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Towards Noncommutative Integrable Equations[1]

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We study an extension of integrable equations which possess the Lax representations to noncommutative spaces. We construct various noncommutative Lax equations by Lax-pair generating technique and Sato theory. Sato theory has revealed essential aspects of the integrability of commutative soliton equations and the noncommutative extension is worth studying. We succeed in deriving various noncommutative hierarchy equations in the framework of Sato theory, which is brand-new. The existence of the hierarchy would suggest a hidden infinite-dimensional symmetry in the noncommutative Lax equations. We finally show that a noncommutative version of Burgers equation is completely integrable because it is linearizable via noncommutative Cole–Hopf transformation. These results are expected to lead to the completion of noncommutative Sato theory.

1 Introduction

The extension of ordinary integrable systems to noncommutative (NC) spaces is one of hot topics in the recent study of integrable systems [1–26]. NC extension in gauge theories corresponds to the presence of background magnetic fields and leads to the discovery of many new physical objects and successful applications to string theories [27]. In particular, NC (anti-)self-dual Yang–Mills (YM) equations are integrable and important [28].

On the other hand, many typical integrable equations such as the Korteweg–de Vries (KdV) equation contain no gauge field and the NC extension of them perhaps might have no physical picture. NC extension of $(1 + 1)$-dimensional nonlinear equations introduces infinite number of time derivatives and it becomes very hard to define the integrability. Nevertheless, some of them actually possess integrable properties, such as the existence of infinite number of conserved quantities [3, 14]. Furthermore, a few of them can be derived from NC (anti-)self-dual YM equations by suitable reductions [8, 24]. This fact may give some physical meanings and good properties to the lower-dimensional NC field equations. Now it is time to study in detail whether they are actually integrable or not.

In this article, we present various NC equations which possess the Lax representations. We mainly discuss the Lax-pair generating technique and applications of Sato theory. Sato theory is one of the most beautiful soliton theories and reveals various integrable aspects of soliton equations, such as the existence of multi-soliton solutions, the structure of the solution space and the hidden symmetry of them. Hence NC extension of Sato theory is worth studying and the present discussions on it are all new. Here we prove the existence of various NC hierarchy equations, which would suggest hidden infinite-dimensional symmetries of NC equations. Finally we discuss the integrability of NC $(1 + 1)$-dimensional Burgers equation and prove that it is completely integrable due to the linearizability.

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2 Noncommutative field equations

NC spaces are defined by the noncommutativity of the coordinates: $[x^i, x^j] = i\theta^{ij}$, where θ^{ij} are real constants and are called the *NC parameters*. NC field theories can be defined by replacement of ordinary products of fields in the commutative theories with the *star-product*. The star-product is defined for ordinary fields on commutative spaces and explicitly given by

$$f \star g(x) := \exp\left(\frac{i}{2}\theta^{ij}\partial_i^{(x')}\partial_j^{(x'')}\right) f(x')g(x'')\Big|_{x'=x''=x}, \tag{1}$$

where $\partial_i^{(x')} := \partial/\partial x'^i$ and so on. The star-product has associativity: $f \star (g \star h) = (f \star g) \star h$ and returns to the ordinary product in the commutative limit: $\theta^{ij} \to 0$. The modification of the product makes the ordinary spatial coordinate "noncommutative", that is, $[x^i, x^j]_\star := x^i \star x^j - x^j \star x^i = i\theta^{ij}$.

We note that the fields themselves take c-number values as usual and the differentiation and the integration for them are well-defined as usual. A nontrivial point is that NC field equations contain infinite number of derivatives in the nonlinear terms. Hence the integrability of the equations is not so trivial as in commutative cases. For detailed discussion on it, see [24].

3 Noncommutative Lax equations

A given NC differential equation is said to have the Lax representation if there exists a suitable pair of operators (L, B) so that the following equation:

$$[\partial_t - B, L]_\star = 0, \tag{2}$$

is equivalent to the given NC differential equation. Here the star-product does not affect the derivative operator, for example, $\partial_t \star \partial_x = \partial_t \partial_x$. The pair of operators (L, B) and the equation (2) are called the *Lax pair* and the *NC Lax equation*, respectively.

On NC spaces, the meaning of Lax representations would be vague [24]. However, they actually have close connections with the bi-complex method [2] which leads to infinite number of conserved quantities, and the (anti)-self-dual YM equation which is integrable in the context of twistor descriptions and ADHM constructions [28].

