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Soliton Systems as ⋆-Deformed Dispersionless Equations

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The formalism of quantization deformation is reviewed and the Weyl–Moyal like deformation is applied to systematic construction of the field and lattice integrable soliton systems from Poisson algebras of dispersionless systems.

1 Introduction

Recently, various aspects of the Moyal deformation theory and its application to the integrable field systems, which leads to the so-called Moyal type Lax dynamics, have become of increasing interest [1–5]. The aim of this paper is to present a complete picture of construction of the field and lattice soliton systems by Weyl–Moyal like deformations from Poisson algebras of underlying dispersionless systems. The Weyl–Moyal like deformation is the special case of the deformation quantization.

In the theory of evolutionary systems (dynamical systems) one of the most important issues is a systematic method for construction of integrable systems. As integrable systems we understand those which have infinite hierarchy of symmetries and conservation laws. It is well known that a very powerful tool, called the classical R-matrix formalism, proved to be very fruitful in systematic construction of the field and lattice soliton systems as well as dispersionless systems (see [6–16] and the references there).

As well known, a quasi-classical limit of field and lattice soliton systems gives related integrable dispersionless systems. We would like to inverse this procedure and construct field and lattice soliton systems from some classes of integrable dispersionless systems through a Weyl–Moyal like deformation quantization procedure. Actually, we will do it at the level of their Lax representations.

2 Classical R-matrix formalism

The crucial point of the formalism is the observation that integrable dynamical systems can be obtained from the Lax equations. Let $\mathfrak{g}$ be a Lie algebra, equipped with the Lie bracket $[\cdot,\cdot]$, for which its dual $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$. So, we assume the existence of a symmetric, non-degenerate and ad-invariant scalar product $(\cdot,\cdot)_{\mathfrak{g}}$. We assume additionally that the scalar product is given by a trace form $\mathrm{tr}:\mathfrak{g}\to\mathbb{R}$, i.e. $(a,b)_{\mathfrak{g}}=\mathrm{tr}\,(ab)$. A linear map $R:\mathfrak{g}\to\mathfrak{g}$, such that the bracket $[a,b]_R := [Ra,b]+[a,Rb]$ is a second Lie product on $\mathfrak{g}$, is called the classical R-matrix. Then, the vector fields on $\mathfrak{g}$

$$L_{t_n} = [R\,(dC_n)\,, L] \qquad n\in\mathbb{Z}, \tag{1}$$

generated by the differentials of Casimir functions $dC_n\in\mathfrak{g}$, such that

$$[dC_n, L] = 0 \qquad \forall\, L\in\mathfrak{g},$$

commute mutually. The hierarchy of evolution equations (1) is the Lax hierarchy with common infinite set of symmetries and conserved quantities. In this sense (1) represents a hierarchy of integrable evolution equations. Moreover, it is known that (1) are tri-Hamiltonian systems [17].

To construct the simplest R-structure let us assume that the Lie algebra $\mathfrak{g}$ can be split into a direct sum of Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$, i.e. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $[\mathfrak{g}_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm$. Denoting the projections onto these subalgebras by $P_\pm$, we define the R-matrix as

$$R = \frac{1}{2}(P_+ - P_-) = P_+ - \frac{1}{2} = \frac{1}{2} - P_-. \tag{2}$$

Following the above scheme, we are able to construct in a systematic way integrable Hamiltonian systems, with infinite hierarchy of involutive constants of motion and infinite hierarchy of related commuting symmetries, once we fix a Lie algebra. For example the Lie algebra of pseudo-differential operators with the commutator leads to construction of soliton systems [6–10]. The Lie algebra of shift operators leads to lattice field systems [11,12]. On the other hand, the Poisson algebras (which are Lie algebras with associative, commutative multiplication) of formal Laurent series lead to the construction of dispersionless systems [13–16].

3 Star products and deformation quantizable Poisson brackets

The idea behind the deformation quantization theory [18–21] is that a classical system can be obtained from quantum system by the quasi-classical limit $\hbar \to 0$, where $\hbar$ is the Planck constant divided by 2π. Therefore, the quantization of classical systems should be done by appropriate deformations depending on a formal parameter $\hbar$.

