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Group Explanation for the Conditional Similarity Reductions of the (2+1)-Dimensional KdV Equation

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The conventional Lie group approach is extended successfully to give out the group explanation to the new *conditional* similarity reductions obtained by modifying the Clarkson and Kruskal's (CK's) direct method for the (2+1)-dimensional Korteweg–de Vries (KdV) equation.

1 Introduction

As is well-known that the classical Lie group approach, the nonclassical Lie group approach, the Clarkson and Kruskal's (CK's) direct method are three powerful methods in finding similarity reductions for a given nonlinear partial differential equation (NPDE) [1–6]. In many cases, the similarity reductions obtained by the classical Lie group approach can also be yielded by the CK's direct method; and those obtained by the CK's direct method while not by the classical Lie group approach can be reobtained by the nonclassical Lie group approach. There have been several modifications of these three methods in the literature [7–12].

Three years ago, we for the first time proposed the modified CK's direct method to construct the so-called *conditional* similarity reductions of the (2+1)-dimensional KdV equation in the integrable case [9]. We call a reduction conditional similarity reduction since one reduction field need to satisfy more than one reduction equation. Since then, similar work has been carried out on several other NPDEs including the nonintegrable (2+1)-dimensional KdV equation [10], the Jimbo–Miwa (JM) equation [11] and the Boussinesq equation [12]. It is noticed that the conditional similarity reductions obtained by means of the modified CK's direct method cannot be recovered by utilizing the classical or even the nonclassical Lie group approach in their present forms. The very reason lies in the fact that the constrained equation introduced in the present nonclassical Lie group approach does not offer an additional conditional reduction equation for the reduction field. Consequently, in order to reobtain the conditional similarity reductions by using the classical Lie group approach or the nonclassical Lie group approach, a conditional equation which will lead to the additional reduction equation must be introduced. In Ref. [13], all the conditional similarity reductions of the JM equation resulting from the modified CK's direct method were retrieved by introducing a conditional equation, the KP equation, to form an equation system and then applying the classical Lie group approach to the system. Thus, the whole group theoretical explanation is given of the conditional similarity reductions for the JM equation. In fact, how to introduce a conditional equation so that the conditional similarity reductions got by the modified CK's direct method can also be yielded by the classical Lie group approach and/or the nonclassical Lie group approach has not yet been precisely known.

The aim of this paper is to report the recent progress on the Lie group approach which gives out the group explanation of the conditional similarity reduction solutions obtained by the modified CK's direct method. The modified CK's direct method and the conditional similarity reduction solutions for the (2+1)-dimensional KdV equation will be reviewed in the next section. In Section 3, the conventional Lie group approach is developed further to give out the full group explanation for the conditional similarity reduction solutions given in Section 2. The last section is a short summary and discussion.

2 Conditional similarity reduction solutions

In the traditional CK's direct method, in order to find the similarity solutions of a general Nth order n-dimensional nonlinear system,

$$E_0\left(x_1, x_2, \ldots, x_n, u, u_{x_i}, u_{x_i x_j}, \ldots, u_{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}}\right) \equiv E_0 = 0, \qquad \sum_{j=1}^{n} i_j = N, \tag{1}$$

one seeks solutions of the kind $u = U(x_1, x_2, \ldots, x_n, P(\xi_1, \xi_2, \ldots, \xi_{n-1}))$, where ξ_j, $j = 1, 2, \ldots$, $n-1$ are all functions with respect to $\{x_1, x_2, \ldots, x_n\}$. For some types of models, the solution u may commonly be simplified to the linear ansatz

$$u = \alpha + \beta P(\xi_1, \xi_2, \ldots, \xi_{n-1}), \tag{2}$$

where α and β are both functions of $\{x_1, x_2, \ldots, x_n\}$. In general, it is known that the ansatz (2) may not be valid for other types of models, say, the Harry–Dym equation [3]. Substituting (2) into equation (1) gives out

$$\sum_{l=1}^{L} r_l(\alpha, \alpha_{x_i}, \beta, \beta_{x_i}, \xi_{x_i}, \ldots) F_l(\xi_j, P, P_{\xi_j}, \ldots) \equiv \sum_{l=1}^{L} r_l F_l = 0. \tag{3}$$

Since the similarity reduction function P satisfies only one reduction equation, equation (3) becomes an $(n-1)$-dimensional PDE $G(\xi_j, P, P_{\xi_j}, \ldots) \equiv G(P) = 0$, only for all the ratios of r_l being functions of $\{\xi_1, \xi_2, \ldots, \xi_{n-1}\}$. Namely, $r_l = r_k \Gamma_l$ should be satisfied for some fixed nonzero r_k, and Γ_l are functions of $\{\xi_1, \xi_2, \ldots, \xi_{n-1}\}$.

In order to find the conditional similarity reductions of (1), we relax the condition that *the similarity reduction function P satisfies only one reduction equation* as that P *is allowed to satisfy more than one reduction equation at the same time*. Based on this idea, we make use of the same reduction ansatz (2) and then separate the resulting equation (3) into m parts

$$\sum_{k=1}^{m} \sum_{l=1}^{M} A_{lk} F_l = 0, \qquad M \geq L, \tag{4}$$

with the condition $\sum\limits_{k=1}^{m} A_{lk} = r_l$, $(l \leq L)$, $\sum\limits_{k=1}^{m} A_{lk} = 0$, $(l > L)$, where F_l for $l > L$ may be some suitable functions of P and its partial differential derivatives with respect to $\{\xi_1, \xi_2, \ldots, \xi_{n-1}\}$. Then we can see that the reduction function P may satisfy m reduction equations $\sum\limits_{l=1}^{M} A_{lk} F_l = 0$, $(k = 1, 2, \ldots, m)$. When applying this modified direct method, one must make sure that all the ratios of A_{lk} are functions of $\{\xi_j\}$ for the *same* k, while cannot be functions of $\{\xi_1, \xi_2, \ldots, \xi_{n-1}\}$ for the *different* k. The reason is that if the ratio of any two A_{lk} is a function of $\{\xi_1, \xi_2, \ldots, \xi_{n-1}\}$, then they can be put into the same part of (4). The arbitrariness of m and the functions F_l for $l > L$ makes it a hard job to cover all the cases of the conditional similarity reductions. Therefore, up to now, we have just considered the case $m = 2$ and $F_l \equiv 0$ for $l > L$. Obviously, many more meaningful conditional similarity reductions may be found for $m > 2$ and/or for $F_l \neq 0$ when $l > L$.

