Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 1, 260–265

Dynamical Symmetries of Autonomous Differential Second Order Equations

Mikhail I. TIMOSHIN

Samara State Pedagogical University, Russia
E-mail: *midvolga@mail.ru*

In the report the possibility of dynamic symmetries to an analytical integration of autonomous ODE of second order is presented. The correlation with known results is pointed out. The concept of a defining system of dynamic symmetries is given. The example indicating to the effectiveness of an offered method is produced.

1 Introduction

The concept of dynamic symmetry is given, for example, in [1]. The differential equation, in this case, is replaced by ODE system of the first order:

$$y'' = f(x, y, y') \;\Leftrightarrow\; \frac{dy}{dx} = z, \qquad \frac{dz}{dx} = f(x, y, z). \tag{1}$$

Then it is possible to consider the question of an infinitesimal transformation

$$X = \xi(x, y, z)\frac{\partial}{\partial x} + \eta(x, y, z)\frac{\partial}{\partial y} + \mu(x, y, z)\frac{\partial}{\partial z}, \tag{2}$$

translating a solution of system (1) again into a solution. That is operator (2) should satisfy the condition

$$[X, A] = \lambda(x, y, z)A, \tag{3}$$

where

$$A = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x, y, z)\frac{\partial}{\partial z}.$$

As against the contact symmetries, set by the characteristic function $\Omega(x, y, z)$

$$X = \Omega_z\frac{\partial}{\partial x} + (z\,\Omega_z - \Omega)\frac{\partial}{\partial y} - (\Omega_x + z\,\Omega_y)\frac{\partial}{\partial z},$$

of the operator of dynamic symmetry (2), the components ξ, η, μ are defined only by condition (3).

2 Dynamical symmetries of autonomous differential second order equations

Let us consider the autonomous differential equation of the second order

$$y'' = f(y, y'). \tag{4}$$

The equation (4) admits point symmetry $X_1 = \frac{\partial}{\partial x}$ and dynamic symmetry X_2. Following the scheme of S. Lie, we shall write out the most general transformations retaining autonomy (4). For this purpose it is necessary that the transformed system should suppose symmetry $\bar{X}_1 = \frac{\partial}{\partial t}$. Thus two cases are possible:

I uncrossed transformation

$$
\begin{array}{ccc}
X_1 & \Leftrightarrow & \bar{X}_1 \\
 & & \\
X_2 & \Leftrightarrow & \bar{X}_2
\end{array}
$$

II crossed transformation

$$
\begin{tikzcd}
X_1 \arrow[dr, leftrightarrow] & \bar{X}_1 \arrow[dl, leftrightarrow] \\
X_2 & \bar{X}_2
\end{tikzcd}
$$

In case of **I** transformations have the form

$$
x = t + \alpha(u, w), \qquad y = \beta(u, w), \qquad z = \gamma(u, w). \tag{5}
$$

In case of **II** transformations will be defined by a choice of the form of the transformed dynamic system. By taking dynamic system

$$
\frac{du}{dt} = R(w)u + P(w), \qquad \frac{dw}{dt} = Q(w), \tag{6}
$$

it is possible to speak about linearizing of the equation (4) according to the circuit **I** or **II**. In case of **I** the class of linearizing equations is most clearly defined

$$
y'' = \begin{vmatrix} \dfrac{d^2\beta}{dt^2} & \dfrac{d\beta}{dt} \\ \dfrac{d^2(t+\alpha)}{dt^2} & \dfrac{d(t+\alpha)}{dt} \end{vmatrix} \Bigg/ \left(\frac{d\,(t+\alpha)}{dt} \right)^3. \tag{7}
$$

The general solution of the equation (7) has the form:

$$
\begin{aligned}
x &= \int \frac{dw}{Q(w)} + \alpha(u, w) + C_1, \qquad y = \beta(u, w), \\
u &= e^{\int \frac{R(w)}{Q(w)} dw} \left(\int \frac{P(w)}{Q(w)} e^{-\int \frac{R(w)}{Q(w)} dw} dw + C_2 \right).
\end{aligned} \tag{8}
$$