Let $\mathcal{A} = \mathcal{C}^\infty(M)$ be the space of all smooth ($\mathbb{R}$ or $\mathbb{C}$ valued) functions on $2n$-dimensional smooth manifold M. Let $\{\cdot,\cdot\}_{\mathrm{PB}}$ be the classical Poisson bracket, which is bilinear, skew-symmetric and satisfies the Jacobi identity. Obviously $\mathcal{A}$ is a commutative, associative algebra over $\mathbb{R}$ or $\mathbb{C}$ with the standard multiplication. Let $\star$ be the deformed associative non-commutative multiplication on $\mathcal{A}$ given by the following formula

$$f \star g = \sum_{k\geq 0} \hbar^k B_k(f,g), \qquad f, g \in \mathcal{A}, \tag{3}$$

where $\hbar$ is the formal parameter and $B_k : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ are bidifferential (bilinear) operators. We also define deformed bracket as a commutator

$$\{f,g\}_\star := \frac{1}{\hbar}(f \star g - g \star f). \tag{4}$$

Definition 1. An associative deformed multiplication $\star$, given by the formula (3), is a formal quantization of the algebra $\mathcal{A}$ and is called the $\star$-product if

1. $\lim\limits_{\hbar\to 0} f \star g = fg$,
2. $c \star f = f \star c = cf$ for $c \in \mathbb{R}$ or $\mathbb{C}$,
3. $\lim\limits_{\hbar\to 0} \{f,g\}_\star = \{f,g\}_{\mathrm{PB}}$.

Lemma 1. *The bracket* (4) *defined by the $\star$-product is bilinear, skew-symmetric and satisfies the Jacobi identity. So, it is well defined Lie bracket.*

The proof is obvious as the Jacobi identity is a consequence of an associativity of multiplication $\star$. Hence, the bracket (4) is called the deformation quantization of the underlying classical Poisson bracket $\{\cdot,\cdot\}_{\mathrm{PB}}$. Let $D : \mathcal{A} \longrightarrow \mathcal{A}$ be a linear automorphism parameterized by $\hbar$, such that

$$Df = \sum_{k\geq 0} \hbar^k D_k f, \qquad D_0 = 1, \tag{5}$$

where D_k are differential operators. Such an automorphism produces a new $\star'$-product in $\mathcal{A}$ in the following way

$$f \star' g := D\left(D^{-1} f \star D^{-1} g\right). \tag{6}$$

Definition 2. Two $\star$-products: $\star$ and $\star'$ are called gauge equivalent or simply equivalent if there exist a linear automorphism $D : \mathcal{A} \longrightarrow \mathcal{A}$ (5) such that (6) is satisfied.

Let us consider now a Weyl–Moyal like deformations. It is well known that an arbitrary classical Poisson bracket can be presented in the following form

$$\begin{aligned}\{f, g\}_{\mathrm{PB}} &= f\left(\sum_{i=1}^{n} Y_i \wedge X_i\right) g = f\left(\sum_{i=1}^{n} (Y_i \otimes X_i - X_i \otimes Y_i)\right) g \\ &= \sum_{i=1}^{n} \left[Y_i(f) X_i(g) - X_i(f) Y_i(g)\right], \end{aligned} \tag{7}$$

where X_i, Y_i, $i = 1, \ldots, n$ are pair-wise commuting vector fields on $2n$ dimensional smooth manifold M and $f, g \in \mathcal{A}$. The Jacobi identity for (7) follows from the commutativity of vector fields X_i, Y_i. In what follows, we will use the Einstein summation convention in the case of repeating indices i, j at the vectors X, Y and a standard convention (with the summation symbols) otherwise.

