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Generalized Binary Darboux-like Theorem for Constrained Kadomtsev–Petviashvili (cKP) Flows

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Generalizations of the classical Darboux theorem are established for pseudo–differential scattering operators of the form $L = \sum\limits_{i=0}^{n} u_i \mathcal{D}^i + \boldsymbol{q}\mathcal{M}\mathcal{D}\boldsymbol{r}^\top$. Iteration of the simplest binary Darboux-like transformations (BDT) leads to a gauge transformed operator with coefficients given by Grammian-type formulas involving a set of eigenfunctions of an operator L and adjoint eigenfunctions (eigenfunctions of the transposed operator L^τ). Nonlinear integrable partial differential equations are associated with the scattering operator L which arise as a symmetry reduction of the matrix KP hierarchy. With a suitable linear time evolution for the eigenfunctions the binary Darboux-like transformation is used to obtain solutions of the integrable equations in terms of Grammian-type determinants.

1 Introduction

Since the mid-1970s, Darboux transformations (DTs) have been shown to be a powerful tool for studying of the nonlinear integrable systems of soliton theory. Darboux's original result [1] is a transformation of the Schrödinger equation $f_{xx} + uf = \lambda f$ with different potentials. With a first eigenfunction for a potential u given, it can be used to map all other eigenfunctions to new eigenfunctions for a Schrödinger operator with a modified potential. Crum [2] considered the iteration of the DT, when a collection of eigenfunctions is given, and Crum's formulae correspond to compact Wronskian representations of the n-soliton solutions of the KdV [3].

Generalizations of these ideas for other spectral equations and their associated soliton systems have been successfully applied to a variety of systems. An excellent overview summarizing various forms of DTs and their applications in soliton theory is the book [3] by Matveev and Salle.

We show that not only differential Lax operators but also pseudo-differential scattering problems may be considered. Our aim is to apply the BDT to Lax operators of the form

$$L = \sum_{i=0}^{n} u_i \mathcal{D}^i + \boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{r}^\top \tag{1}$$

(where $\boldsymbol{q} = (q_1, \dots, q_l)$, $\boldsymbol{r} = (r_1, \dots, r_l)$, $\mathcal{M}$ is $(l \times l)$-matrix of constants) and obtain Grammian type solution formulas for the associated nonlinear evolution equations.

Scattering operators of the class (1) were encountered in the context of reductions of the KP hierarchy [4–12], which generalizes a reduction concept originally used to reduce soliton systems in $1 + 1$ dimensions to finite dimensional equations [13–15].

In Section 2 we briefly review the essentials of pseudo-differential symbols to be used for the description of the (constrained) KP hierarchy. We review the fact that for each pair of eigenfunctions φ and adjoint eigenfunctions ψ (eigenfunctions of the transposed operator) the squared eigenfunction can be integrated to a potential $\Omega[\psi, \varphi]$ satisfying $\Omega_x[\psi, \varphi] = \psi^\top \varphi$ and a corresponding time evolution [10, 16]. With these potentials a proper spectral problem can be formulated for pseudo-differential operators of the form (1).

In Section 3 general formulation for the BDT for Lax pairs is presented in terms of pseudo-differential symbols. The resulting Lax pairs are characterized by a gauge operator. Also we show that the BDT will leave the class of operators (1) invariant, if the triggering eigenfunctions are solutions of a spectral problem involving the potential associated with eigenfunctions and adjoint eigenfunctions. This, in combinations with the result on the temporal aspects of the BDTs, provides an efficient tool to generate exact solutions of the nonlinear equations associated with operators of the form (1).