Before consideration of type **II** transformation we shall present additional information concerning dynamic symmetries. In the book [1] it is pointed out that if the symmetry X of dynamic system of the equations determined by the operator A is known, then

$$
\hat{X} = X + \rho(x, y, z)A \tag{9}
$$

with any function $\rho(x, y, z)$, is also a dynamic symmetry. That is explained by the fact that the following

$$
[\hat{X}, A] = \lambda A - (A\rho)A = \hat{\lambda} A
$$

takes place. Using ratio (9) it is always possible to proceed from the operator X to the operator $\widehat{X}$ at which the coefficient at $\frac{\partial}{\partial x}$ will be equal to zero. In the case of point symmetry

$$
X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \cdots \tag{10}
$$

and a differential equation of the second order $y'' = f(x, y, y')$ we shall obtain

$$
\hat{X} = (\eta - y'\xi)\frac{\partial}{\partial y} + \cdots . \tag{11}
$$

In the book [2] the proof by Boyer (1967) that the operators (10), (11) are equivalent is presented. By repeating Boyer's reasonings it is easy to show that in case of dynamic system

$$\frac{dy}{dx} = \varphi(x,y,z), \qquad \frac{dz}{dx} = f(x,y,z), \qquad A = \frac{\partial}{\partial x} + \varphi(x,y,z)\frac{\partial}{\partial y} + f(x,y,z)\frac{\partial}{\partial z},$$

generators

$$X = \xi(x,y,z)\frac{\partial}{\partial x} + \eta(x,y,z)\frac{\partial}{\partial y} + \mu(x,y,z)\frac{\partial}{\partial z} \qquad \text{and} \qquad \hat{X} = (\eta - \varphi\xi)\frac{\partial}{\partial y} + (\mu - f\xi)\frac{\partial}{\partial z}$$

are equivalent. Let us consider now linearization of the second type. By taking as new variables the first integrals of the system (6)

$$s = ue^{-\int \frac{R(w)}{Q(w)}dw} - \int \frac{P(w)}{Q(w)} e^{-\int \frac{R(w)}{Q(w)}dw} dw, \qquad v = \int \frac{dw}{Q(w)} - t,$$

we shall obtain the following system

$$\frac{ds}{dt} = 0, \qquad \frac{dv}{dt} = 0. \tag{12}$$

The system (12) has the dynamic symmetry

$$X = \xi(t,s,v)\frac{\partial}{\partial t} + \eta(s,v)\frac{\partial}{\partial s} + \mu(s,v)\frac{\partial}{\partial v}$$

with any functions $\xi(t,s,v)$, $\eta(s,v)$, $\mu(s,v)$. Going back to variables t, u, w we shall obtain the dynamic symmetry of system (6)

$$X = (\xi + \mu)\frac{\partial}{\partial t} + \left[(\xi + \mu)(R(w)u + P(w)) + \eta e^{\int \frac{R(w)}{Q(w)}dw}\right]\frac{\partial}{\partial u} + (\xi + \mu)Q(w)\frac{\partial}{\partial w}.$$

Going from the found symmetry to the equivalent operator, we obtain dynamic symmetry in the following form

$$\hat{X} = \eta e^{\int \frac{R(w)}{Q(w)}dw}\frac{\partial}{\partial u}. \tag{13}$$

The most general transformation reducing the generator (13) to $\frac{\partial}{\partial x}$ has the form

$$x = \alpha(s,v) + \Phi(t,w), \qquad y = \Psi(t,w), \qquad z = \Omega(t,w), \tag{14}$$

where $\alpha_s = \frac{1}{\eta(s,v)}$. The transformation (14) allows to proceed from system (6) to system

$$\frac{dy}{dx} = \left(\frac{d}{dt}\Psi(t,w)\right) \Bigg/ \left(\frac{d}{dt}\Phi(t,w)\right), \qquad \frac{dz}{dx} = \left(\frac{d}{dt}\Omega(t,w)\right) \Bigg/ \left(\frac{d}{dt}\Phi(t,w)\right). \tag{15}$$

To bring system (15) into correspondence with equation (4) it is necessary to put

$$z = \Omega(t,w) = \left(\frac{d\Psi}{dt}\right) \Bigg/ \left(\frac{d\Phi}{dt}\right). \tag{16}$$

Thus we obtain the second class of linearizing autonomous ODE of the second order as

$$y'' = \begin{vmatrix} \dfrac{d^2\Psi}{dt^2} & \dfrac{d\Psi}{dt} \\ \dfrac{d^2\Phi}{dt^2} & \dfrac{d\Phi}{dt} \end{vmatrix} \Bigg/ \left(\frac{d\Phi}{dt}\right)^3. \tag{17}$$