Infinitely many Weyl–Moyal like deformed multiplications are of the following form [17]

$$f \star^{\alpha} g = f \exp\left[\frac{\hbar}{2}\left((2-\alpha)\, Y_i \otimes X_i - \alpha X_i \otimes Y_i\right)\right] g. \tag{8}$$

If the classical Poisson bracket (7) is a canonical one, i.e. $Y_i = \partial_{p_i}$, $X_i = \partial_{x_i}$ ($\partial_x = \frac{\partial}{\partial x}$, $\partial_p = \frac{\partial}{\partial p}$), then for $\alpha = 1$ the product (8) is the Groenewold product [22] and the deformed bracket (4) is the well known Moyal bracket [23] and for $\alpha = 0$ the product (8) is the Kupershmidt–Manin (KM) product and the deformed bracket (4) is the KM bracket [24, 25]. Notice that all (8) are well defined quantization deformation of (7). Moreover, they are gauge equivalent by the linear automorphism $D : \mathcal{A} \longrightarrow \mathcal{A}$

$$D^{\alpha} = \exp\left[-\alpha \frac{\hbar}{2} Y_i X_i\right], \qquad \alpha \in \mathbb{R}. \tag{9}$$

Then, $f \star^{\alpha'} g := D^{\alpha'-\alpha}\big(D^{-(\alpha'-\alpha)} f \star^{\alpha} D^{-(\alpha'-\alpha)} g\big)$. Now, we impose the Lie algebra structure on the algebra $\mathcal{A}$, denoting it by $\mathcal{A}_{\alpha} = (\mathcal{A}, \star^{\alpha})$, with the commutator $\{f, g\}_{\star^{\alpha}} := \frac{1}{\hbar}(f \star^{\alpha} g - g \star^{\alpha} f)$. Obviously, the automorphism (9) induces the isomorphisms between the Lie algebras $D^{\alpha'-\alpha} : \mathcal{A}_{\alpha} \longrightarrow \mathcal{A}_{\alpha'}$. We will call the Lie algebras $\mathcal{A}_{\alpha}$ gauge equivalent as one can choose freely the algebra one wants to work with.

4 Poisson algebras of formal Laurent series

Consider the simplest possible case of $\dim M = 2$, when M is parametrized by a pair of coordinates (x, p). The Poisson bracket on $\mathcal{A}$ can be introduced in infinitely many ways as

$$\{f, g\}_{\mathrm{PB}}^{r} := p^{r}\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}\right), \qquad r \in \mathbb{Z}. \tag{10}$$

Moreover, in $\mathcal{A}$ there exists the following Poisson subalgebra of formal Laurent series (Lax polynomials)

$$A = \left\{ L = \sum_{i \in \mathbb{Z}} u_i(x) p^i \right\}, \tag{11}$$

where the coefficients u_i are smooth functions of x. An appropriate non-degenerate symmetric ad-invariant product on A is given by a trace form $\operatorname{Tr} L = \int_{\Omega} \operatorname{res}_r L \; dx$, where $\operatorname{res}_r L \equiv$

$u_{r-1}(x)$ [15]. We construct the simplest R-matrix, through a decomposition of A into a direct sum of Lie subalgebras. For a fixed r let $A_{\geqslant -r+k} = \{\sum_{i\geqslant -r+k} u_i(x)p^i\}$ and $A_{<-r+k} = \{\sum_{i<-r+k} u_i(x)p^i\}$. As presented in [15], $A_{\geqslant -r+k}, A_{<-r+k}$ are Lie subalgebras in the following cases: $k=0$, $r=0$; $k=1,2$, $r\in\mathbb{Z}$ and $k=3$, $r=2$[1]. Then, the R-matrix (2) is given by $R=\frac{1}{2}(P_{\geqslant -r+k} - P_{<-r+k})$. Hence, the hierarchy of evolution equations (1) for Casimir functionals

$$C_n(L) = \frac{1}{n+1}\operatorname{Tr}\left(L^{n+1}\right), \qquad dC_n(L) = L^n,$$

is given by

$$L_{t_q} = \{(L^q)_{\geqslant -r+k}, L\}^r_{\mathrm{PB}} = -\{(L^q)_{<-r+k}, L\}^r_{\mathrm{PB}}, \qquad L\in A, \tag{12}$$

which are Lax hierarchies. Notice that (12) are multi-Hamiltonian systems [16]. To construct (1+1)-dimensional dispersionless systems we have to choose properly restricted Lax operators L in such way that L contains the finite number of dynamical fields and gives consistent equations (12). They are given in the form [16]

$$\begin{aligned} k=0,\ r=0 &: \quad L = p^N + u_{N-2}p^{N-2} + \cdots + u_1 p + u_0,\\ k=1,\ r\in\mathbb{Z} &: \quad L = p^N + u_{N-1}p^{N-1} + \cdots + u_{1-m}p^{1-m} + u_{-m}p^{-m},\\ k=2,\ r\in\mathbb{Z} &: \quad L = u_N p^N + u_{N-1}p^{N-1} + \cdots + u_{1-m}p^{1-m} + p^{-m}. \end{aligned} \tag{13}$$