2 The algebra ζ of the pseudo-differential operators. Integrable hierarchies in $2+1$ and $1+1$ dimensions

We consider the algebra

$$\zeta = \left\{ \sum_{i \ll \infty} u_i(x) \mathcal{D}^i \right\}, \tag{2}$$

of pseudo-differential operators of a "space" variable $x \in \mathbb{R}$ with coefficients u_i being (in general case) $(N \times N)$-matrix-valued functions of x. For positive powers $m \in \mathbb{Z}_+$ of the differential operator $\mathcal{D}$, the algebraic multiplication with a multiplication operator represented by a function $u = u(x)$ is given by the usual Leibnitz rule

$$\mathcal{D}^m u := \sum_{j \geq 0} \left(\begin{array}{c} m \\ j \end{array} \right) \frac{\partial^j u}{\partial x^j} \mathcal{D}^{m-j}, \qquad \left(\begin{array}{c} m \\ j \end{array} \right) = \frac{m(m-1)\cdots(m-j+1)}{j!}, \qquad 0! = 1$$

with the same definition for negative powers, so that in particular $\mathcal{D}^{-1} u := \sum_{j \geq 0} (-1)^j u^{(j)} \mathcal{D}^{-1-j}$, we obtain a well defined associative algebraic structure on ζ (2).

The integrable scalar KP hierarchy can be formulated through linear system [14]

$$f_{t_n} = B_n\{f\}, \qquad g_{t_n} = -B_n^\tau\{g\}, \qquad n \in \mathbb{N}, \qquad t_1 := x, \quad t_2 := y, \quad t_3 := t, \quad \ldots \tag{3}$$

with an infinite hierarchy of operators $B_n := (L_{\mathrm{KP}}^n)_+$, where subscript $+$ denotes the projection of the powers of the micro-differential Lax operator

$$L := L_{KP} = \mathcal{D} + \sum_{i \geq 1} U_i \mathcal{D}^{-1} \tag{4}$$

onto its differential part and B_n^τ is a transposed operator. The first of the operators B_n are computed as

$$B_1 = \mathcal{D}, \qquad B_2 = \mathcal{D}^2 + 2U_1, \qquad B_3 = \mathcal{D}^3 + 3U_1\mathcal{D} + 3U_{1x} + U_2.$$

The Lax equation and the implying compatibility conditions of (3) are as follow:

$$L_{t_n} = [B_n, L], \qquad n \in \mathbb{N}, \tag{5}$$

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0. \tag{6}$$

They yield the field equations along with differential relations for the fields U_i such as

$$U_2 = -\frac{1}{2} u_x + \frac{1}{2} \partial_x^{-1} u_y,$$

$$U_3 = \frac{1}{4} u_{xx} - \frac{1}{2} u_y - \frac{1}{2} u^2 + \frac{1}{4} \partial_x^{-2} u_{yy}, \qquad \partial_x^{-1} f := \int^x f(s)\, ds,$$

where the leading coefficient $u := U_1$ plays a distinguished role, since it satisfies the KP equation (9) ($n = 2$, $k = 3$ in the (6)) and its higher flows. Consequently all equations of this hierarchy can be expressed via the single field $u := U_1$.

We will refer to solutions of (3) as eigenfunctions and adjoint eigenfunctions respectively.

It has been observed that the product qr of an eigenfunction q and an adjoint eigenfunction r represents a conserved covariant of the KP hierarchy. Hence, one may impose constraints on the KP flows by expressing the dynamical field u via qr [4, 5, 8, 9]. In terms of the Lax operator (4) these constraints are characterized by the requirement that the negative differential orders of a power L^k have the specific form

$$\left(L^k\right)_- = q\mathcal{D}^{-1}r,$$

or, in general,

$$\left(L^k\right)_- = \sum_{i=1}^{m} q_i\mathcal{D}^{-1}r_i := \boldsymbol{q}\mathcal{D}^{-1}\boldsymbol{r}^\top, \tag{7}$$

where m pairs of (adjoint) eigenfunctions are considered [6, 7]. For given k this constraint leaves the coefficients $u := U_1$, $U_2, \ldots, U_{k-1}$ in (4) and q_i, r_i, $i = \overline{1,m}$ as independent fields, whereas $U_k, \ldots, U_{k+1}, \ldots$ become differential expressions of these functions. One may replace L by the new Lax operator

$$L_{k\text{-cKP}} := L^k = \mathcal{D}^k + ku\mathcal{D}^{k-2} + \sum_{i=0}^{k-3} u_i\mathcal{D}^i + \boldsymbol{q}\mathcal{D}^{-1}\boldsymbol{r}^\top, \tag{8}$$