Some types of the conditional similarity reductions of

$$u_{xt} - u_{xxxy} - 4u_x u_{xy} - 4u_{xx} u_y = 0, \tag{5}$$

which is the potential form ($v = u_x$) of the (2+1)-dimensional KdV equation

$$v_t - v_{xxy} - 4v v_y - 4v_x \partial_x^{-1} v_y = 0.$$

have been discussed in detail via the modified CK's direct method in [9]. Here, we will list the general known conditional similarity solution of (5) which reads

$$u = \frac{\eta_t}{4\eta_y}x + \frac{1}{4}\int\left(\frac{\sigma_t\eta_y - \sigma_y\eta_t}{\theta\eta_y} + \theta(\omega_{14}\sigma + B\sigma_y) + \gamma_{14}\sigma\eta_y\right)\mathrm{d}y + \theta P, \qquad \xi = \theta x + \sigma,$$

where the function $P = P(\xi, \eta)$ satisfies the conditional similarity reduction equations (i.e. (116) and (117) of Ref. [9])

$$P_{\xi\xi\eta} + 4P_\xi P_\eta + \gamma_{14}(\xi P_\xi - P) + \gamma_{29}P + (\gamma_{21}\eta + \gamma_{20})\xi + f_1 = 0,$$
$$P_{\xi\xi\xi} + 4P_\xi^2 + \omega_{14}(\xi P_\xi - P) + BP_\xi + \omega_{29}P + (\omega_{21}\eta + \omega_{20})\xi + f_0 = 0,$$

with $B = B(\eta)$, $f_1 = f_1(\eta)$ and $f_0 = f_0(\eta)$ being arbitrary functions of η. For the other functions ($\theta = \theta(t)$, $\sigma = \sigma(y,t)$, $\eta = \eta(y,t)$) and constants (γ_{14}, γ_{29}, γ_{21}, γ_{20}, ω_{14}, ω_{29}, ω_{21}, ω_{20}), seven possible selections were given in [9] which will be explicitly written down again in Table 1 later.

3 Group explanation of the conditional similarity reduction solutions

In order to give out the whole group theoretical explanation of the conditional similarity reductions obtained via the modified CK's direct method described in the last section, we have to extend the present classical Lie group approach and the nonclassical Lie group approach.

Simply speaking, the extended classical and nonclassical conditional Lie group approaches are realized if we introduce some constrained equations when solving the model equation and then applying the standard group approach to the formed equation system. As a first attempt, we worked on the JM equation [13] where the conditional similarity reductions yielded by the modified CK's direct method have been recovered totally. However, the constrained equation in this case comes quite specifically so that it cannot be a good candidate for all the NPDEs in observation. Consequently, a more systemic way should be established to find out a common equation which can then be considered for a class of NPDEs. Starting from this standpoint, a more general conditional equation for the integrable (2+1)-dimensional KdV equation is imported when we try to give out the group explanation of the results [9] obtained via the modified CK's direct method, which read

$$u_{xxx} + A_1u_{xxy} + A_2u_xu_y + A_3u_x + A_4u_y + A_5u_x^2 + A_6u + A_7 = 0, \tag{6}$$

with the coefficients A_i, $(i = 1, 2, \ldots, 7)$ being suitable functions with respect to the space-time variables $\{x, y, t\}$ to be determined later. Writing down the constrained equation (6) is based on the fact that the orders of the differentiations and the nonlinearity of the conditional reductions obtained by the modified CK's direct method in [9] are not higher than those of the original system (5). Like the conventional nonclassical Lie group approach, when using the extended nonclassical Lie group approach, we have to utilize the following constraint condition

$$Xu_x + Yu_y + Tu_t - U = 0, \tag{7}$$

where X, Y, T, U of $\{x, y, t, u\}$ are called the infinitesimals of the transformation functions related to the infinitesimal transformations $\{x, y, t, u\} \to \{x, y, t, u\} + \epsilon\{X, Y, T, U\}$ with ϵ being the group parameter, under which the model equation and the constrained equations should be form invariant.

Ignoring the concrete calculations, we directly give out the whole group explanation in Table 1 where the first column is the selections corresponding to the known conditional similarity solutions in Ref. [9], the second column is the solutions for the coefficients of the constrained equation (6) and the last column is the concrete form of the related infinitesimal transformations.

Table 1. Group explanation for the results obtained by the modified CK's direct method.

