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On Applications of Non-Point and Discrete Symmetries for Reduction of the Evolution-Type Equations

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Combination of non-point and conditional symmetries of the evolution-type equations for construction of its exact solutions are presented. Also we analyzed discrete symmetries of Maxwell equations in vacuum and decoupled ones to the four independent equations that can be solved independently.

1 Introduction

Generalization of Lie classical approach to non-Lie approach that take into consideration conditional symmetries, non-point symmetries, supersymmetries, parasupersymmetries, discrete symmetries opens us additional ways of investigation of partial differential equations (PDEs). For instance, it is possible to seek invariance algebra of DE in a class of high-order differential operators or even of integro-differential operators. In this paper we consider two possibilities for expansion of applications of symmetries to (PDEs) by using non-point symmetries and discrete symmetries. First we consider a problem of construction of ansatzes not for the dependent variable $u(t,x)$ (as it usually used by non-classical method, by method of nonlinear separation and by method of generalized symmetry) but for its *derivatives*:

$$\frac{\partial u}{\partial t} = R_1(t,x,u,\varphi_1(\omega),\varphi_2(\omega)),$$
$$\frac{\partial u}{\partial x} = R_2(t,x,u,\varphi_1(\omega),\varphi_2(\omega)), \qquad \omega=\omega(t,x),$$

that reduce our evolution equation (F is an arbitrary smooth function)

$$u_t = F(u_{xx}) \tag{1}$$

to a system of two DE for function $\varphi_1(\omega)$ and $\varphi_2(\omega)$. The main idea we exploit here is to find symmetries of the system of equations

$$v^2_1 = v^1_2,$$
$$v^2_2 = \Phi\big(v^1\big), \qquad \Phi = F^{-1}, \qquad v^1 \equiv \frac{\partial u}{\partial t}, \qquad v^2 \equiv \frac{\partial u}{\partial x}, \tag{2}$$

(lower indices designate the respect derivatives, i.e., $v^1_2 = \frac{\partial v^1}{\partial x_2}$, $v^2_1 = \frac{\partial v^2}{\partial x_1}$ and $x_1 \equiv t$, $x_2 \equiv x$) corresponding to (1) instead of symmetries of initial equation (1). Taking into account the above designations, it is evident that second equation of system (2) corresponds to (1), and the first one is the compatibility condition. More detailed explanation that (1) yields system (2) see in [1–3].

We note that by investigating the symmetry properties of system (2) we can find symmetry operators that would not correspond to any infinitesimal operators of group of point transformations of (1) and thus we can extend the class of symmetry operators of (1).

Another goal of our paper is to show an effective way of simplifying a given problem using discrete symmetries of DE. A successful using of discrete symmetries for investigation of PDE can be found in [5–9].

2 Symmetry classification of evolution equation

Here we investigate the system of (2) and find the maximal invariance algebra of the system. We prove that

Theorem 1. *System* (2) *is invariant with respect to a local Lie algebra generated by operator* ($\xi^1_{v_1}$, $\xi^1_{v_2}$, $\xi^2_{v_1}$, $\xi^2_{v_2}$ *are not equal to zero simultaneously):*

$$X = \xi^1(x_1,x_2,v_1,v_2)\frac{\partial}{\partial x_1} + \xi^2(x_1,x_2,v_1,v_2)\frac{\partial}{\partial x_2} + \eta^1(x_1,x_2,v_1,v_2)\frac{\partial}{\partial v^1} + \eta^2(x_1,x_2,v_1,v_2)\frac{\partial}{\partial v^2},$$

if function $\Phi(v^1)$ *is a solution of equation:*

$$a(\Phi_{v^1}v^1)_{v^1} = b\Phi\Phi_{v^1} + c\Phi_{v^1v^1} + d\Phi_{v^1}, \tag{3}$$

where a, b, c, d *are arbitrary real constants and* $\Phi_{v^1} \equiv \frac{d\Phi}{dv^1}$, $\Phi_{v^1v^1} \equiv \frac{d^2\Phi}{dv_1^2}$.