where $u_0, \ldots, u_{k-3}$ are differential expressions of $u, U_2, \ldots, U_{k-1}$. Thus the fields $u_0, \ldots, u_{k-3}, u$ may be regarded as new independent fields related to $u = U_1, U_2, \ldots, U_{k-1}$ by a coordinate transformation. The constraints (7) may be regarded as multi-component symmetry reductions of the KP hierarchy (5)–(6), since

$$u_{t_k} = \operatorname{res} L_{t_k} = \operatorname{res}\left[\left(L^k\right)_+, L\right] = \operatorname{res}\left(L^k\right)_x, \qquad \operatorname{res}\left(\sum_{i\ll\infty} a_i\mathcal{D}^i\right) := a_{-1},$$

so that (7) implies the relation [6, 7]

$$u_{t_k} = \left(\boldsymbol{q}\boldsymbol{r}^\top\right)_x$$

between the k-th flow of the KP hierarchy and the symmetry generated by square eigenfunctions.

Remark 1. If $\boldsymbol{q} = (q_1, \ldots, q_l)$ are eigenfunctions then the product $\boldsymbol{q}\mathcal{M}$ with a constant $(l \times l)$-matrix $\mathcal{M}$ is a vector of the eigenfunctions for KP hierarchy (5)–(6) too.

Remark 2. If we eliminate U_i, $i \geq 2$ from (5), the remaining equations for the function $u := U_1$ in (6) represent the $(2+1)$-dimensional KP equation

$$4u_{t_3} = u_{xxx} + 12uu_x + 3\partial_x^{-1}u_{t_2t_2} \tag{9}$$

and its higher versions.

Remark 3. The so-called "k-constrained KP (k-cKP) hierarchy" (scalar) is the ordinary KP hierarchy (5)–(6) restricted to a pseudo-differential operator $L_{k\text{-cKP}}$ of the form (8)

$$L := L^k = \mathcal{D}^k + \sum_{i=0}^{k-2} u_i\mathcal{D}^i + \boldsymbol{q}\mathcal{D}^{-1}\boldsymbol{r}^\top, \qquad u_{k-2} = ku,$$

$$L_{t_n} = [B_n, L], \qquad B_n := \left(L^{\frac{n}{k}}\right)_+, \qquad n = 2, 3, \ldots,$$

where u_i, $\boldsymbol{q} = (q_1, \dots, q_l)$, $\boldsymbol{r} = (r_1, \dots, r_l)$ are all functions of $x := t_1$ and higher time evolution variables $t_2 := y$, $t_3 = t$, ... In particular, the constraints

$$u_x = \left(\boldsymbol{q}\mathcal{M}\boldsymbol{r}^\top\right)_x, \qquad u_y = \left(\boldsymbol{q}\mathcal{M}\boldsymbol{r}^\top\right)_x \qquad \text{and} \qquad u_t = \left(\boldsymbol{q}\mathcal{M}\boldsymbol{r}^\top\right)_x,$$

are known to produce vector versions of $(1+1)$-dimensional of the AKNS ($k = 1$), the Yajima–Oikawa ($k = 2$) and the Melnikov hierarchy ($k = 3$), respectively [6, 7].

3 BDT for the $(N \times N)$-matrix cKP hierarchy

Theorem 1. *Let φ, ψ be $(N \times K)$ matrix solutions of the scattering problem*

$$\sum_{i=0}^{n} u_i \varphi^{(i)} + \boldsymbol{q}\mathcal{M}_0 \Omega[\boldsymbol{r}, \varphi] := L\{\varphi\} = \varphi\Lambda,$$

and the transposed scattering problem

$$\sum_{i=0}^{n} (-1)^i \left(u_i^\top \psi\right)^{(i)} - \boldsymbol{r}\mathcal{M}_0^\top \Omega[\boldsymbol{q}, \psi] := L^\tau\{\psi\} = \psi\tilde{\Lambda},$$

where Λ, $\tilde{\Lambda}$ are constant spectral matrices and $\Omega[\boldsymbol{r}, \varphi]$, $\Omega[\boldsymbol{q}, \psi]$ are functions satisfying $\Omega_x[\boldsymbol{r}, \varphi] = \boldsymbol{r}^\top \varphi$, $\Omega_x[\boldsymbol{q}, \psi] = \boldsymbol{q}^\top \psi$. Then