	Selections	Parameters	Infinitesimal Transformations
1	$\gamma_{14} = c_1, \gamma_{29} = c_1 - 1$ $\gamma_{20} = c_0,\ \gamma_{21} = -\frac{1}{2}(c_1 + 1)$ $\omega_{14} = \omega_{29} = \omega_{21} = \omega_{20} = 0$ $\theta = c_2 t^{-1/(3+c_1)}$ $\eta = -c_2^{-2}\frac{1}{3+c_1}t^{-(1+c_1)/(3+c_1)}y$ $+c_5 t^{-(1+c_1)/(3+c_1)} + \frac{2c_0}{1+c_1}$ $+c_6 t^{2/(3+c_1)}$ $\sigma \equiv \sigma(y,t)$ arbitrary	$A_1 = A_2 = A_4 = A_5 = A_7 = 0$ $A_3 = -\theta^2(2C_1\eta - B - 4c_0)$ $+2c_2^2 c_6 C_2$ $A_6 = -\frac{1}{4}c_6 c_2^2\theta^2 C_2(2C_1\eta - 4c_0$ $-B) + \frac{1}{4}\theta^4[(C_1\eta - 2c_0)(C_1\eta$ $-2c_0 - B) - 4f_0] + \frac{1}{4}c_2^4 c_6^2 C_2^2$ where $C_1 \equiv 1 + c_1,\ C_2 \equiv 3 + c_1$	$X = -\sigma_t + \frac{1}{C_2 t}[c_2 t^{-1/C_2}x - \sigma_y(C_1 y$ $-c_2^2 C_2(c_5 C_1 - 2c_6 t))]$ $Y = t^{-1/C_2}[-\frac{c_2 C_1(c_5 c_2^2 C_2 - y)}{C_2 t} + 2c_6 c_2^3]$ $T = c_2 t^{-1/C_2}$ $U = \frac{\sigma_y}{4C_2^2 t^2}[c_2^2 C_2 B t^{\frac{C_1}{C_2}}(2c_2^2 c_6 C_2 t - c_5 c_2^2$ $C_1 C_2 + C_1 y) + 2C_1^2 y^2 - 4c_2^2 C_1 C_2(c_5 C_1$ $-2c_6 t)y - 8c_2^4 c_6 t C_2^2(c_5 C_1 - c_6 t) + 2c_2^4$ $c_5^2 C_1^2 C_2^2] + \frac{c_2}{4C_2 t^2}t^{-1/C_2}[C_1(x-1)$ $(c_5 c_2^2 C_2 - y) + 2t(2\alpha - c_6 c_2^2 C_2 x - 2u)]$ $+c_2\alpha_t t^{-1/C_2} + \frac{1}{4C_2^2 t^2}(c_1\sigma - 2C_2\sigma_t t)$ $[c_2^2 C_2(c_5 C_1 - 2c_6 t) - C_1 y]$ $\alpha = -\frac{1}{4c_2 C_2}t^{1/C_2}[(1 + 2c_1)t^{-1}\int \sigma \mathrm{d}y$ $-C_2\int \sigma_t \mathrm{d}y - c_2^2 C_2 t^{-2/C_2}\int \sigma_y B\mathrm{d}y$ $+\sigma(C_1(c_5 c_2^2 C_2 - y)t^{-1} - 2c_2^2 c_6 C_2)]$
2	$\gamma_{14} = -3, \gamma_{29} = -4$ $\gamma_{20} = c_0, \gamma_{21} = 1$ $\omega_{14} = \omega_{29} = \omega_{21} = \omega_{20} = 0$ $\theta = \exp(c_2 t)$ $\eta = (c_2 y + c_3 t + c_4)e^{-2c_2 t} - c_1$ $\sigma \equiv \sigma(y,t)$ arbitrary	$A_1 = A_2 = A_4 = A_5 = A_7 = 0$ $A_3 = \theta^2(4\eta + 4c_1 + B) - \frac{2c_3}{c_2}$ $A_6 = -\frac{1}{4c_2^2}[-2c_2^2\theta^4((\eta + c_1)$ $(2\eta + 2c_1 + B) - 2f_0)$ $+c_2 c_3\theta^2(4\eta + 4c_1 + B) - c_3^2]$	$X = -\sigma_t - 2I\sigma_y - c_2 x\exp(c_2 t)$ $Y = 2c_2 y e^{c_2 t} + (2c_4 + 2c_3 t - \frac{c_3}{c_2})e^{c_2 t}$ $T = \exp(c_2 t)$ $U = \frac{1}{2}I[2\sigma_t - 3c_2\sigma + \sigma_y(B\exp(2c_2 t) + 4I)]$ $-(\frac{1}{2}c_3 - c_2 u + c_2\alpha - \alpha_t)\exp(c_2 t)$ $\alpha = -\frac{\exp(-c_2 t)}{4}[\int(5c_2\sigma - \sigma_t)\mathrm{d}y + 2I\sigma]$ $+\frac{\exp(c_2 t)}{4}\int \sigma_y B\mathrm{d}y$ where $I \equiv c_2 y + c_3 t + c_4 - \frac{c_3}{2c_2}$
3	$\gamma_{14} = c_1, \gamma_{29} = c_1 - 1$ $\gamma_{20} = \gamma_{21} = \omega_{14} = \omega_{29} = \omega_{21} = 0$ $\theta = c_2 t^{-1/(3+c_1)}$ $\eta = -\frac{c_2^{-2}y}{3+c_1}t^{(-1-c_1)/(3+c_1)} + \eta_0$ $\sigma = \xi_0 + \frac{y}{4c_2^2 c_3}t^{2/(3+c_1)}$ $[(3 + c_1)t\eta_{0tt} + 2(1 + c_1)\eta_{0t}]$ $-\frac{(1+c_1)y^2}{4c_2^4 c_3(3+c_1)^2}t^{-2(1+c_1)/(3+c_1)}$ $\xi_0 \equiv \xi_0(t),\ \eta \equiv \eta_0(t)$ arbitrary	$A_1 = A_2 = A_4 = A_5 = A_7 = 0$ $A_3 = -\theta^2[2C_1\eta' - B]$ $+2c_2^{C_2}C_2\eta_{0t}\theta^{-C_1}$ $A_6 = \frac{\theta^4}{4}[C_1\eta'(C_1\eta' - B) + 4c_3\xi$ $-4f_0] + \frac{1}{4}c_2^{2C_2}C_2^2\eta_{0t}^2\theta^{-2C_1}$ $-\frac{1}{4}c_2^{C_2}C_2\eta_{0t}\theta^{1-C_1}[2C_1\eta' - B]$ $\eta' \equiv \eta - \eta_0,\ C_1 \equiv 1 + c_1$ $C_2 \equiv 3 + c_1$	$X = \frac{c_2 A x}{(3+c_1)t} - \frac{1}{2c_3}(3 + c_1)(1 + c_1)t\eta_{0t}^2$ $-\frac{1}{4c_3}(3 + c_1)^2 t^2\eta_{0tt}\eta_{0t} - \xi_{0t}$ $-\frac{1}{A^2}[\frac{(2+c_1)(1+c_1)y\eta_{0t}}{2c_2^2 c_3(3+c_1)t} + \frac{(2+c_1)y\eta_{0tt}}{c_2^2 c_3}$ $+\frac{(3+c_1)ty\eta_{0ttt}}{4c_2^2 c_3}]$ $Y = \frac{c_2 A C_1 y}{C_2 t} + c_2^3 A^3 C_2\eta_{0t}t$ $T = c_2 A$ $U \equiv U_{\mathrm{III}}$ $\alpha \equiv \alpha_{\mathrm{III}}$ where $A \equiv t^{-\frac{1}{3+c_1}}$