We notice that Theorem 1 defines us a class of equations (1) admitting Lie symmetries. Fixing values of a, b, c, d in (3) we find various explicit form of $\Phi\big(v^1\big)$. We consider only such forms of $\Phi\big(v^1\big)$ that cannot be transformed to each other by point transformations of equivalence of (1).

Theorem 2. *The maximal invariance algebra of system* (2) *is a 5-dimensional Lie algebra:*

1. $\langle P_1,P_2,P_3,D_1,Q_1\rangle$ *if* $\Phi = \dfrac{1}{\alpha\ln v^1}$, $\alpha = \text{const}$;
2. $\langle P_1,P_2,P_3,D_1,Q_2\rangle$ *if* $\Phi = \dfrac{1}{e^{v^1}-c}$, $c = \text{const}$;
3. $\langle P_1,P_2,P_3,D_1,Q_3\rangle$ *if* $\Phi = \dfrac{1}{1-\big(v^1\big)^r}$, $r\in\mathbb{R}$, $r\neq 0,\pm 1$;
4. $\langle P_1,P_2,P_3,D_1,Q_4\rangle$ *if* $\Phi = \tan v^1$;
5. $\langle P_1,P_2,P_3,D_1,Q_5\rangle$ *if* $\Phi = \tan\big(\alpha\ln v^1\big)$, $\alpha = \text{const}$.

Theorem 3. *The maximal invariance algebra of system* (2) *is a 7-dimensional Lie algebra:*

$$\langle P_1,P_2,P_4,P_5,D_2,Q_7,Q_8\rangle \quad \textit{if} \quad \Phi = \frac{1}{1-(v^1)^{-3}}$$

(it is a particular case of 3 from Theorem 2 when $r=-3$*).*

Theorem 4. *The maximal invariance algebra of system* (2) *is an infinite-dimensional Lie algebra:*

$$\langle P_1,P_2,P_3,D_1,Q_6\rangle \quad \textit{if} \quad \Phi = \frac{1}{v^1}.$$

The basis elements of the Lie algebras have the following form:

$$P_1 = \frac{\partial}{\partial x_1}, \quad P_2 = \frac{\partial}{\partial x_2}, \quad P_3 = \frac{\partial}{\partial v^2}, \quad P_4 = x_2\frac{\partial}{\partial x_3} + \frac{\partial}{\partial v^2}, \ P_5 = \frac{\partial}{\partial x_3},$$
$$D_1 = 2x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + v^2\frac{\partial}{\partial v^2}, \ D_2 = 2x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 2x_3\frac{\partial}{\partial x_3} + v^2\frac{\partial}{\partial v^2},$$
$$Q_1 = -x_1\frac{\partial}{\partial x_1} + v^2\frac{\partial}{\partial x_2} + v^1\frac{\partial}{\partial v^1}, \quad Q_2 = \left(x_2 + 2cv^2\right)\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial v^1} - v^2\frac{\partial}{\partial v^2},$$
$$Q_3 = -(r+1)x_1\frac{\partial}{\partial x_1} - rv^2\frac{\partial}{\partial x_2} + v^1\frac{\partial}{\partial v^1} - rv^2\frac{\partial}{\partial v^2},$$
$$Q_4 = -2v^2\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial v^1} + 2x_2\frac{\partial}{\partial v^2}, \quad Q_5 = -x_1\frac{\partial}{\partial x_1} - \alpha v^2\frac{\partial}{\partial x_2} + v^1\frac{\partial}{\partial v^1} + \alpha x_2\frac{\partial}{\partial v^2},$$
$$Q_6 = -B_{v^2}\frac{\partial}{\partial x_2} + B_{x_1}\frac{\partial}{\partial v^1},$$
$$Q_7 = 2x_1\frac{\partial}{\partial x_1} + 3v^2\frac{\partial}{\partial x_2} + \left[\frac{3}{2}\left(v^2\right)^2 + 3x_3\right]\frac{\partial}{\partial x_3} + v^1\frac{\partial}{\partial v^1} + 3v^2\frac{\partial}{\partial v^2},$$
$$Q_8 = \left[3\left(v^2\right)^2 - 2x_3\right]\frac{\partial}{\partial x_2} + 2\left(v^2\right)^3\frac{\partial}{\partial x_3} + +2v^1v^2\frac{\partial}{\partial v^1} + 2(v^2)^2\frac{\partial}{\partial v^2},$$