1. *Let f be a solution of the spectral problem $\sum\limits_{i=0}^{n} u_i f^{(i)} + \boldsymbol{q}\mathcal{M}_0 \Omega[\boldsymbol{r}, f] := L\{f\} = f\hat{\Lambda}$ with some constant spectral matrix $\hat{\Lambda}$ and functions $\Omega_x[\boldsymbol{r}, f] = \boldsymbol{r}^\top f$.*

Then $F := W\{f\} = f - \Phi\Omega[\psi, f]$ satisfies the spectral problem

$$\sum_{i=0}^{n} \hat{u}_i F^{(i)} + \Phi\mathcal{M}\Omega[\Psi, F] + \hat{\boldsymbol{q}}\mathcal{M}_0 \Omega[\hat{\boldsymbol{r}}, F] = F\hat{\Lambda}, \tag{10}$$

where $\Omega[\Psi, F]$ and $\Omega[\hat{\boldsymbol{r}}, F]$ are functions satisfying $\Omega_x[\Psi, F] = \Psi^\top F$ and $\Omega_x[\hat{\boldsymbol{r}}, F] = \hat{\boldsymbol{r}}^\top F$.

2. *The gauge transformed operator $\hat{L} := WLW^{-1}$ with $W = I - \varphi(C + \Omega[\psi, \varphi])^{-1}\mathcal{D}^{-1}\psi^\top$, where C is a constant $(K \times K)$ matrix, has the form*

$$\hat{L} = \sum_{i=0}^{n} \hat{u}_i \mathcal{D}^i + \Phi\mathcal{M}\mathcal{D}^{-1}\Psi^\top + \hat{\boldsymbol{q}}\mathcal{M}_0\mathcal{D}^{-1}\hat{\boldsymbol{r}}^\top, \tag{11}$$

with $\mathcal{M} = C\Lambda - \tilde{\Lambda}^\top C$, $\Phi = \varphi(C + \Omega[\psi, \varphi])^{-1}, \Psi^\top = (C + \Omega[\psi, \varphi])^{-1}\psi^\top$, $\hat{\boldsymbol{q}} = \boldsymbol{q} - \Phi\Omega[\psi, \boldsymbol{q}]$, $\hat{\boldsymbol{r}} = \boldsymbol{r} - \Psi\Omega[\varphi, \boldsymbol{r}]$. The coefficients $\hat{u}_i$, $i = \overline{0, n}$ can be expressed in the form $\hat{u}_n = u_n$, $\hat{u}_i = u_i + P_i(u_{i+1}, \dots, u_n, \varphi, \psi)$ with suitable differential expressions P_i of the indicated arguments.

3. *If we have a Lax equation $L_t = [M, L](:= ML - LM)$, $M = \sum\limits_{i=0}^{m} v_i \mathcal{D}^i$ and φ, ψ are solutions of the evolution equations*

$$\varphi_t = M\{\varphi\}, \qquad \psi_t = -M^\tau\{\psi\} \tag{12}$$

then BDT triggered by φ and ψ will map any eigenfunction f satisfying both equations

$$f_t = M\{f\}, \qquad L\{f\} = f\hat{\Lambda} \tag{13}$$

to a new eigenfunction F for the transformed Lax pair $\hat{L}$ (11), $\hat{M} = -W(\partial_t - M)W^{-1} + \partial_t$, which satisfies both the new evolution equations $F_t = \hat{M}\{F\}$ as well as the new spectral problem $\hat{L}\{F\} = F\hat{\Lambda}$ (10).