5 Weyl–Moyal like deformation of Poisson algebras of formal Laurent series

The Poisson brackets (10) on $\mathcal{A}$ can be presented in the following form

$$\{f,g\}^r_{\mathrm{PB}} = f\,(p^r\partial_p \wedge \partial_x)\,g, \qquad r\in\mathbb{Z}. \tag{14}$$

Notice that this is a special case of (7), when $Y_1 = p^r\partial_p$ and $X_1=\partial_x$ with $[Y_1,X_1]=0$. For a fixed r, the Poisson bracket (14) on $\mathcal{A}$ can be quantized in infinitely many equivalent ways via the $\star^\alpha$-product (8)

$$f\star^\alpha g = f\exp\left[\frac{\hbar}{2}\left((2-\alpha)\,p^r\partial_p\otimes\partial_x - \alpha\partial_x\otimes p^r\partial_p\right)\right]g.$$

One finds that $(p^r\partial_p)^s p^m = c^m_s(r)\ p^{m-s(1-r)}$, $s\in\mathbb{Z}_+$, where for $k\in\mathbb{Z}$

$$c_s^{k(1-r)}(r) = \begin{cases} (1-r)^s\dfrac{k!}{(k-s)!} & \text{for } k\geq s \text{ and } r\neq 1,\\ 0 & \text{for } s>k\geq 0 \text{ and } r\neq 1,\\ (-1+r)^s\dfrac{(s-k-1)!}{s!} & \text{for } k<0 \text{ and } r\neq 1, \end{cases}$$

for $m\neq k(1-r)$: $c^m_s(r) = m(m-(1-r))\cdots(m-(s-1)(1-r))$ and for an arbitrary $m\in\mathbb{Z}$: $c^m_s(1)=m^s$.

We can quantize separately the Poisson subalgebra A (11) to the following Lie subalgebras $A_\alpha = (A,\star^\alpha)\subset\mathcal{A}_\alpha$. Obviously, the Lie algebras A_α for a fixed value of r are gauge equivalent under the isomorphism

$$D^{\alpha'-\alpha}: A_\alpha \longrightarrow A_{\alpha'}, \qquad D^{\alpha'-\alpha} = \exp\left[(\alpha-\alpha')\frac{\hbar}{2}p^r\partial_p\partial_x\right]. \tag{15}$$

[1] In [15–17] there was omitted the case $r=2$, $k=3$, but it does not lead to any new dispersionless systems as it is related to the case $r=0$, $k=0$ by transformation $p\to p^{-1}$. In further considerations we will skip this case.

Let $L = \sum_{m=-\infty}^{+\infty} u_m p^m \in A_\alpha$ and $L' = \sum_{n=-\infty}^{+\infty} v_n p^n \in A_{\alpha'}$. Then $L' = D^{\alpha'-\alpha} L$ and fields u_m, v_n are mutually related in the following way $v_n = \sum_{s\geq 0} ((\alpha-\alpha')\frac{\hbar}{2})^s \frac{1}{s!}\ c_s^{s(1-r)+n}(r)\ (u_{s(1-r)+n})_{sx}$.