where $B\left(x_1, v^2\right)$ is an arbitrary solution of equation:

$$B_{v^2v^2} + B_{x_1} = 0$$

(we designate $B_{x_1} \equiv \frac{\partial B}{\partial x_1}$, $B_{v^2v^2} \equiv \frac{\partial^2 B}{\partial v^2 \partial v^2}, \ldots$).

The Theorems are proved by means of the Lie algorithm [11, 12]. It is worth noting that operators Q_a ($a = \overline{1,8}$) correspond to non-point symmetry operators of (1). The cases 4 and 5 of Theorem 2 have been obtained under classical symmetry investigation in [13] and under studying of potential symmetries in [14].

3 Ansätze and exact solutions

Let us use operators of non-point symmetries Q_a, $a = \overline{1,8}$ for reducing of (2) to a system of two ordinary differential equations. It is obvious that we can use also the first prolongation operators of point symmetry of (1). In order to construct an ansatz reducing given equation to system of two ordinary DEs we should use two-dimensional subalgebra of invariance algebra of equation. At the same time according to [15] the obtained solution will be an invariant solution in the Lie sense and can be obtained by classical Lie method. It follows that we should use operators of non-point symmetries to obtain new results.

We note that (1) connected with nonlinear heat equation:

$$\chi_t + (C(\chi)\chi_x)_x = 0, \tag{4}$$

Indeed, by double differentiation of (1) with respect to x we obtain equation (4), where $\chi = u_{xx}$, $C(\chi) = F'(\chi)$.

Thus, it is possible to find exact solutions for function χ instead of function u. The ansatz

$$v^1 = \frac{\varphi_1\left(v^2\right)}{x_1}, \qquad v^2 = \frac{x_2}{\varphi_2\left(v^2\right) - \ln x_1}$$

constructed by the operator Q_1 reduces system (2) to the following ordinary differential equations ($\alpha = 1$):

$$\frac{d\varphi_1}{dv^2} = v^2, \qquad \ln\varphi_1 - \varphi_2 = v^2\frac{d\varphi_2}{dv^2}.$$

Integrating the latter system we find exact solution of (4) (when $C(\chi) = -\frac{1}{\chi^2}\exp\left(\frac{1}{\chi}\right)$) in the form:

$$\exp\left(\frac{1}{\chi}\right) = \frac{\theta^2+c}{2t}, \qquad \theta = \frac{x}{\ln\frac{\theta^2+c}{2te^2} + \frac{2\sqrt{c}}{\theta}\arctan\frac{\theta}{\sqrt{c}} + \frac{c_1}{\theta}}, \quad \text{if} \quad c>0;$$

$$\exp\left(\frac{1}{\chi}\right) = \frac{\theta^2+c}{2t}, \qquad \theta = \frac{x}{\ln\frac{\theta^2+c}{2te^2} + \frac{\sqrt{-c}}{\theta}\ln\left|\frac{\theta-\sqrt{-c}}{\theta+\sqrt{-c}}\right| + \frac{c_1}{\theta}}, \quad \text{if} \quad c<0,$$

c, c_1 are constants, $\theta \equiv v^2$. In an analogous way we can find exact solutions for the other cases listed in Theorem 2. Here we represent operators, corresponding ansatzes and reduced systems of ODEs:

Operator	Ansätze	Reduced system
Q_1+P_3	$v^1 = \frac{\varphi_1(\omega)}{x_1}$ $v^2 = \varphi_2(\omega) - \ln x_1$ $\omega = x_2 - \frac{(v^2)^2}{2}$	$\frac{d\varphi_1}{d\omega} = -1$ $(\ln\varphi_1 - \varphi_2)\frac{d\varphi_2}{d\omega} = 1$
Q_1+D_1	$v^1 = x_1\varphi_1(\omega)$ $v^2 = x_1\varphi_2(\omega)$ $\omega = \frac{x_2}{v^2} - \ln v^2$	$\frac{d\varphi_1}{d\omega} = {\varphi_2}^2$ $\varphi_2 + \left(1+\omega+\ln\frac{\varphi_2}{\varphi_1}\right)\frac{d\varphi_2}{d\omega} = 0$
Q_2-D_1	$v^1 = \varphi_1(\omega) - \ln(x_1\varphi_2(\omega))$ $v^2 = x_1\varphi_2(\omega)$ $\omega = x_2 + c(v^2)^2$	$\exp(\varphi_1)\frac{d\varphi_1}{d\omega} = \varphi_2$ $\varphi_2\frac{d\varphi_1}{d\omega} - \frac{d\varphi_2}{d\omega} = {\varphi_2}^2$
Q_3	$v^1 = x_1^{\frac{-1}{(r+1)}}\varphi_1(\omega)$ $v^2 = x_1^{\frac{r}{r+1}}\varphi_2(\omega)$ $\omega = x_2 - v^2$	$\frac{r+1}{r}\varphi_2 = \frac{d\varphi_1}{d\omega}$ $\varphi_1^r\frac{d\varphi_2}{d\omega} = -1$
Q_3+rD_1	$v^1 = x_1^{\frac{1}{(r-1)}}\varphi_1(v^2)$ $v^2 = x_2 - x_1^{\frac{r}{r-1}}\varphi_2(v^2)$	$\frac{r}{1-r}\varphi_2 = \frac{d\varphi_1}{dv^2}$ $\varphi_1^{-r}\frac{d\varphi_2}{dv^2} = -1$
Q_4	$v^1 = \varphi_2(x_1) - \arcsin\frac{x_2}{\sqrt{\varphi_1(x_1)}}$ $v^2 = \sqrt{\varphi_1(x_1) - x_2^2}$	$\frac{d\varphi_1}{dx_1} = -2$ $\tan\varphi_2 = 0$
Q_5	$v^1 = \frac{\varphi_1(\omega)}{x_1}$ $v^2 = \sqrt{\varphi_2(\omega) - x_2^2}$ $\omega = \ln x_1 - \arcsin\frac{x_2}{\sqrt{\varphi_2}}$	$\frac{d\varphi_1}{d\omega} = \frac{1}{2}\frac{d\varphi_2}{d\omega}$ $\frac{d\varphi_2}{d\omega} = \frac{2\tan\{\ln(\varphi_1-\omega)\}\varphi_2}{1-\tan\{\ln(\varphi_1-\omega)\}}$

We stress that the obtained ansatzes cannot be found by classical point symmetries.

4 Discrete symmetries of Maxwell equations

Let us consider the Maxwell equations for vacuum in Dirac form [4]:

$$L\Psi = (p_0 - \vec{\alpha}\vec{p})\Psi = 0 \tag{5}$$

where Ψ is a 4-component function:

$$\Psi = \begin{pmatrix} 0 \\ E_1 - iH_1 \\ E_2 - iH_2 \\ E_3 - iH_3 \\ E_4 - iH_4 \end{pmatrix}, \qquad \alpha_a\alpha_b + \alpha_b\alpha_a = 2\delta_{ab}$$

Equation 5 admits the following operators of discrete symmetry (ODS):

$$Q_1 = \alpha_1 R_{23}, \qquad Q_2 = \alpha_2 R_{13}, \qquad Q_3 = \alpha_3 R_{12}, \qquad Q_4 = CT$$

(where R_{ab}: $x_a \to -x_a$, $x_b \to -x_b$, $a \neq b$, $a, b = 1, 2, 3$, C is operator of complex conjugation, T is operator of time inversion) as much as the invariance condition $[L, Q]\,\Psi = 0$ holds.