Now let us construct NC Lax equations by the *Lax-pair generating technique*. The technique is a method to find a corresponding B-operator for a given L-operator and based on the following ansatz for the B-operator:

$$B = \partial_i^n L^m + B'. \tag{3}$$

Then the problem reduces to that for the B'-operator which is determined by hand so that the Lax equation should be a differential equation without bare differential ∂_i.

In order to explain the steps, for example, let us consider the KdV equation on NC $(1+1)$-dimensional space-time where the coordinate and the noncommutativity are denoted by (x, t) and $[t, x] = i\theta$, respectively.

NC KdV equation [3, 8, 17, 21]. The L-operator for KdV equation is given by $L_{\mathrm{KdV}} = \partial_x^2 + u(x,t)$. The ansatz for the operator B is of the following type:

$$B_{\mathrm{KdV}} = \partial_x L_{\mathrm{KdV}} + B' = \partial_x^3 + u\partial_x + u_x + B', \tag{4}$$

where $u_x := \partial u/\partial x$. The Lax equation (2) leads to the equation for the unknown operator B':

$$[\partial_x^2 + u, B'] = u_x \partial_x^2 + u_x \star u - u_t. \tag{5}$$

Here the bare derivative term $u_x\partial_x^2$ is troublesome. In order to eliminate it, let us take the following ansatz for B': $B' = X\partial_x + Y$, where X and Y are polynomial of $u, u_x, u_t, u_{xx} := \partial^2 u/\partial x^2$ etc. The Lax equation (2) reduces to

$$(2X-u)_x\partial_x^2 + (X_{xx} + [u,X] + 2Y_x)\partial_x + (Y_{xx} + [u,Y] - X \star u_x + u_t - u_x \star u) = 0. \qquad (6)$$

The condition that the coefficients of ∂_x^2 and ∂_x should vanish yields differential equations for X and Y, which are easily solved by $X = \frac{1}{2}u$, $Y = -\frac{1}{4}u_x$. Now the Lax equation (2) becomes a differential equation, that is, the NC KdV equation:

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u \star u)_x, \qquad (7)$$

where $(u\star u)_x := u_x \star u + u \star u_x$. The nonlinear term becomes symmetric and the equation shows just a conservation law. In the star-product formalism, the spatial integration is well-defined. Hence the spatial integrations of current densities are conserved quantities as in commutative cases [24]. Here $Q := \int dxu$ is conserved, that is, $\partial_t Q = 0$.

In this way, we can generate a wide class of Lax equations on NC (2+1) and (1+1)-dimensional space-times. In particular, this method is suitable for higher-dimensional extension both on commutative spaces [29] and NC spaces [17]. Here we present some results of (2+1)-dimensional NC Lax equations where the coordinate is denoted by (x, y, t). The noncommutativity is basically introduced in space-space directions: $[x, y] = i\theta$. For more discussions and examples, see [17,21].

NC Kadomtsev–Petviashvili (KP) equation [11, 17, 21]

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u \star u)_x + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}\left[u, \partial_x^{-1}u_y\right]_\star, \qquad (8)$$

where $\partial_x^{-1}f(x) := \int^x dx' f(x')$. The Lax pair is given by

$$\begin{aligned} L_{\rm KP} &= \partial_x^2 + u(x,y,t) + \partial_y =: L'_{\rm KP} + \partial_y, \\ B_{\rm KP} &= \partial_x L'_{\rm KP} + X\partial_x + Y = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x - \frac{3}{4}\partial_x^{-1}u_y. \end{aligned} \qquad (9)$$

There is seen to be a nontrivial deformed term $[u, \partial_x^{-1}u_y]_\star$ in the equation (8), which vanishes in the commutative limit.

NC Bogoyavlenski–Calogero–Schiff (BCS) equation [17, 21]

$$u_t = \frac{1}{4}u_{xxy} + \frac{1}{2}(u \star u)_y + \frac{1}{4}u_x \star (\partial_x^{-1}u_y) + \frac{1}{4}(\partial_x^{-1}u_y) \star u_x + \frac{1}{4}[u, \partial_x^{-1}[u, \partial_x^{-1}u_y]_\star]_\star, \qquad (10)$$

whose Lax pair and the ansatz are

$$\begin{aligned} L_{\rm BCS} &= \partial_x^2 + u(x,y,t), \\ B_{\rm BCS} &= \partial_y L_{\rm BCS} + X\partial_x + Y = \partial_x^2\partial_y + u\partial_y + \frac{1}{2}(\partial_x^{-1}u_y)\partial_x + \frac{3}{4}u_y - \frac{1}{4}\partial_x^{-1}[u, \partial_x^{-1}u_y]_\star. \end{aligned} \qquad (11)$$

This time, a nontrivial term is found even in the B-operator. In commutative limit, this coincides with the BCS equation which has multi-soliton solutions [30].