Proof. 1. Let $\hat{L} := WLW^{-1}$ and $L\{f\} = f\hat{\Lambda}$ then $F := W\{f\} = f - \Phi\Omega[\boldsymbol{r}, f]$ satisfies

$$\hat{L}\{F\} := WLW^{-1}\{W\{f\}\} = WL\{f\} = W\{f\hat{\Lambda}\} = W\{f\}\hat{\Lambda} = F\hat{\Lambda},$$

and an explicit form of the operator $\hat{L}$ is given in the second part of the Theorem 1.

2. For proving the second part of the Theorem 1 we use helpful formulas

$$\begin{aligned}
&1)\ \Psi^\top\Phi = (C+\Omega[\psi,\varphi])^{-1}\psi^\top\varphi(C+\Omega[\psi,\varphi])^{-1} = [(C+\Omega[\psi,\varphi])^{-1}]_x \\
&\quad\Rightarrow \Omega[\Psi,\Phi] := -(C+\Omega[\psi,\varphi])^{-1}, \qquad (14)\\
&2)\ \Psi^\top F = (C+\Omega[\psi,\varphi])^{-1}\psi^\top[f - \varphi(C+\Omega[\psi,\varphi])^{-1}\Omega[\psi,f]] = [(C+\Omega[\psi,\varphi])^{-1}\Omega[\psi,f]]_x \\
&\quad\Rightarrow \Omega[\Psi,F] := -(C+\Omega[\psi,\varphi])^{-1}\Omega[\psi,f], \qquad (15)\\
&3)\ (I+\varphi\mathcal{D}^{-1}\Psi^\top)\{F\} := F + \varphi\Omega[\Psi,F] := W\{f\} + \varphi(C+\Omega[\psi,\varphi])^{-1}\Omega[\psi,f] \\
&\qquad = f - \Phi\Omega[\psi,f] + \Phi\Omega[\psi,f]] = f \\
&\quad\Rightarrow W^{-1} = I + \varphi\mathcal{D}^{-1}\Psi^\top := I + \varphi\Omega[\Psi,\cdot] \qquad (16)
\end{aligned}$$

and W^{-1} (16) is the inverse operator to $W = I - \Phi\mathcal{D}^{-1}\psi^\top := I - \Phi\Omega[\psi,\cdot]$.

Analogously to the statement of part 1 of Theorem 1 we prove the next statement.

4) Let g be a solution of the spectral problem $L^\tau\{g\} = g\tilde{\hat{\Lambda}}$ with some constant spectral matrix $\tilde{\hat{\Lambda}}$

$$\sum_{i=0}^{n}(-1)^i\big(u_i^\top g\big)^{(i)} - \boldsymbol{r}\mathcal{M}_0^\top\Omega[\boldsymbol{q},g] = g\tilde{\hat{\Lambda}} \qquad (17)$$

and functions $\Omega_x[\boldsymbol{q},g] = \boldsymbol{q}^\top g$.

Then

$$G := W^{-1,\tau}\{g\} = (W^\tau)^{-1}\{g\} = \big(I - \Psi\mathcal{D}^{-1}\varphi^\top\big)\{g\} = g - \Psi\Omega[\varphi,g]$$

satisfies the spectral problem

$$\hat{L^\tau}\{G\} = G\tilde{\hat{\Lambda}}, \qquad \hat{L^\tau} := W^{-1,\tau}L^\tau W^\tau = (\hat{L})^\tau. \qquad (18)$$

$$\begin{aligned}
5)\ G^\top F &= [g^\top - \Omega[g,\varphi](C+\Omega[\psi,\varphi])^{-1}\psi^\top][f - \varphi(C+\Omega[\psi,\varphi])^{-1}\Omega[\psi,f]] \\
&= g^\top f - g^\top\varphi(C+\Omega[\psi,\varphi])^{-1}\Omega[\psi,f] - \Omega[g,\varphi](C+\Omega[\psi,\varphi])^{-1}\psi^\top f \\
&\quad + \Omega[g,\varphi](C+\Omega[\psi,\varphi])^{-1}\Omega[\psi,f](C+\Omega[\psi,\varphi])^{-1}\psi^\top f \\
&= [\Omega[g,f] - \Omega[g,\varphi](C+\Omega[\psi,\varphi])^{-1}\Omega[\psi,f]]_x \\
&\Rightarrow \Omega[G,F] := \Omega[g,f] - \Omega[g,\varphi](C+\Omega[\psi,\varphi])^{-1}\Omega[\psi,f]. \qquad (19)
\end{aligned}$$