4	$\gamma_{14} = -3, \gamma_{29} = -4, \omega_{20} = c_3$ $\gamma_{20} = \gamma_{21} = \omega_{14} = \omega_{29} = \omega_{21} = 0$ $\theta = \exp(c_2 t)$ $\eta = c_2 y\exp(-2c_2 t) + \eta_0$ $\sigma = -\frac{y\exp(-2c_2 t)}{4c_3 c_2}(4c_2\eta_{0t} + \eta_{0tt})$ $+\frac{c_2^2 y^2}{2c_3}\exp(-4c_2 t) - \xi_0$ $\xi_0 \equiv \xi_0(t),\ \eta \equiv \eta_0(t)$ arbitrary	$A_1 = A_2 = A_4 = A_5 = A_7 = 0$ $A_3 = \theta^2(4\eta - 4\eta_0 - 2c_2^{-1}\eta_{0t} + B)$ $A_6 = \frac{\theta^4}{4c_2^2}[-c_2\eta_{0t}(4\eta - 4\eta_0 + B)$ $+2c_2^2(\eta - \eta_0)B + 4c_2^2\eta(\eta - 2\eta_0)$ $+4c_2^2(\eta_0^2 + c_3\xi - f_0) + \eta_{0t}^2]$	$X = \frac{y\exp(-2c_2 t)}{4c_2 c_3}(4c_2^2\eta_{0t} + 4c_2\eta_{0tt}$ $+\eta_{0ttt}) + \xi_{0t} - c_2 x\exp(c_2 t)$ $-\frac{\eta_{0t}\eta_{0tt} + 4c_2\eta_{0t}^2}{4c_3 c_2^2}$ $Y = 2c_2 y\exp(c_2 t) - \frac{1}{c_2}\eta_{0t}\exp(3c_2 t)$ $T = \exp(c_2 t)$ $U \equiv U_{\mathrm{IV}}$ $\alpha \equiv \alpha_{\mathrm{IV}}$
5	$\gamma_{14} = 1, \gamma_{29} = 0$ $\gamma_{21} = c_2,\ \gamma_{20} = c_3$ $\omega_{14} = \omega_{29} = c_1$ $\omega_{21} = c_4, \omega_{20} = c_5$ $\eta = \theta^{-3}\theta_t y + \eta_0$ $\sigma = (\frac{\theta_{tt}}{\theta^2\theta_t c_1} - \frac{5\theta_t}{c_1\theta^3}) + \xi_0$ $\xi_0 \equiv \xi_0(t)$ arbitrary while $\eta_0 \equiv \eta_0(t)$ and $\theta \equiv \theta(t)$ are given by equations (8) and (9).	$A_1 = A_2 = A_4 = A_5 = A_7 = 0$ $A_3 = \theta^2(6\eta' + c_1\xi + B) - 2\theta^3\theta_t^{-1}$ $\eta_{0t} - 2\theta^3\theta_{tt}\theta_t^{-2}\eta'$ $A_6 = \frac{\theta^4}{4}[9\eta'^2 - 4f_0 + 3(c_1\xi + B)\eta'$ $+4(c_4\eta + c_5)\xi] + \frac{1}{4}\theta^6\theta_t^{-4}\theta_{tt}\eta'$ $(\theta_{tt}\eta' + 2\theta_t\eta_{0t}) + \frac{1}{4}\theta^5\theta_t^{-2}[\theta\eta_{0t}^2$ $-(6\eta' + c_1\xi + B)\theta_{tt}\eta']$ $-\frac{1}{4}\theta^5\theta_t^{-1}\eta_{0t}[6\eta' + c_1\xi + B]$ where $\eta' \equiv \eta - \eta_0$	$X = -\theta_t x + \frac{(2\theta_{tt}^2\theta - \theta_t^2\theta_{tt} - \theta\theta_t\theta_{ttt})y}{c_1\theta^3\theta_t^2}$ $+\frac{\theta_{tt}\eta_{0t}\theta - 5\theta_t^2\eta_{0t} - c_1\xi_{0t}\theta_t^2}{c_1\theta_t^2}$ $Y = -\theta_t^{-1}\left[(\theta_{tt}\theta - 3\theta_t^2)y + \eta_{0t}\theta^4\right]$ $T = \theta$ $U \equiv U_{\mathrm{V}}$ $\alpha = \frac{\theta\theta_{tt} - 5\theta_t^2}{4c_1\theta^2\theta_t}\int B\mathrm{d}y + \frac{y^2}{8c_1\theta^5\theta_t^2}[4\theta_t^2(5\theta_t^2$ $-2\theta_{tt}\theta) + \theta^2(\theta_{ttt}\theta_t - \theta_{tt}^2)]$ $-\frac{y}{4c_1\theta^2\theta_t^2}[\theta\theta_{tt}(\theta\eta_{0t} - c_1\theta_t\xi_0)$ $-\theta_t^2(c_1\theta\xi_{0t} - 4c_1\theta_t\xi_0 + 5\theta\eta_{0t})]$
6	$\gamma_{14} = c_3, \gamma_{29} = c_2, \gamma_{21} = c_5$ $\omega_{14} = c_1 + c_2, \gamma_{20} = c_6$ $\omega_{29} = c_1, \omega_{20} = c_4$ $\omega_{21} = c_7 = \frac{c_5(c_1 - c_2)}{c_2 - c_3}$	$A_2 = 4A_1, A_5 = 4, A_6 = 0$ $A_1 = -\frac{c_2\theta^3 t}{b_2}$ $A_3 = \frac{t}{c_1^2 b_2\theta^2\eta_2^2}\{-c_1\eta_{1t}\eta_2\theta(3c_2\theta_t$ $+2c_1 b_8\eta_2\theta^3) - c_2(c_2 - 2c_1)\eta'\eta_2\theta_t^2$	$X = -b_1 b_2 t^{b_2 - 1}x - \xi_{0t} - \frac{b_2 c_2(2b_2 b_4 t + b_3 b_6)}{c_1 b_5 t^{2b_2 + 2}}$ $Y = \frac{b_1 t^{b_2 - 1}}{b_5}[b_5 b_6 y - 2c_2 b_2 b_4 b_1^2 t - c_2 b_3 b_6 b_1^2]$ $T = b_1 t^{b_2}$ $U \equiv U_{\mathrm{VI}}$