On the other hand one can show the following relations

$$u \star^\alpha v = uv, \qquad p^m \star^\alpha p^n = p^{m+n}, \tag{16}$$

$$p^m \star^\alpha u = \sum_{s\geq 0} \frac{\hbar^s}{s!} u_{sx} \star^\alpha (p^r \partial_p)^s p^m, \qquad u \star^\alpha p^m = \sum_{s\geq 0} \frac{\hbar^s}{s!} (p^r \partial_p)^s p^m \star^\alpha u_{sx}. \tag{17}$$

As all relations (16)–(17) have the same form independently of α we skip this index in further considerations. Hence, we can quantize separately the algebra A to the following special algebra of Lax operators:

$$\mathfrak{a} = \left\{ L = \sum_{i\in\mathbb{Z}} u_i(x) \star p^i \right\}. \tag{18}$$

It is obviously associative algebra under commutation rules (17). The algebra $\mathfrak{a}$ in the case of $r = 0$ was considered for the first time in [4]. Then, the Lie-bracket on $\mathfrak{a}$ is given by

$$\begin{aligned} \{u \star p^m, v \star p^n\}_\star &= \frac{1}{\hbar} (u \star p^m \star v \star p^n - v \star p^n \star u \star p^m) \\ &= \sum_{s=0}^{\infty} \frac{\hbar^{s-1}}{s!} \left[c_s^m(r)\ u v_{sx} - c_s^n(r)\ u_{sx} v\right] \star p^{m+n-s(1-r)}. \end{aligned} \tag{19}$$

Notice that the algebra $\mathfrak{a}$ differs from that defined in the third section, where we introduced deformation quantization, as in (18) we also deformed the Lax polynomials. Let us remark that the algebra $\mathfrak{a}$ is naturally isomorphic to the algebra A_0 as $u \star^0 p^m = up^m$. Hence, in the further considerations we will concentrate only on the algebra $\mathfrak{a}$, as the results for the algebras A_α for all values of α can be obtained simply from A_0. The second reason is that $\mathfrak{a}$ can be considered as the generalization of the algebra of the pseudo-differential operators and the algebra of the shift operators in the following sense. Let us consider the case of $r = 0$, then the rules (17) take the particular form

$$p^m \star u = \sum_{s=0} \hbar^s \binom{m}{s} u_{sx} \star p^{m-s}, \qquad u \star p^m = \sum_{s=0} (-\hbar)^s \binom{m}{s} p^{m-s} \star u_{sx}$$

and the Lie bracket (19) is

$$\{u \star p^m, v \star p^n\} = \sum_{s=0}^{\infty} \hbar^{s-1} \left[\binom{m}{s} u v_{sx} - \binom{n}{s} u_{sx} v \right] \star p^{m+n-s}.$$

Notice, that the algebra $\mathfrak{a}$ for fixed $r = 0$ is isomorphic to the algebra of pseudo-differential operators. In the case of $r = 1$ the rules (17) become

$$\begin{aligned} p^m \star u(x) &= \sum_{s=0} \frac{\hbar^s}{s!} m^s \left(u(x)\right)_{sx} \star p^m = u(x + m\hbar) \star p^m, \\ u(x) \star p^m &= \sum_{s=0} \frac{(-\hbar)^s}{s!} m^s p^m \star \left(u(x)\right)_{sx} = p^m \star u(x - m\hbar), \end{aligned}$$

where we use the formula for Taylor expansion. The Lie bracket (19) is

$$\{u(x) \star p^m, v(x) \star p^n\}_\star = \frac{1}{\hbar} \left[u(x) v(x + m\hbar) - u(x + n\hbar) v(x)\right] \star p^{m+n}.$$

The algebra $\mathfrak{a}$ for a fixed $r = 1$ is isomorphic to the algebra of shift operators.

The following property for $\mathfrak{a}$ is very important.

Lemma 2 ([17]). *An appropriate symmetric, non-degenerate and* ad*-invariant scalar product on* $\mathfrak{a}$ *is given by a trace form* $(L_1, L_2)_{\mathfrak{a}} := \operatorname{Tr}(L_1 \star L_2)$*, where* $\operatorname{Tr} L = \int_\Omega \operatorname{res}_r L \, dx$ *and* $\operatorname{res}_r L \equiv u_{r-1}(x)$*.*

To proof Lemma 2 one needs the following useful relation: $c_s^m(r) = (-1)^s \, c_s^{(s-1)(1-r)-m}(r)$. Notice that one can simply rewrite the trace formula from $\mathfrak{a} \cong A_0$ to A_α as the trace formula is invariant under isomorphism (15).