The general solution of the equation (17) looks as follows:

$$x = C_1 + \Phi(t, w), \qquad y = \Psi(t, w), \qquad t = C_2 + \int \frac{dw}{Q(w)}. \tag{18}$$

Comparing the general solutions (8), (18) it is easy to notice that they will coincide if in (8) functions α, β are considered as depending on arguments s, w: $\alpha = \alpha(s, w)$, $\beta = \beta(s, w)$, and in (18) functions Φ, Ψ depending on arguments v, w: $\Phi = \Phi(v, w)$, $\Psi = \Psi(v, w)$. It makes possible to speak not about two classes of autonomous ODE of the second order but about one. Of three functions Ω, Ψ, Φ only two are independent, the third function is determined from a condition (16). By taking as arguments variables t, v it is easy to obtain well known results. With the help of functions $\Omega(t, v)$, $\Psi(t, v)$ the autonomous differential equation of the second order is determined as

$$\Omega \frac{d\Omega}{dt} = \frac{d\Psi}{dt} f(\Psi, \Omega). \tag{19}$$

In spite of the fact that at transition from the equation (4) to the equation (19) there was a reduction of the order, it does not always facilitate a task of integration of the differential equations.

The group approach with the help of criterion (3) allows to receive additional equations of the second order with respect to required functions Ω, Ψ, Φ. In the coordinate form the criterion (3) is equivalent to the system of two equations of the form:

$$\begin{aligned} &\mu - \eta_x - z\eta_y - f\eta_z + z\xi_x + z^2\xi_y + zf\xi_z = 0, \\ &\eta f_y + \mu f_z - \mu_x - z\mu_y - f\mu_z + f\xi_x + zf\xi_y + f^2\xi_z = 0. \end{aligned} \tag{20}$$

The transformation (14) allows to write out the type of dynamic symmetries in variables x, y, z:

$$X = (\Phi_t - \alpha_v)\frac{\partial}{\partial x} + \Psi_t \frac{\partial}{\partial y} + \Omega_t \frac{\partial}{\partial z}.$$

Replacing in the adduced out system variables y, z with the help of the formulas (14), and adding the equations

$$\Psi_t + \Psi_w Q - (\Phi_t + \Phi_w Q)\Omega = 0, \qquad \Omega_t + \Omega_w Q - (\Phi_t + \Phi_w Q) f(\Psi, \Omega) = 0, \tag{21}$$

we can receive a system of four equations with three unknown functions. As against classical system of the determining equations used while finding point symmetries, the constructed system of the equations is not linear. However, it is easy to notice, that with the help of the equations (21) equations with partial derivatives can be reduced to the ordinary differential equations of the second order, in which one of variables plays a role of parameter. It makes possible to apply the well elaborated algorithms of point symmetries to investigation of the constructed system of the determining equations of dynamic symmetries.

3 Example

As a testing example underlining efficiency of the offered technique, we shall consider the equation of the second order having the form

$$y''(A + By') + y'^3 C + y'^2 D + y' E = 0 \tag{22}$$

where

$$A = r_3 l^2 k - r_1 k^3 - r_3 h^2 k^3 - r_2 k^3 h - k^2 g_1 l,$$

$$
\begin{aligned}
B &= 2r_3kl - r_2k^2 - 2r_3hk^2 - k^2g_1,\\
C &= r_2kk_y - r_3k_yl - r_3kl_y + r_3h_yk^2 + 2r_3hkk_y + g_1kk_y,\\
D &= r_1k^2k_y + 2r_3h_yk^2l + g_1kk_yl + r_2kk_yl + 2r_3hkk_yl + r_2hk^2k_y + r_3h^2k^2k_y\\
&\quad - g_1k^3h_y - r_2k^2l_y - 2r_3hk^2l_y - r_3l^2k_y,\\
E &= r_1k^2k_yl + r_2k^2k_yhl + r_3l^2k^2h_y - g_1k^3lh_y + r_3h^2k^2k_yl - r_3h^2k^3l_y - r_1k^3l_y - r_2k^3hl_y,
\end{aligned}
$$

in which k, l, h are arbitrary functions of argument y, and r_1, r_2, r_3, g_1 are arbitrary constants.