4 Noncommutative hierarchy equations and Sato theory

In this section, we derive various noncommutative hierarchy equations in the framework of Sato theory [31] introducing the pseudo-differential operator.

Let us introduce the following Lax operator as a (first-order) pseudo-differential operator:

$$L = \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + u_4\partial_x^{-3} + \cdots, \qquad u_k = u_k(t_1, t_2, t_3, \ldots). \qquad (12)$$

The action of the operator ∂_x^n on a multiplicity operator f is given by

$$\partial_x^n \cdot f := \sum_{i\geq 0} \begin{pmatrix} n \\ i \end{pmatrix} (\partial_x^i f)\partial^{n-i} := \sum_{i\geq 0} \frac{n(n-1)\cdots(n-i+1)}{i(i-1)\cdots 1}(\partial_x^i f)\partial^{n-i}. \tag{13}$$

We note that the definition can be extended to negative n. The examples are,

$$\begin{aligned}
\partial_x^{-1} \cdot f &= f\partial_x^{-1} - f_x\partial_x^{-2} + f_{xx}\partial_x^{-3} - \cdots, \\
\partial_x^{-2} \cdot f &= f\partial_x^{-2} - 2f_x\partial_x^{-3} + 3f_{xx}\partial_x^{-4} - \cdots, \\
\partial_x^{-3} \cdot f &= f\partial_x^{-3} - 3f_x\partial_x^{-4} + 6f_{xx}\partial_x^{-5} - \cdots,
\end{aligned}$$

where ∂_x^{-1} in the RHS acts as an integration operator $\int^x dx$. Products of pseudo-differential operators are also well-defined and the total set of pseudo-differential operators forms an operator algebra. For more on pseudo-differential operators and Sato theory, see e.g. [32].

The Lax representation for a hierarchy in Sato's framework is defined as

$$[\partial_{t_m} - B_m, L]_\star = 0, \qquad m = 1, 2, \ldots, \tag{14}$$

where B_m is given here by

$$B_m := (\underbrace{L \star \cdots \star L}_{m \text{ times}})_{\geq 0} =: (L^m)_{\star\geq 0}. \tag{15}$$

The suffix "≥ 0" represents the positive and 0-th power part of L^m. The examples are

$$B_1 = \partial_x, \quad B_2 = \partial_x^2 + 2u_2, \quad B_3 = \partial_x^3 + 3u_2\partial_x + 3(u_3 + u_{2x}), \quad \ldots. \tag{16}$$

The noncommutativity is introduced for infinite number of "time variables" $(t_1, t_2, \ldots)$. As it can be taken arbitrarily, we do not fix the noncommutativity here.

NC KP hierarchy. The hierarchy (14) gives rise to NC KP hierarchy which contains the NC KP equation (8). The coefficients of each powers of (pseudo-)differential operators yield infinite series of NC "evolution equations", that is, for $m = 1$

$$\partial_x^{1-k}) \quad u_{kt_1} = u_{kx}, \quad k = 2, 3, \ldots \quad \Rightarrow \quad t_1 \equiv x, \tag{17}$$

for $m = 2$

$$\begin{aligned}
&\partial_x^{-1}) \quad u_{2t_2} = u_{2xx} + 2u_{3x}, \\
&\partial_x^{-2}) \quad u_{3t_2} = u_{3xx} + 2u_{4x} + 2u_2 \star u_{2x} + 2[u_2, u_3]_\star, \\
&\partial_x^{-3}) \quad u_{4t_2} = u_{4xx} + 2u_{5x} + 4u_3 \star u_{2x} - 2u_2 \star u_{2xx} + 2[u_2, u_4]_\star, \quad \ldots
\end{aligned} \tag{18}$$

and for $m = 3$

$$\begin{aligned}
&\partial_x^{-1}) \quad u_{2t_3} = u_{2xxx} + 3u_{3xx} + 3u_{4x} + 3u_{2x} \star u_2 + 3u_2 \star u_{2x}, \\
&\partial_x^{-2}) \quad u_{3t_3} = 3u_{4xx} + 6u_2 \star u_{3x} + 3u_{2x} \star u_3 + 3u_3 \star u_{2x} + 3[u_2, u_4]_\star, \quad \ldots.
\end{aligned} \tag{19}$$

These just imply the NC KP equation (8) with $2u_2 \equiv u$, $t_2 \equiv y$, $t_3 \equiv t$. Important point is that infinite kind of fields $u_3, u_4, u_5, \ldots$ are represented in terms of one kind of field $2u_2 \equiv u$ as is seen in equation (18). This guarantees the existence of NC KP hierarchy and the infinite differential equations are called the *NC (KP) hierarchy equations*.

