F_t + 2F_xF - F_{xx} = 0,$$
$$F_t + 2F_xF + F_{xx} = 4\frac{\sin x}{\cos^3 x}.$$

There is the general solution of the system $F = \mathrm{tg}x$. It gives us the following substitution:

$$u = \frac{2v_x\cos^2 x - 4 - v^2\cos^2 x + 2v\sin x\cos x}{\cos x(-v\cos x + 2\sin x)}. \tag{12}$$

This expression can be applied to generation of solutions of the equation (11). Thus, the partial solution of the homogeneous Burgers equation:

$$v = \frac{-4x}{2t + x^2}, \tag{13}$$

generates the following solution of the equation (11):

$$u = -2\frac{2\cos^2 x + 2t + x^2 + 2x\sin x\cos x}{\cos x(2x\cos x + 2t\sin x + x^2\sin x)}. \tag{14}$$

Example 2. It is an example of application of the non-local transformation (8) for a partial case of equation (1): $A = 1, B = t$. Then we have the equation:

$$u_t + uu_x - u_{xx} = 4x + 4t + 2, \tag{15}$$

and corresponding invariant transformation:

$$\overset{(2)}{u} = \frac{2\overset{(1)}{u}_x - 4 - \overset{(1)}{u}{}^2 + 2\overset{(1)}{u}x + 2\overset{(1)}{u}t}{-\overset{(1)}{u} + 2x + 2t}. \tag{16}$$

One of the similarity solutions of the equation (15) (see Appendix 1) is

$$u = -2 - 2x - 2t + e^{2t}\mathrm{tg}\left(\frac{1}{4}(2x + 1 + 2t)\,e^{2t}\right).$$

Having substituted it into (16) we find a new solution of the equation (15):

$$\overset{(2)}{u} = -\frac{(6t + 6x + 4)\,\mathrm{tg}\left(\frac{e^{2t}}{4}(2x + 1 + 2t)\right)e^{2t} - 12 - 8x^2 - 12x - 12t - 8t^2 - 16xt}{-2 - 4x - 4t + e^{2t}\mathrm{tg}\left(\frac{e^{2t}}{4}(2x + 1 + 2t)\right)}.$$

Using this algorithm we get a chain of solutions:

$$\begin{aligned} &-2x - 2t - 2 \to -2\frac{3 + 2x^2 + 4xt + 3x + 2t^2 + 3t}{2x + 2t + 1} \\ &\quad \to -2\frac{4x^3 + 8x^2 + 12x^2t + 12xt^2 + 16xt + 15x + 15t + 4t^3 + 7 + 8t^2}{3 + 4x^2 + 8xt + 4x + 4t^2 + 4t} \to \cdots. \end{aligned}$$

2 Linearization of the inhomogeneous Burgers equation

We are looking for a non-local transformation

$$u = f(v, v_x), \tag{17}$$

of equation (1), where $v = v(x,t)$ is a smooth function, which yields equation:

$$v_t - v_{xx} + \varphi = 0. \tag{18}$$

Here $\varphi = \varphi(x,t,v)$ is an arbitrary smooth function. To obtain this transformation we substitute (17) and its differential prolongations into equation (1). For differential prolongations of the equation (18) we obtain the determining correlation:

$$f_v\varphi + f_{v_x}\varphi_x + f_{v_x}\varphi_v v_x + ff_v v_x + ff_{v_x}v_{xx} - f_{vv}{v_x}^2 - 2v_x f_{vv_x}v_{xx} - f_{v_xv_x}{v_{xx}}^2 + c = 0.$$

Having split these expressions with respect to the derivative v_{xx}, we obtain the system of equations:

$$\begin{aligned} &f_{v_xv_x} = 0, \\ &2f_{vv_x}v_x - ff_{v_x} = 0, \\ &f_v\varphi + f_{v_x}\varphi_x + f_{v_x}\varphi_v v_x + ff_v v_x + c - f_{vv}{v_x}^2 = 0. \end{aligned}$$

The nontrivial solution is

$$f = -2\frac{v_x}{v + c_1}, \tag{19}$$

$$c = -2F_x, \tag{20}$$

$$\varphi = -F(v + c_1). \tag{21}$$

Here c_1 is an arbitrary constant and $F = F(x,t)$ is an arbitrary smooth function. Thus we obtain the Cole–Hopf substitution with the parameter c_1.