6) For functions h_i, $i = \overline{1,4}$ and the differential operator $A \in \zeta_+$ next formulas hold (see e.g. [16, 17])

$$Ah_1\mathcal{D}^{-1}h_2 = \big(Ah_1\mathcal{D}^{-1}h_2\big)_+ + A\{h_1\}\mathcal{D}^{-1}h_2, \qquad (20)$$

$$h_1\mathcal{D}^{-1}h_2A = \big(h_1\mathcal{D}^{-1}h_2A\big)_+ + h_1\mathcal{D}^{-1}[A^\tau\{h_2\}]^\tau, \qquad (21)$$

$$h_1\mathcal{D}^{-1}h_2h_3\mathcal{D}^{-1}h_4 = h_1\Omega[h_2^\top, h_3]\mathcal{D}^{-1}h_4 - h_1\mathcal{D}^{-1}\Omega[h_2^\top, h_3]h_4, \qquad (22)$$

and for the operator

$$L = \sum_{i=0}^{n}u_i\mathcal{D}^i + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top := L_+ + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top,$$

using formulas (16), (20), (21) by direct calculations we obtain

$$
\begin{aligned}
L \to WLW^{-1} &= \big(I - \Phi\mathcal{D}^{-1}\psi^\top\big)\big(L_+ + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\big)\big(I + \varphi\mathcal{D}^{-1}\Psi^\top\big) \\
&= \big(L - \Phi\mathcal{D}^{-1}\psi^\top L\big)\big(I + \varphi\mathcal{D}^{-1}\Psi^\top\big) = L + L\varphi\mathcal{D}^{-1}\Psi^\top - \Phi\mathcal{D}^{-1}\psi^\top L \\
&- \Phi\mathcal{D}^{-1}\psi^\top L\varphi\mathcal{D}^{-1}\Psi^\top = L_+ + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top + L_+\varphi\mathcal{D}^{-1}\Psi^\top \\
&+ \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\varphi\mathcal{D}^{-1}\Psi^\top - \Phi\mathcal{D}^{-1}\psi^\top L_+ - \Phi\mathcal{D}^{-1}\psi^\top\boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top \\
&- \Phi\mathcal{D}^{-1}\psi^\top L_+\varphi\mathcal{D}^{-1}\Psi^\top - \Phi\mathcal{D}^{-1}\psi^\top\boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\varphi\mathcal{D}^{-1}\Psi^\top \\
&= L_+ + \big(L_+\varphi\mathcal{D}^{-1}\Psi^\top\big)_+ - \big(\Phi\mathcal{D}^{-1}\psi^\top L_+\big)_+ - \big(\Phi\mathcal{D}^{-1}\psi^\top L_+\varphi\mathcal{D}^{-1}\Psi^\top\big)_+ \\
&+ \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top + L_+\{\varphi\}\mathcal{D}^{-1}\Psi^\top + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\varphi\mathcal{D}^{-1}\Psi^\top - \Phi\mathcal{D}^{-1}(L_+^\tau\{\psi\})^\top \\
&- \Phi\mathcal{D}^{-1}\psi^\top\boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top - \big(\Phi\mathcal{D}^{-1}\psi^\top L_+\varphi\mathcal{D}^{-1}\Psi^\top\big)_- \\
&- \Phi\mathcal{D}^{-1}\psi^\top\boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\varphi\mathcal{D}^{-1}\Psi^\top = \hat{L}_+ + \hat{L}_-,
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{L}_+ &= L_+ + \big(L_+\varphi\mathcal{D}^{-1}\Psi^\top\big)_+ - \big(\Phi\mathcal{D}^{-1}\psi^\top L_+\big)_+ - \big(\Phi\mathcal{D}^{-1}\psi^\top L_+\varphi\mathcal{D}^{-1}\Psi^\top\big)_+ \\
&:= L_+ + U_{+1} - U_{+2} - U_{+3} = \sum_{i=0}^{n} \hat{u}_i\mathcal{D}^i, \qquad \hat{u}_n = u_n.
\end{aligned} \tag{23}
$$

We obtain the exact form of the differential operators U_{1+}, U_{2+}, U_{3+} after using the Leibnitz rule for a composition of pseudo-differential symbols. These operators are obtained in [16] in explicit forms.