	$-\frac{c_1(c_2^2+2c_1c_2-3c_1c_3)}{2(c_2-c_1)}$ $\theta = b_1 t^{b_2}, \qquad \sigma = \frac{\theta_t}{c_1\theta^3} y + \xi_0$ $\eta = \eta_2 y + \eta_1 \equiv \frac{3-5b_6}{2c_2 b_1^2 t^{b6}} y$ $+\{\frac{2b_5 b_7}{c_1 c_2^2 b_2 b_6} + \frac{b_3 t + b_4}{t^{b6}}\}$ $b_2 = \frac{c_1(c_3-c_2)}{4c_1c_2-5c_1c_3+c_2^2}$ $b_5 = 5b_2+1,\ b_6 = 2b_2+1$ $b_7 = c_2 c_8 b_2 - c_6 c_1 b_5$ $\xi_0 \equiv \xi_0(t)$ arbitrary	$+c_1^2 c_2 \theta^4 \xi_{0t} \eta_2^2 + c_1 c_2 \theta_t \theta (\eta' 4 \eta_{2t}$ $+\eta_2^2 \theta^2 (c_2 \xi_0 + c_1 \xi_0 - c_3 \eta' + B))$ $-c_1^2 c_2 \eta_2 \theta^4 \eta_{2t} (\xi_0 - \xi) + c_1 \eta' \eta_2 \theta$ $(2c_1 \theta^3 \eta_{2t} b_8 - c_2 \theta_{tt}) + c_1^2 \eta_2^3 \theta^6$ $(b_8(c_1\xi + B) + c_2(c_3\xi_0 - c_2\xi))\}$ $A_4 = \frac{c_2\theta^3 t(\eta_{1t}\eta_2 - \eta_{2t}\eta')}{b_2\eta_2^2}$ $A_7 \equiv A_{\rm VI7}$ where $\eta' \equiv \eta_1 - \eta,\ b_8 \equiv c_3 - c_2$	$\alpha = -\frac{c_2 y}{4c_1c_2b_1b_5t^{b_2+1}}[c_1b_5((3b_2+1)\xi_0 - \xi_{0t}t)$ $+c_2 b_2 t^{-b_6}(2b_2b_4t + b_3b_6)] - \frac{b_2 y^2(3b_2+1)}{8c_1b_1^3t^{3b_2+2}}$ $+\frac{b_2}{4c_1b_1t^{b_2+1}}\int B\mathrm{d}y$
7	$\gamma_{14} = \gamma_{29} = \omega_{29} = c_1$ $\omega_{14} = 2c_1, \omega_{20} = c_4, \omega_{21} = c_7$ $\gamma_{21} = c_5, \gamma_{20} = c_6$ $\eta = \eta_2 y + \eta_1$ $\sigma = \frac{\theta_t}{c_1\theta^3} y + \xi_0$ $\xi_0 \equiv \xi_0(t)$ arbitrary, $\eta_1 \equiv \eta_1(t)$ $\eta_2 \equiv \eta_2(t)$ and $\theta \equiv \theta(t)$ are determined by Eqs. (10)–(12)	$A_2 = 4A_1, A_5 = 4, A_6 = 0$ $A_1 = -\frac{c_1\theta^4}{\theta_t}$ $A_3 = \frac{\eta'}{3c_1\theta\theta_t\eta_2}(27c_1\theta_t^2 + 4\theta^3\eta_2$ $(2c_1^2\theta_t + c_7\theta_t + c_1c_5\theta^3\eta_2))$ $-\frac{3\eta_{1t}}{\eta_2} + \theta^2(B - c_1\xi_0 + 3c_1\xi)$ $+\frac{c_1\theta^3\xi_{0t}}{\theta_t}$ $A_4 = -\frac{c_1\theta^3\eta'(c_1\eta_2\theta^3 + 3\theta_t)}{\eta_2\theta_t}$ $+\frac{c_1\theta^4\eta_{1t}}{\theta_t\eta_2}$ $A_7 \equiv A_{\rm VII7}$, where $\eta' \equiv \eta_1 - \eta$	$X = -\theta_t x - \xi_{0t} + \frac{2y}{3c_1^2\theta^4}(2c_1c_5\theta^6\eta_2^2 + 3c_1\theta_t^2$ $+\theta^3\theta_t\eta_2(c_1^2 + 2c_7)) + \frac{\theta_t\eta_{1t}}{c_1\eta_2\theta^3}$ $Y = -\frac{\theta(\eta_{2t}y + \eta_{1t})}{\eta_2}, \qquad T = \theta, \qquad U \equiv U_{\rm VII}$ $\alpha = \frac{y^2}{24\ c_1^2\theta^5}[\theta_t\theta^3\eta_2(3c_1c_3 - 2c_1^2 - 4c_7)$ $+3(c_2 - c_1)\theta_t^2 - 4c_1c_5\theta^6\eta_2^2] + \frac{y}{4c_1\theta^4\eta_2}$ $[-\eta_{1t}\theta_t + c_1\eta_2\theta^3(c_3\xi_0\theta^2\eta_2 + \xi_{0t})$ $+(c_1 + c_2)\theta_t\theta^2\xi_0\eta_2] + \frac{\theta_t}{4c_1\theta^2}\int B\mathrm{d}y.$