6 R-matrix formalism and Lax hierarchies for Lie algebra $\mathfrak{a}$

To construct the integrable field systems one has to split the algebra $\mathfrak{a}$ into a direct sum of Lie subalgebras. Observing (19) one finds that in general it can be done only for $r = 0$, $r = 1$ or $r = 2$. Now, we decompose $\mathfrak{a}$ for $r = 0, 1, 2$ into a direct sum of Lie subalgebras in the following way. Let $\mathfrak{a}_{\geqslant -r+k} = \{\sum_{i \geqslant -r+k} u_i(x) \star p^i\}$ and $\mathfrak{a}_{<-r+k} = \{\sum_{i<-r+k} u_i(x) \star p^i\}$. Then, $\mathfrak{a}_{\geqslant -r+k}$ and $\mathfrak{a}_{<-r+k}$ are Lie subalgebras in the case of $r = 0$ for $k = 0, 1, 2$; in the case of $r = 1$ for $k = 1, 2$ and in the case of $r = 2$ for $k = 1, 2, 3$[2]. Hence, the R-matrix (2) is given by $R = \frac{1}{2}(P_{\geqslant -r+k} - P_{<-r+k})$. Then,

The hierarchy of evolution equations are generated by the Casimir functionals

$$C_n(L) = \frac{1}{n+1} \operatorname{Tr}\left(L^{n+1}\right), \qquad dC_n(L) = L^n, \qquad L^n = L \star L \star \cdots \star L,$$

and for appropriate k has the form

$$L_{t_n} = \{(L^n)_{\geqslant -r+k}, L\}_\star = -\{(L^q)_{<-r+k}, L\}_\star \tag{20}$$

which are Lax hierarchies. Notice, that (20) are in general tri-Hamiltonian systems [17].

The Lie algebras A_α can be decomposed into direct sum of Lie subalgebras exactly in the same way as $A_0 \cong \mathfrak{a}$. Hence, the R-matrix is invariant under transformations (15). Moreover, as transformations (15) are Lie algebra isomorphisms the Lax hierarchies (20) are also invariant with respect to them.

To construct (1+1)-dimensional closed systems with a finite number of fields we have to choose properly restricted Lax operators L which give consistent Lax equations (20). In the case of $r = 0$ the admissible simplest restricted Lax operators are given in the form

$$\begin{aligned}
k = 0: &\quad L = p^N + u_{N-2} \star p^{N-2} + \cdots + u_1 \star p + u_0,\\
k = 1: &\quad L = p^N + u_{N-1} \star p^{N-1} + \cdots + u_0 + p^{-1} \star u_{-1},\\
k = 2: &\quad L = u_N \star p^N + u_{N-1} \star p^{N-1} + \cdots + u_0 + p^{-1} \star u_{-1} + p^{-2} \star u_{-2}.
\end{aligned} \tag{21}$$

In the case of $r = 1$ the admissible simplest restricted Lax operators are

$$\begin{aligned}
k = 1: &\quad L = p^N + u_{N-1} \star p^{N-1} + \cdots + u_{1-m} \star p^{1-m} + u_{-m} \star p^{-m},\\
k = 2: &\quad L = u_N \star p^N + u_{N-1} \star p^{N-1} + \cdots + u_{1-m} \star p^{1-m} + p^{-m}.
\end{aligned} \tag{22}$$

Comparing the Lax operators related to soliton systems (21) and (22) with the Lax operators related to the dispersionless systems (13) one observes that dispersionless systems for $r \neq 0, 1$ and some for $r = 0$ do not have counterpart soliton systems in the quantization scheme considered.

We will now display examples of some field and lattice soliton systems calculated in proposed quantization scheme. We consider the Lax hierarchy (20) with little changed numerations of

[2] In [17] there was omitted the case $r = 2$, $k = 1, 2, 3$, but it does not lead to any new soliton systems as it is related to the case $r = 0$, $k = 2, 1, 0$, respectively, by transformation $p \to p^{-1}$. In further considerations we will skip the case of $r = 2$.

evolution variables: $L_{t_n} = \left\{\left(L^{\frac{n}{N}}\right)_{\geqslant -r+k}, L\right\}_\star$, where N is the highest order of the Lax operator L. We will exhibit the first nontrivial equation of the Lax hierarchy. The advantage of the use of $\mathfrak{a}$ algebra is that during whole calculations there is no need of using the $\star^\alpha$-product in explicit form and we only use the commutation relations (17). As a result, one gets the same equations and Poisson structures as these obtained from quantized algebra A_0. The Hamiltonian systems related to quantized algebras A_α are simply reconstructed via the linear transformation (15). Such a procedure of calculations is applied in present examples and we have written down only the final results for A_α.