The integral part $\hat{L}_-$ of the operator $\hat{L}$ may be constructed after long transformation using formulas (20)–(22) and the definition of the functions Φ and Ψ.

As a result we obtain

$$
\begin{aligned}
\hat{L}_- = {} & (L\{\varphi\} - \Phi\Omega[\psi, L\{\varphi\}])\mathcal{D}^{-1}\Psi^\top - \Phi\mathcal{D}^{-1}(L^\tau\{\psi\} - \Psi\Omega[\varphi, L^\tau\{\psi\}])^\top + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top \\
& - \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}(\Psi\Omega[\varphi,\boldsymbol{r}])^\top - \Phi\Omega[\psi,\boldsymbol{q}]\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top + \Phi\Omega[\psi,\boldsymbol{q}]\mathcal{M}_0\mathcal{D}^{-1}(\Psi\Omega[\varphi,\boldsymbol{r}])^\top.
\end{aligned}
$$

From the formulas

$$
\Omega[\psi, L\{\varphi\}] = \Omega[\psi, \varphi\Lambda] = \Omega[\psi,\varphi]\Lambda, \quad \Omega[\varphi, L^\tau\{\psi\}] = \Omega[\varphi, \psi\tilde{\Lambda}] = \Omega[\varphi,\psi]\tilde{\Lambda} = (\tilde{\Lambda}^\top\Omega[\varphi,\psi])^\top
$$

and the definition of the functions

$$
\Phi := \varphi(C + \Omega[\psi,\varphi])^{-1}, \qquad \Psi := \psi\big(C^\top + \Omega[\varphi,\psi]\big)^{-1}
$$

it follows

$$
\hat{L}_- = \Phi\big(C\Lambda - \tilde{\Lambda}^\top C\big)\mathcal{D}^{-1}\Psi^\top + \big(\boldsymbol{q} - \Phi\Omega[\psi,\boldsymbol{q}]\big)\mathcal{D}^{-1}\big(\boldsymbol{r} - \Psi\Omega[\varphi,\boldsymbol{r}]\big)^\top, \tag{24}
$$

thus the gauge transformed operator $\hat{L} := \hat{L}_+ + \hat{L}_-$ (23), (24) has the form (11) and F solves the new spectral problem (10) with the potentials $\Omega[\Psi, F]$ (15) and

$$
\Omega[\hat{\boldsymbol{r}}, F] = \Omega[\boldsymbol{r}, f] - \Omega[\hat{\boldsymbol{r}},\varphi](C + \Omega[\psi,\varphi])^{-1}\Omega[\psi, f], \qquad \Omega_x[\hat{\boldsymbol{r}}, F] = \hat{\boldsymbol{r}}^\top F. \tag{25}
$$

Corollary 1. 1. *From formulas* (17)–(18) *we see that function* $G = W^{-1,\tau}\{g\}$ *solves the new adjoint spectral problem*

$$
\hat{L^\tau}\{G\} := \sum_{i=0}^{n}(-1)^i\big(\hat{u}_i^\top G\big)^{(i)} - \Psi\mathcal{M}\Omega[\Phi, G] - \hat{\boldsymbol{r}}\mathcal{M}_0\Omega[\hat{\boldsymbol{q}}, G] = G\tilde{\hat{\Lambda}}
$$

with the potentials $\Omega[\Phi, G]$, $\Omega[\hat{\boldsymbol{q}}, G]$, *which can be calculated similarly to formulas* (14)–(15), (24)*:*