Putting the constraint $L^l = B_l$ on the NC KP hierarchy (14), we get infinite set of NC hierarchies. Then the following NC hierarchy:

$$\left[\partial_{t_m} - B_m, L^l\right]_\star = 0, \tag{20}$$

becomes a NC differential equation. We can easily show

$$\frac{\partial u}{\partial t_{Nl}} = 0, \qquad N = 1, 2, \ldots \tag{21}$$

because of $B_{Nl} = L^{Nl}$. The reduced NC hierarchy is called the *l-reduction* of NC KP hierarchy.

NC KdV hierarchy (2-reduction of NC KP). Taking the constraint $L^2 = B_2$, we get the KdV hierarchy. The Lax equation:

$$\frac{\partial u}{\partial t_m} = \left[B_m, L^2\right]_\star, \tag{22}$$

gives rise to the m-th KdV hierarchy equation which becomes, for example, the (third) NC KdV equation (7) with $t_3 \equiv t$ and the 5-th NC KdV equation [17]:

$$u_{t_5} = \frac{1}{16}u_{xxxxx} + \frac{5}{16}(u \star u_{xxx} + u_{xxx} \star u) + \frac{5}{8}(u_x \star u_x + u \star u \star u)_x. \tag{23}$$

In Lax-pair generating technique, the B_m-operator is given by the ansatz $B_m = \partial_x^{m-2} L_{\mathrm{KdV}} + B'$.

NC Boussinesq hierarchy (3-reduction of NC KP). The 3-reduction $L^3 = B_3$ yields the NC Boussinesq hierarchy. The Lax equation:

$$\frac{\partial u}{\partial t_m} = \left[B_m, L^3\right]_\star, \tag{24}$$

leads to the m-th Boussinesq hierarchy equation which is, for $m = 3$, the NC Boussinesq equation [17]:

$$u_{tt} = \frac{1}{3}u_{xxxx} + (u \star u)_{xx} + ([u, \partial_x^{-1} u_t]_\star)_x. \tag{25}$$

In this way, we can generate infinite set of the l-reduced NC hierarchies. Furthermore, if we take other set-up for the definition of pseudo-differential operator L and "time-evolution" operator B_m, we can get many other hierarchies as follows.

NC modified KdV (mKdV) hierarchy. If we take

$$L = \partial_x + u_1 + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots, \qquad B_m = (L^m)_{\star \geq 1}, \tag{26}$$

and put the constraint $L^2 = B_2$, infinite kind of fields $u_2, u_3, \ldots$ are represented in terms of one kind of field $2u_1 \equiv v$. We can easily see $\partial_{t_{2N}} v = 0$. The set of Lax equations:

$$\left[B_m - \partial_{t_m}, L^2\right]_\star = 0 \tag{27}$$

yields NC mKdV hierarchy which implies the NC mKdV equation for $m = 3$ with $t_3 \equiv t$:

$$\frac{\partial v}{\partial t} = \frac{1}{4}v_{xxx} - \frac{3}{8}v \star v_x \star v + \frac{3}{8}[v, v_{xx}]_\star. \tag{28}$$

We note that the NC mKdV equation (28) is different from that in [3] from NC KdV equation (7) via NC Miura map $u = -v \star v - v_x$. The NC mKdV hierarchy can be considered as the 2-reduction of NC mKP hierarchy. However the existence of the NC mKP hierarchy seems to be nontrivial because it cannot be represented in terms of one kind of field. (The NC mKP equation given in early versions of [21] is incorrect.) For the same reason, the higher reduction of the NC mKP hierarchy seems to be hard to obtain.