$$u = -2\frac{v_x}{v + c_1}. \tag{22}$$

So the inhomogeneous Burgers equation may be transformed into the linear equation with variable coefficients [2, 4]:

$$v_t - F(v + c_1) - v_{xx} = 0.$$

Here the function F is obtained from the equation (20). In the case $c_1 = 0$, we obtain the transformation into the homogeneous equation:

$$v_t - Fv - v_{xx} = 0. \tag{23}$$

Theorem 1. *Formula of a nonlinear superposition for inhomogeneous Burgers equation can be written in the following way:*

$$\begin{aligned} &\overset{(3)}{u} = -2\partial_x \ln\left(\overset{(1)}{\tau} + \overset{(2)}{\tau}\right), \\ &-2\partial_x \ln \overset{(k)}{\tau} = \overset{(k)}{u}, \qquad k = 1, 2, \\ &-2\partial_t \ln \overset{(k)}{\tau} = \overset{(k)}{u}_x - \frac{1}{2}\overset{(k)}{u}^2 + \psi, \qquad \psi_x = c(x,t) \end{aligned} \tag{24}$$

Here $\overset{(1)}{u}$, $\overset{(2)}{u}$ *are known solutions, and* $\overset{(3)}{u}$ *is the new one.*

Proof. Let $\overset{(1)}{\tau}$, $\overset{(2)}{\tau}$ be solutions of equation (23). Then $\overset{(3)}{\tau} = \overset{(1)}{\tau} + \overset{(2)}{\tau}$ is a new solution of equation (23). By using the substitution $u = -2\partial_x \ln(\tau)$ we can find $\overset{(3)}{u}$:

$$\overset{(3)}{u} = -2\partial_x \ln\left(\overset{(3)}{\tau}\right) = -2\partial_x \ln\left(\overset{(1)}{\tau} + \overset{(2)}{\tau}\right).$$

On the other side $\overset{(k)}{\tau}$, $k = 1, 2$ are connected with $\overset{(k)}{u}$, $k = 1, 2$ in the following way:

$$-2\partial_x \ln \overset{(k)}{\tau} = \overset{(k)}{u}, \qquad -2\partial_t \ln \overset{(k)}{\tau} = \overset{(k)}{u}_x - \frac{1}{2}\overset{(k)}{u}^2 + \psi, \qquad \psi_x = c(x,t), \qquad k = 1, 2.$$

So superposition formula (24) is obtained. ■

Example 3. We can use superposition formula for equation (15). There are two solutions of equation (15):

$$\overset{(1)}{u} = -2 - 2x - 2t + e^{2t}\mathrm{tg}\left(\frac{1}{4}(2x + 1 + 2t)e^{2t}\right),$$

$$\overset{(2)}{u} = -2x - 2t - 2,$$

The formula (24) gives us a third one:

$$\overset{(3)}{u} = -2 - 2x - 2t + \frac{\mathrm{tg}\left(\left(\frac{x}{2} + \frac{1}{4} + \frac{t}{2}\right)e^{2t}\right)e^{2t}}{1 + e^{\frac{e^{4t}}{16}}\sec^2\left(\left(\frac{x}{2} + \frac{1}{4} + \frac{t}{2}\right)e^{2t}\right)}.$$

3 Lie symmetries for the inhomogeneous Burgers equation

To apply the classical method [3] to (1) we require the infinitesimal operator to be of this form:

$$X = \xi_0(x,t,u)\partial_t + \xi_1(x,t,u)\partial_x + \eta(x,t,u)\partial_u.$$

The invariance condition for equation (1) yields an overdetermined system of differential equations for the coordinates of X. Having solved the system of equations we obtain the following expressions for infinitesimals and function $c(x,t)$:

$$\begin{gathered}
\xi_0 = F_1, \qquad \xi_1 = \frac{1}{2}F_1'x + F_2, \qquad \eta = -\frac{1}{2}uF_1' + \frac{1}{2}F_1''x + F_2', \\
c = \left(\frac{1}{2}\int F_1'''F_1 \int F_2F_1^{-3/2}dtdt - \frac{1}{2}\int F_1'''F_1dt \int F_2F_1^{-3/2}dt \right. \\
\left. + \int F_1^{1/2}F_2''dt + F_3(\omega)\right)F_1^{-3/2} + \frac{x}{2}\int F_1'''F_1dtF_1^{-2}, \\
\omega = \left(x - F_1^{1/2}\int F_2F_1^{-3/2}dt\right)F_1^{-1/2}.
\end{gathered}$$

Here $F_1 = F_1(t)$, $F_2 = F_2(t)$, $F_3 = F_3(\omega)$ are arbitrary smooth functions. If $F_1 = 0$ we have:

$$c = \frac{F_2''x}{F_2} + F_3,$$

Obviously, the connection between the right hand side of the equation (1) and its Lie symmetries exists.

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