Note. The concrete forms of $U_{\rm III}$, $\alpha_{\rm III}$, $U_{\rm IV}$, $\alpha_{\rm IV}$, $U_{\rm V}$, $U_{\rm VI}$, $A_{\rm VI7}$, $U_{\rm VII}$, $A_{\rm VII7}$ and equations (8)–(12) are all given out in the Appendix.

Hence, the full group theoretical explanation is given out for all the conditional similarity reductions of the integrable (2+1)-dimensional KdV equation obtained by the modified CK's direct method in Ref. [9]. It is worth pointing out that the coefficients of the conditional equation used in giving out the group explanation of the results resulting from the modified CK's direct method are permitted to be functions with respect to the space-time arguments. In the case that all the coefficients in this added equation are fixed to be constants, we are still able to find many conditional symmetry reductions by utilizing the extended classical and nonclassical Lie group approaches [14] which are definitely recoverable by the modified direct method though the related work has not yet been carried out.

4 Summary and discussions

By extending both the CK's direct method and the Lie group approach, we can obtain the conditional similarity reductions. The crucial point for this kind of reduction is that the one reduction field of a given NPDE needs to satisfy more than one reduction equation. From the detailed description in the last two sections, one can notice much freedom or arbitrariness in both the modified CK's direct method and the extended Lie group approach. For the modified CK's direct method, we need to separate the resulting equation (3). Obviously, the division is quite arbitrary. Moreover, the values of F_l for $l > L$ can be set arbitrarily. While for the extended Lie group approach, the choice of the constrained equation is rather arbitrary. Many different useful results will be generated with different selections of the conditional equations. Anyhow, we do believe that the conditional similarity reductions obtained from the modified CK's direct method can definitely be reobtained from the extended Lie group approach and vice versa through ascertaining or balancing the arbitrariness between these two methods.

Furthermore, on the other hand, one can also see that the constrained equations introduced in the extended Lie group approach have little relation with the model equation under investigation. However, as for the nonintegrable JM equation, the conditional equation forms an integrable system with the JM equation. Therefore, probably we can decrease the arbitrariness by considering a constrained equation which has some possible relation with the studied equation especially for the nonintegrable models. All in all, the search of a method to introduce a constrained equation for the model equation into the extended classical and nonclassical Lie group approaches is still in progress.

Appendix

$$-\frac{1}{2}c_1\theta^2\theta_{tt}^2 + \left(\frac{3}{4}c_1 + c_4\right)\theta\theta_t^2\theta_{tt} + \frac{1}{4}\theta_t c_1\theta^2\theta_{ttt} + \left(c_1c_1 - \frac{3}{2}c_1 - 5c_4\right)\theta^4 = 0, \tag{8}$$

$$\begin{aligned}&(4c_4\theta_t\theta\theta_{tt} - 20c_4\theta^3 + 4c_1c_2\theta_t^3)\eta_0 + \theta^2\theta_t c_1\eta_{0tt}\\ &\quad + (6c_1\theta\theta_t^2 - 2c_1\theta^2\theta_{tt})\eta_{0t} - 20c_5\theta_t^3 + 4c_1c_3\theta_t^3 + 4c_5\theta\theta_t\theta_{tt} = 0.\end{aligned} \tag{9}$$

$$\theta\eta_{2t} - c_1\theta^3\eta_2^2 - 3\eta_2\theta_t = 0, \tag{10}$$

$$c_1\theta\eta_{1tt} - 2c_1\eta_{1t}(c_1\eta_2\theta^3 + 3\theta_t) + 4\theta^2\eta_2\theta_t(c_8 + c_7\eta_1) + 4c_1\theta^5\eta_2^2(c_6 + c_5\eta_1) = 0, \tag{11}$$

$$3c_1\theta\theta_{tt} - 12c_1\theta_t^2 - (c_1^2 - 4c_7)\eta_2\theta^3\theta_t + 4c_1c_5\eta_2^2\theta^6 = 0. \tag{12}$$

$$\begin{aligned}U_{\mathrm{III}} &= c_2A\alpha_t + \frac{c_1C_1^2y^3}{16c_3(c_2C_2tA)^4} - \frac{c_1C_1\xi_0y}{4C_2^2t^2} + \frac{1}{4c_3}c_2^2C_1C_2^2A^2t^2\eta_{0t}^3 + \frac{C_1y^2\eta_{0ttt}}{8c_3c_2^2A^2} + \frac{C_1\xi_{0t}y}{2C_2t}\\ &\quad + \frac{\eta_{0t}^2}{8c_3}[c_2^2C_2^3A^2t^3\eta_{0tt} + c_2^2C_1C_2A^2Bt + 3C_1(2+c_1)y] + \frac{c_2A}{4C_2t^2}[C_1y(1-x)\\ &\quad - 4(u-\alpha)t] + \eta_{0t}\left[\frac{C_2t}{16c_3}(\eta_{0tt}(c_2^2C_2A^2Bt + 9(2+c_1)y) + 2C_2\eta_{0ttt}yt) + \frac{1}{2}c_2^2C_2A^2\xi_{0t}t\right.\\ &\quad \left. + \frac{c_1C_1By}{8c_3C_2t} - \frac{1}{4}c_2^3(C_2x + C_1)A^3 - \frac{1}{4}c_2^2c_1A^2\xi_0 + \frac{C_1(2c_1^2 + 11c_1 + 8)y^2}{16c_3(c_2C_2At)^2}\right]\\ &\quad - \frac{C_1^2By^2}{8c_3c_2^2C_2^3A^2t^3} + \frac{\eta_{0tt}}{16c_3}\left[C_1By - 4c_2^3c_3C_2A^3t + \frac{C_1(16+7c_1)y^2}{C_2c_2^2A^2t}\right],\end{aligned}$$