NC Burgers hierarchy [24]. If we take

$$L = \partial_x + u_1 + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots, \qquad B_m = (L^m)_{\star \geq 1}, \tag{29}$$

and put the constraint $L = B_1 =: \partial_x + v$, the hierarchy:

$$\partial_{t_m} v = [B_m, L]_\star \tag{30}$$

yields the NC Burgers hierarchy which implies the NC Burgers equation:

$$\frac{\partial v}{\partial t_2} = [B_2, L]_\star = \left[\partial_x^2 + 2v\partial_x, \partial_x + v\right]_\star = v_{xx} + 2v \star v_x. \tag{31}$$

and the third-order NC Burgers equation:

$$\frac{\partial v}{\partial t_3} = [B_3, L]_\star = v_{xxx} + 3v \star v_{xx} + 3v_x \star v_x + 3v \star v \star v_x \tag{32}$$

and so on. The nonlinear terms are not symmetric, which will be proved to be a key point in the linearization in the next section. The B_m-operator is given in Lax-pair generating technique by $B_m = \partial_x^{m-1} L + B'$. This time, the generated equations contain some parameters and thus covers wider class of Lax equations [24].

5 Integrability of noncommutative Burgers equation

In this section, we discuss the integrability of the Burgers equation (31) on NC (1+1)-dimensional space-time where the coordinate and the noncommutativity are denoted by $(x, t) \equiv (t_1, t_2)$ and $[t, x] = i\theta$, respectively.

In commutative case, it is well known that the Burgers equation is linearized by the Cole–Hopf transformation. The discussion can be extended to noncommutative case [24, 25]. NC Burgers equation (31) can be linearized by the following noncommutative analogue of the Cole–Hopf transformation: $v = \psi^{-1} \star \psi_x$. The linearized equation is a (NC) diffusion equation: $\psi_t = \psi_{xx}$. The naive solution of the above NC diffusion equation is

$$\psi(t, x) = 1 + \sum_{i=1}^{N} h_i e^{k_i^2 t} \star e^{\pm k_i x} = 1 + \sum_{i=1}^{N} h_i e^{\frac{i}{2} k_i^3 \theta} e^{k_i^2 t \pm k_i x}, \tag{33}$$

where h_i, k_i are complex constants. In the commutative limit, this reduces to the N-shock wave solution in fluid dynamics. The final form in (33) shows that the N-shock wave solution is deformed by $e^{\frac{i}{2} k_i^3 \theta}$ due to the noncommutativity. The explicit representation in terms of v is hard to obtain because the derivation of ψ^{-1} in the NC Cole–Hopf transformation is nontrivial. However we can discuss the asymptotic behaviors at $t \to \pm\infty$ and actually see the effect of the NC deformation. In fact, the exact solutions for $N = 1, 2$ are obtained in [25] and nontrivial effects of the NC-deformation are reported. We note that NC one shock-wave solutions can always reduce to the commutative ones because $f(t - x) \star g(t - x) = f(t - x)g(t - x)$ [24].

The results show that the NC Burgers equation (31) is completely integrable even though the NC Burgers equation contains infinite number of time-derivatives in the nonlinear term. The linearized equation is a differential equation of first order with respect to time and the initial value problem is well-defined. This is a surprising result. The (NC) diffusion equation can be solved for arbitrarily boundary conditions by the Fourier transformation. Furthermore, we note that the form of the nonlinear term in the NC Burgers equation (31) is crucial for the linearization. If it becomes symmetric like the NC KdV equation (7), the linearization is proved to be impossible [24].

6 Conclusion and discussion

In this article, we presented various NC Lax equations and proved the existence of many NC hierarchies. The NC extension of Sato theory is new. We also confirmed that NC Burgers equation is linearizable and completely integrable even though it is a differential equation of infinite order with respect to time. The linearized equation is a (NC) diffusion equation and can be solved in usual ways.

NC extension of Ward conjecture [33] would be very interesting [21] though we have omitted it in this article because of limitations of space. Some NC equations are actually derived from NC (anti-)self-dual YM equations by reduction [8, 24] and embedded [9, 12, 34] in $N = 2$ string theory [35]. This guarantees that NC integrable equations would have a physical meaning and might be helpful to understand new aspects of the corresponding string theory.

The next step is NC extension of Hirota's bilinearization [36]. This could be realized as a simple generalization of the Cole–Hope transformation whose extension to NC spaces are already successful as shown in Section 5. Hirota's bilinearization leads to the theory of tau-functions. Sato theory is based on the existence of hierarchies and tau-functions. We have just revealed the existence of hierarchies and the completion of NC Sato theory would be drawing near at hand.

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