It is easy to verify that these ODS satisfy the following commutation relations ($a = 1, 2, 3$):

$$[Q_1, Q_2] = 2iQ_3, \qquad [Q_1, Q_3] = 2iQ_2, \qquad [Q_2, Q_3] = 2iQ_1, \qquad [Q_a, Q_4] = 0$$

and form $AO(3) \oplus Q_4$ algebra. Finding the diagonalizing operators for Q_1 and Q_2 we reduce the equation (5) to four independent one-component equations. Diagonalizing operator of Q_1 has the form:

$$U = U_1 U_2 U_3, \qquad U_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad U_1^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U_2 = \frac{(1 + i\sigma_1)}{\sqrt{2}} \otimes I_2, \qquad U_2^{-1} = \frac{(1 - i\sigma_1)}{\sqrt{2}} \otimes I_2,$$

$$U_3 = \left(R_+^{23} + i\alpha_2 R_-^{23}\right), \qquad U_3^{-1} = \left(R_+^{23} - i\alpha_2 R_-^{23}\right), \qquad R_\pm^{23} = \frac{1 \pm R_{23}}{2}, \qquad \Psi' = U\Psi.$$

Thus the corresponding reducing equation has the form:

$$L'\Psi' = U_1 U_2 U_3 L U_3^{-1} U_2^{-1} U_1^{-1} \Psi' = \{p_0 - ip_2 R_{23} \mp p_1 R_{23} \pm p_3\}\Psi' = 0.$$

In a similar way we find that the diagonalizing operator of Q_2

$$U = U_1 U_2, \quad U_1 = \frac{1}{2}\begin{pmatrix} i & i & 1 & 1 \\ -1 & 1 & i & -i \\ -1 & -1 & -i & -i \\ i & -i & -1 & 1 \end{pmatrix}, \qquad U_1^{-1} = \frac{1}{2}\begin{pmatrix} -i & -1 & -1 & -i \\ -i & 1 & -1 & i \\ 1 & -i & i & -1 \\ 1 & i & i & 1 \end{pmatrix},$$

$$U_2 = (R_+ + i\alpha_1 R_-), \qquad U_2^{-1} = (R_+ - i\alpha_1 R_-)$$

reduces the corresponding equation (5) to the following uncoupled equations:

$$L'\Psi' = U_1 U_2 L U_2^{-1} U_1^{-1} \Psi' = \{p_0 - ip_1 R_{13} \mp p_2 R_{13} \mp p_3\}\Psi' = 0.$$

5 Conclusion

We have shown how we can use non-Lie symmetries (discrete, conditional and non-point ones) for reducing a given problem to a simpler one and to construct its exact solutions. It enabled us to find new classes of nonlinear heat equations for which the reduction method can be successfully applied. With our symmetries equation (1) can be reduced to system of ODEs. We are then able to derive the exact solutions of the heat equation (4) with nonlinearities:

$$C_1(\chi) = -\frac{2}{\chi^5}\exp\left(\frac{1}{\chi}\right), \qquad C_2(\chi) = \frac{1}{\chi(C\chi + 1)}, \quad C = \text{const},$$

$$C_3(\chi) = \frac{1}{r}\left\{\frac{(\chi - 1)^{(1-r)/r}}{\chi^{\frac{2+r}{r}}}\right\}, \qquad C_4(\chi) = \frac{1}{1 + \chi^2}, \qquad C_5(\chi) = \frac{\arctan\chi\exp(\arctan\chi)}{1 + \chi^2}.$$

As far as we know, this is the first application of the non-point and conditional symmetries of an evolution equation for construction its exact solutions. In our opinion using of generalized conditional symmetry in the proposed approach is very promising. It would be interesting to develop this technique for searching exact solutions of DEs with supersymmetries and extended supersymmetries.

Thus extending of symmetry ideas (non-point symmetry, conditional symmetry, supersymmetry) enables us to obtain new results which are important from both mathematical and physical point of view.

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