$$\begin{aligned}\alpha_{\mathrm{III}} &= \frac{C_1C_2\eta_{0t}^2yt}{8c_2c_3A} + \frac{\eta_{0t}}{16c_2c_3A}\left[C_2^2\eta_{0tt}yt^2 + 2C_1\int B\mathrm{d}y + \frac{2C_1y^2}{c_2^2C_2A^2t}\right] + \frac{C_2\eta_{0ttt}y^2t}{32c_2^3c_3A^3} - \frac{c_1\xi_0y}{4c_2C_2At}\\ &\quad + \frac{\eta_{0tt}}{32c_2c_3A}\left[2C_2t\int B\mathrm{d}y + \frac{y^2(3c_1+8)}{c_2^2A^2}\right] - \frac{C_1\int yB\mathrm{d}y}{8c_3c_2^3C_2^2A^3t^2} + \frac{c_1C_1y^3}{48c_3c_2^5C_2^3A^5t^3} + \frac{\xi_{0t}y}{4c_2A},\end{aligned}$$

$$\begin{aligned}\alpha_{\mathrm{IV}} &= -\frac{(4c_2\eta_{0t} + \eta_{0tt})e^{-c_2t}}{16c_2c_3}\int B\mathrm{d}y + \frac{c_2^2e^{-3c_2t}}{4c_3}\int yB\mathrm{d}y - \frac{c_2^3y^3e^{-5c_2t}}{8c_3} + \frac{y^2e^{-3c_2t}}{32c_2c_3}\\ &\quad \times(-\eta_{0ttt} + 8c_2^2\eta_{0t} - c_2\eta_{0tt}) + \frac{ye^{-c_2t}}{16c_3c_2^2}(-4c_3c_2^2\xi_{0t} + 12c_2^3c_3\xi_0 + 4c_2\eta_{0t}^2 + \eta_{0t}\eta_{0tt}),\end{aligned}$$

$$\begin{aligned}U_{\mathrm{IV}} &= (c_2u - c_2\alpha + \alpha_t)e^{c_2t} + \frac{y^2e^{-2c_2t}}{8c_3}(4c_2^3B + 7c_2^2\eta_{0t} - 5c_2\eta_{0tt} - 2\eta_{0ttt})\\ &\quad + \frac{e^{3c_2t}}{4c_2}(2c_2\eta_{0t} + \eta_{0tt}) + \frac{e^{2c_2t}}{16c_2^3c_3}[Bc_2\eta_{0t}(4c_2\eta_{0t} + \eta_{0tt}) - 2\eta_{0t}(4c_2\eta_{0t}^2 + \eta_{0t}\eta_{0tt}\\ &\quad + 6c_2^3c_3\xi_0 - 4c_3c_2^2\xi_{0t})] - \frac{3c_2^4y^3e^{-4c_2t}}{4c_3} + \frac{y}{16c_3c_2^2}[8c_3c_2^3(3c_2\xi_0 - 2\xi_{0t})\\ &\quad - 2c_2^2B(6c_2\eta_{0t} + \eta_{0tt}) + \eta_{0t}(12c_2^2\eta_{0t} + 9c_2\eta_{0tt} + 2\eta_{0ttt})],\end{aligned}$$

$$\begin{aligned}U_{\mathrm{V}} &= \frac{y^2(\theta_{tt}\theta - 3\theta_t^2)}{4c_1\theta_t^3\theta^5}(3\theta_{tt}^2\theta^2 - 2\theta_t\theta_{ttt}\theta^2 + 7\theta_t^2\theta_{tt}\theta - 20\theta_t^4) - \frac{x(\theta_t^2 - \theta_{tt}\theta)}{4\theta\theta_t^2}(3\theta_t^2y - \theta\theta_{tt}y\\ &\quad - \eta_{0t}\theta^4) - \frac{y}{4c_1\theta^2\theta_t^3}[\theta_t^5(12c_1\xi_0 - 3c_1\theta + 15B) - \theta\eta_{0t}(10\theta_t^4 - 9\theta\theta_t^2\theta_{tt} + 5\theta^2\theta_{tt}^2\\ &\quad - 2\theta^2\theta_t\theta_{ttt}) + \theta^2\theta_t\theta_{tt}^2(c_1\theta + B + c_1\xi_0) - \theta\theta_t^2\theta_{tt}(-3c_1\theta\theta_t - 2c_1\theta\xi_{0t} + 7c_1\theta_t\xi_0 + 8\theta_tB)\\ &\quad - c_1\theta\theta_t^2(6\theta_t^2\xi_{0t} + \theta^2\theta_{ttt})] + \frac{1}{4c_1\theta_t^3}[\theta^2\theta_t\eta_{0t}(-\theta\theta_{tt}(c_1\theta + B + c_1\xi_0) - 2c_1\theta\theta_t\xi_{0t} + (3c_1\theta\\ &\quad + 5B + 4c_1\xi_0)\theta_t^2) + 4c_1\theta\theta_t^3\alpha_t + 2\theta^3\eta_{0t}^2(\theta\theta_{tt} - 5\theta_t^2) + c_1\theta_t^2(\theta^4\eta_{0tt} - 4\theta_t^2\alpha)] + \theta_tu,\end{aligned}$$