Example 1. The case of $r = 0$ and $k = 0$. The dispersionless Boussinesq system is given in the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x \\ -\frac{2}{3}uu_x \end{pmatrix}. \tag{23}$$

It has the following Lax representation [16] $L_{t_2} = \left\{\left(L^{\frac{2}{3}}\right)_{\geqslant 0}, L\right\}^0_{\mathrm{PB}}$ for Lax operator in the form $L = p^3 + up + v \in A$. The quantization procedure leads now to the following Lax operator $L = p^3 + u \star p + v \in \mathfrak{a}$. Then, one can derive the Boussinesq system from $L_{t_2} = \{(L^{\frac{2}{3}})_{\geqslant 0}, L\}_\star$. Now, by the transformation to the algebras A_α one finds the following systems

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x + (\alpha - 1)\,\hbar u_{2x} \\ -\frac{2}{3}uu_x + (1-\alpha)\,\hbar v_{2x} - \left(\frac{\alpha^2}{2} - \alpha + \frac{2}{3}\right)\hbar^2 u_{3x} \end{pmatrix}. \tag{24}$$

In the case of $\alpha = 0$ (24) is obviously the standard case of Boussinesq system obtained from Gel'fand–Dikii hierarchy, for $\alpha = 1$ it is the Moyal case. The limit $\hbar \to 0$ of (24) gives (23) as it should be.

Example 2. The case of $r = 1$ and $k = 1$. The dispersionless Toda system has the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} v_x \\ u_x v \end{pmatrix}. \tag{25}$$

The Lax representation for (25) is [16] $L_{t_2} = \{(L^2)_{\geqslant 0}, L\}^1_{\mathrm{PB}}$, where the Lax operator is $L = p + u + vp^{-1}$. Then, the quantization scheme leads to the following Lax operator in $\mathfrak{a}$ in the form $L = p + u(x) + v(x) \star p^{-1}$. One derives the Toda system from $L_{t_2} = \{(L^2)_{\geqslant 0}, L\}_\star$. Next, by the transformation to A_α one finds

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \frac{1}{\hbar}\begin{pmatrix} v\left(x + \left(1 - \frac{\alpha}{2}\right)\hbar\right) - v\left(x - \frac{\alpha}{2}\hbar\right) \\ v(x)\left[u\left(x + \frac{\alpha}{2}\hbar\right) - u\left(x - \left(1 - \frac{\alpha}{2}\right)\hbar\right)\right] \end{pmatrix}. \tag{26}$$

The case of $\alpha = 0$ of (26) is the standard case of Toda system, the case of $\alpha = 1$ is the Moyal case. Notice that in our construction Toda equation depends on continuous coordinate x contrary to a standard case when x is integer.

7 Conclusions

In a previous paper [16] we have constructed Lax representations for a wide class of dispersionless systems, derived from classical R-matrix theory. The number of the constructed dispersionless systems is much greater then the number of known soliton systems (dispersive integrable systems). So, the question rises whether for any dispersionless Lax hierarchy one can construct a related soliton hierarchy. We have tried to obtain an answer to this question via the procedure of deformation quantization for Poisson algebras of dispersionless systems and appropriate R-matrix theory. We have managed to quantize all Poisson algebras (with arbitrary r (10)) but the R-matrix after quantization, at least of the form (2), exists only in the case of two algebras, i.e. for $r = 0$ ($r = 2$) and $r = 1$, respectively. The first case leads to soliton field systems related to the algebra of pseudo-differential operators, and the second one leads to lattice soliton systems related to the algebra of shift operators. In that sense, although we have failed to construct new soliton equations through presented deformation procedure, nevertheless we have found a unified procedure of the construction of field and lattice Hamiltonian soliton systems in one scheme.

For more details and proofs we send the reader to [17].

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