$$\Omega_x[\Phi, G] := \Phi^\top G = \left[\left(C^\top + \Omega[\varphi,\psi]\right)^{-1}\Omega[\varphi, g]\right]_x$$

$$\Rightarrow\ \Omega[\Phi, G] = \left(C^\top + \Omega[\varphi,\psi]\right)^{-1}\Omega[\varphi, g], \tag{26}$$

$$\Omega[\hat{\boldsymbol{q}}, G] = \Omega[\boldsymbol{q}, g] - \Omega[\boldsymbol{q},\psi]\left(C^\top + \Omega[\varphi,\psi]\right)^{-1}\Omega[\varphi, g]. \tag{27}$$

2. *The functions* F, G *are integrated in the potential* $\Omega[G, F]$ *according to formula* (19)*.*

3. *Let us have a differential operator* $M \in \zeta_+$*. Suppose that equations* (12)–(13) *hold and the adjoint eigenfunction* g *satisfies the linear evolution equation*

$$g_t = -M^\tau\{g\}, \tag{28}$$

then F, G *solve the new linear evolution systems*

$$F_t = \hat{M}\{F\}, \qquad G_t = -\hat{M}^\tau\{G\}, \tag{29}$$

where $\hat{M} = -W(\partial_t - M)W^{-1} + \partial_t = \sum\limits_{i=0}^{m} \hat{v}_i \mathcal{D}^i$*.*

A proof of this fact and exact formulas for the coefficients $\hat{v}_i$, $i = \overline{0, m-1}$; $\hat{v}_m = v_m$ can be found in [16]. From the compatibility conditions $L_t = [M, L]$, $L_t^\tau = [L^\tau, M^\tau]$ for the linear systems $L\{f\} = f\hat{\Lambda}$, $f_t = M\{f\}$ and $L^\tau\{g\} = g\tilde{\hat{\Lambda}}$, $g_t = -M^\tau\{f\}$ respectively compatibility conditions for the new linear equations (10), (18), (29) follow:

$$\hat{L}_t = [\hat{M}, \hat{L}], \qquad \hat{L_t^\tau} = [\hat{L^\tau}, \hat{M^\tau}].$$

From the Lagrange formula (see [16]) for the systems (13), (28)

$$(g^\top f)_t = \left[\sum_{i=1}^{m}\sum_{j=1}^{i-1}(-1)^j\left(g^\top v_i\right)^{(i)} f^{(i-j-1)}\right]_x \tag{30}$$

we obtain a possibility to express all potentials (14), (15), (19), (25), (26), (27) in terms of fixed potential

$$\Omega[g, f] = \int_{(x_0,t_0)}^{(x,t)} P[g, f]\, dx + Q[g, f]\, dt, \qquad \Omega[g, f](x_0, t_0) = 0, \tag{31}$$

where $P[g, f] = \Omega_x[g, f] = g^\top f$, $Q[g, f]_x = P_t[g, f]$ (see (30)). ■

4 Conclusion remarks

I. In paper [18] a general $(N \times N)$-matrix operator W (16) of the BDT was obtain as a composition of K simplest Darboux-type transformations and K simplest inverse Darboux-type transformations. In this paper it is shown too that the operator W admits a so-called canonical factorization by the K simplest elementary Darboux-like transformations.

II. We used a particular case of the general dressing operator W for integration of the Lax–Zakharov–Shabat equations in a matrix algebra of differential [16] and pseudo-differential (in the scalar case cKP hierarchy, see [17, 19]) operators [20]. This particular case corresponds to a $(K \times K)$-matrix potential function Ω (31) of the form

$$\Omega[\psi, \varphi] = \mathcal{D}^{-1}\left\{\psi^\top\varphi\right\} := C + \int_{\pm\infty}^{x} \psi^\top(s,t)\varphi(s,t)\, ds.$$

Wide classes of exact solutions for the integrable systems of the soliton theory in terms of Grammian–type determinants were obtained in these papers.

III. The general BDT Theorem 1 will be applied to constructing the solutions of matrix cKP hierarchy in forthcoming publications.

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