$$
\begin{aligned}
A_{\mathrm{VI7}} = {}& -\frac{t}{4b_2c_1^2\theta^2\eta_2^4}\{-c_1\eta_{1t}^2\eta_2^2\theta(2c_2\theta_t + c_1b_8\eta_2\theta^3) - c_2\eta_2\eta_{2t}\eta'(c_1^2\eta_2\theta^4\xi_{0t} - (c_2-2c_1)\eta'\theta_t^2) \\
& + \eta_2\eta_{1t}[c_2(2c_1-c_2)\theta_t^2\eta_2\eta' + c_1^2\theta^6\eta_2^3(b_8(B+c_1\xi) + c_2(c_3\xi_0 - c_2\xi)) \\
& + c_1^2\theta^4\eta_2(2b_8\eta_{2t}\eta' + c_2\eta_{2t}(\xi-\xi_0) + c_2\xi_{0t}\eta_2) + c_1c_2\theta^3\theta_t\eta_2^2(B + (c_2+c_1)\xi_0 - c_3\eta') \\
& + c_1c_2\theta\eta'(5\theta_t\eta_{2t} - \theta_{tt}\eta_2)] - c_1c_2\theta\theta_t\eta'\eta_{2t}[3\eta'\eta_{2t} + \eta_2^2\theta^2((c_2+c_1)\xi_0 - c_3\eta' + B)] \\
& + c_1\eta_{2t}\eta'\eta_2\theta[c_2\eta'\theta_{tt} - c_1\eta_2^2\theta^5(b_8(B+c_1\xi) + c_2(c_3\xi_0 - c_2\xi))] - 4c_1^2\eta_2^5\theta^8(b_8f_0 - c_2f_1 \\
& + (c_8b_8 - c_2c_6)\xi + (c_7b_8 - c_2c_5)\xi\eta) + c_1^2\eta_2\theta^4\eta_{2t}^2\eta'(-b_8\eta' + c_2(\xi_0 - \xi))\}, \\
U_{\mathrm{VI}} = {}& -\frac{b_1xt^{b_2-2}}{4b_5}[b_6b_5y - c_2b_1^2(b_3b_6 + 2b_2b_4t)] + \frac{y}{4c_1b_5t^2}[c_1b_5b_6(2\xi_{0t}t - (3b_2+1)\xi_0 + b_1t^{b_2}) \\
& + c_2b_2b_5b_6B - c_2b_2(b_2+1)t^{-2b_2-1}(b_3b_6 + 2b_2b_4t)] + b_1t^{b_2-1}(b_2u + b_1\alpha_t t) \\
& - \frac{c_2b_1^2}{4c_1b_5t^2}(2b_2b_4t + b_3b_6)[2c_1\xi_{0t}t + b_2B - c_1\xi_0(3b_2+1)] - \frac{b_2b_6(3b_2+1)y^2}{4c_1b_1^2t^{2b_2+3}} \\
& + \frac{b_1}{4c_1b_5^2}[2c_2^2b_1b_2b_3b_6t^{-2b_2-3}(b_3b_6 + 4b_2b_4t) + 8c_2^2b_1b_2^3b_4^2t^{-b_6} \\
& - c_1b_5t^{b_2-2}(c_2b_1^2b_3b_6 + 4b_2b_5\alpha t)], \\
A_{\mathrm{VII7}} = {}& \frac{c_1\theta^2\xi_{0t}}{4\theta_t\eta_2}[\eta'(3\theta_t + c_1\eta_2\theta^3) - \theta\eta_{1t}] + \frac{\theta_t\eta'}{4c_1\eta_2^2\theta}[3c_1(B\theta^2\eta_2 + 4\eta_{1t}) + \theta^2\eta_2\eta'(4c_7 + 11c_1^2) \\
& - 3c_1^2\theta^2\eta_2(\xi_0 - 3\xi)] - \frac{\theta^2\eta_{1t}}{12c_1\eta_2}[(4c_7 + 11c_1^2)\eta' + 3c_1(B - c_1\xi_0 + 3c_1\xi)] \\
& + \frac{\theta^4\eta'}{12}[(5c_1^2 + 4c_7 + 12c_5)\eta' + 3c_1(B - c_1\xi_0 + 3c_1\xi)] - \frac{\theta^5}{3\theta_t}\big[c_5\eta'(\eta_{1t} - c_1\theta^2\eta_2\eta') \\
& + 3c_1\theta^2\eta_2(c_6\xi + c_5\xi\eta + f_1)\big] + \frac{9(\eta_1-\eta)^2\theta_t^2 + \theta^2\eta_{1t}^2}{2\eta_2^2\theta^2}, \\
U_{\mathrm{VII}} = {}& -\frac{x}{4\theta\eta_2}(5\theta_t + c_1\eta_2\theta^3)(3\eta_2\theta_t y + c_1\eta_2^2\theta^3 y + \theta\eta_{1t}) - \eta_2\theta^5(c_5\eta_1 + c_6) + \theta\alpha_t + \theta_t(u - \alpha) \\
& + \frac{\theta^2}{4c_1}[c_1\eta_{1t}(c_1\theta - c_3\xi_0) - 4\theta_t(c_8 - c_7\eta_1)] - \frac{\eta_{1t}}{4c_1\eta_2^2\theta^3}[c_1\theta^3\eta_2(2\xi_{0t} - 3\theta_t) - 2\theta_t\eta_{1t} \\
& + \theta_t\theta^2\eta_2(B + (c_1+c_2)\xi_0)] + \frac{(c_1\eta_2\theta^3 + 3\theta_t)y^2}{12c_1^2\theta^5}[4(c_1^2 + 2c_7)\theta^3\theta_t\eta_2 \\
& - c_1\theta^3\eta_2(3c_3\theta_t - 8c_5\theta^3\eta_2) - 3(c_2 - 3c_1)\theta_t^2] - \frac{y}{12c_1^2\eta_2\theta^4}\{3c_1^2\theta^9\eta_2^3(4c_5 - c_1^2) \\
& + 3c_1^3c_3\theta^8\eta_2^3\xi_0 + 3\eta_{1t}\theta_t^2(c_2 - 9c_1) - 2c_1\theta^6\eta_2^2(9c_1^2\theta_t - 3c_1^2\xi_{0t} - 6c_7\theta_t + 4c_5\eta_{1t}) \\
& + 3c_1^2\theta_t\theta^5\eta_2^2((c_1+c_2)\xi_0 + 3c_3\xi_0 + B) + 9c_1\theta_t^2\theta^2\eta_2((c_1+c_2)\xi_0 + B) \\
& + \theta_t\theta^3\eta_2[c_1^2(18\xi_{0t} - 27\theta_t - 10\eta_{1t}) + \eta_{1t}(3c_1c_3 - 8c_7)]\}.
\end{aligned}
$$

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