Using the standard method of reduction to form (19), from the equation (22) we obtain the Abel equation of the second type

$$
\frac{d\Omega}{dy}(A + B\Omega) + \Omega^2C + \Omega D + E = 0. \tag{23}
$$

The equation (22) is linearized with the help of transformation

$$
u = \frac{k}{z+l} - \frac{r_3+1}{g_1}, \qquad w = \frac{z}{k} + h. \tag{24}
$$

Thus we obtain equation

$$
\left(r_1 + r_2w + r_3w^2\right)\frac{du}{dw} = g_1u + 1,
$$

whose general solution looks as follows:

$$
(u + 1/g_1) - C_2e^{\int \frac{g_1\,dw}{r_1+r_2w+r_3w^2}} = 0. \tag{25}
$$

For representation of the general solution of the equation (22) as (8), it is necessary to define the form of function $\alpha(u,w)$. Using ratio (5), and considering the function α on arguments (y, C_2) we obtain expression

$$
\alpha(y, C_2) = \int\left(\frac{1}{z} + \frac{kl_y - k_yl - h_yk^2}{(kr_2 - 2r_3l + 2r_3hk + kg_1)z - l^2r_3 + k^2r_1 + lkg_1 + r_3h^2k^2 + k^2hr_2}\right)dy.
$$

In this case the integral is calculated with the assumption that variable z is function from y determined by ratios (24), (25). Thus the general solution of equation (22) has the form

$$
\begin{aligned}
&x = \alpha(y, C_2) + \int\frac{dw}{r_1 + r_2w + r_3w^2} + C_1,\\
&\frac{k}{z+l} - \frac{r_3}{g_1} + C_2e^{\int \frac{g_1\,dw}{r_1+r_2w+r_3w^2}} = 0, \qquad w = \frac{z}{k} + h.
\end{aligned}
$$

Let us also note that last two equations give the general solution of Abel equation (23). It is interesting to notice that in variable y, $w(y)$ equation (23) takes the form

$$
\begin{aligned}
&r_1 + r_2w + r_3w^2\\
&\qquad + \frac{dw}{dy}\left\{\frac{w\left(2r_3k(l-kh) - k^2(g_1+r_2)\right) - r_1k^2 + r_3(l-kh)^2 + kg_1(kh-l)}{k_yl - kl_y + h_yk^2}\right\} = 0,
\end{aligned}
$$

and the standard transformation is brought to the canonic form:

$$
\frac{d\vartheta}{d\tau}\vartheta + \vartheta + \frac{3\tau}{16} - \frac{3^{\frac{2}{3}}\left(2g_1^2\lambda^{\frac{4}{3}} - r_3\lambda^{\frac{1}{3}}\right)}{24\tau^{\frac{1}{3}}r_3^2\lambda^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}}{144\tau^{\frac{5}{3}}r_3^2\lambda^{\frac{2}{3}}} = 0, \tag{26}
$$

where

$$\lambda = \frac{r_3}{g_1^2 + 4r_3r_1 - r_2^2}.$$

In directory [3] the transformation of equation of type (26) into another Abel equation and then to Riccati equation is presented. The obtained equation was integrated by Koyalovich [4]. The author of the directory asserts that the form of the obtained solution is very bulky and consequently it was not presented. In [3] solutions of special cases of the equation (26) through Bessel's functions are also given.

4 Concluding remark

In the conclusion we shall note that the proposed approach to investigation of integrated cases of the autonomous differential equations of the second order can be considered as some analogue of the Galois theory for polynomials. In investigating the solubility of the equation

$$P_n = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

the various ratios between roots of x_i equation are considered, i.e. the equation is replaced with a system of the equations for roots $F_1(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = 0$, …, $F_m(x_{i_1}, x_{i_2}, \ldots, x_{i_r}) = 0$, which can be solved easier than the initial equation.

In the proposed method the question of integration of differential equations is actually reduced to investigation of the system of differential equations

$$F_i(x, y, z, x_t, y_t, z_t, x_w, y_w, z_w, x_{tw}, y_{tw}, z_{tw}, x_{tt}, y_{tt}, z_{tt}, x_{ww}, y_{ww}, z_{ww}) = 0,$$

where $i = 1, 2, 3, 4$. And while investigating the constructed system, the standard algorithms of the group analysis can be used.

[1] Stephani H., Differential equations: their solution using symmetries, Cambridge University Press, 1989.

[2] Bluman G. and Kumei S., Symmetries and differential equations, Berlin, Springer, 1989.

[3] Zaitsev V.F. and Polyanin A.D., Directory on ordinary differential equations: precise solutions, Moscow, Fismatgis, 1995.

[4] Koyalovich B.M., Investigatios of differential equation $ydy - ydx = Rdx$, Saint Petersburg, Printshop of Academy of Sciences, 1894.