Classical Dynamical Yang–Baxter Equations and Quasi-Poisson Homogeneous Spaces

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In this paper we provide a connection between the solutions of the classical dynamical Yang–Baxter equation (with not necessary Abelian base) and quasi-Poisson homogeneous spaces of quasi-Poisson Lie groups.

1 Introduction

This paper is a continuation of [6]. Let us recall the main result of [6]. Let G be a Lie group, $\mathfrak{g} = \operatorname{Lie} G$, $U \subset G$ a connected closed Lie subgroup such that the corresponding subalgebra $\mathfrak{u} \subset \mathfrak{g}$ is reductive in $\mathfrak{g}$ (i.e., there exists an $\mathfrak{u}$-invariant subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{m}$), and $\Omega \in (\mathfrak{u}\otimes\mathfrak{u}) \oplus (\mathfrak{m}\otimes\mathfrak{m})$ a symmetric tensor. Take a solution $\rho \in \mathfrak{g}\otimes\mathfrak{g}$ of the classical Yang–Baxter equation such that $\rho + \rho^{21} = \Omega$ and consider the corresponding Poisson Lie group structure π_ρ on G. Assuming additionally that

$$\rho + s \in \frac{\Omega}{2} + \left(\bigwedge\nolimits^2 \mathfrak{m}\right)^{\mathfrak{u}} \tag{1}$$

for some element $s \in \bigwedge^2 \mathfrak{g}$ that satisfies a certain "twist" equation, we establish a 1-1 correspondence between the moduli space of classical dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$ with the symmetric part $\frac{\Omega}{2}$ and the set of all structures of Poisson homogeneous (G, π_ρ)-spaces on G/U. We emphasize that the first example of such a correspondence was found by Lu in [8].

In this paper we generalize the main result of [6]. We replace Poisson Lie groups (resp. Poisson homogeneous spaces) by quasi-Poisson Lie groups (resp. quasi-Poisson homogeneous spaces), but even in the Poisson case our result (see Theorem 2) is stronger than in [6]: condition (1) is relaxed now. We hope that now we present this result in its natural generality.

The paper is organized as follows. In Section 2 we present the definitions of classical dynamical r-matrices, quasi-Poisson Lie groups and their quasi-Poisson homogeneous spaces, and then formulate and prove the main result of this paper, Theorem 2. In Section 3 we consider an example: the case of quasi-triangular (in the strict sense) classical dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$, where $\mathfrak{g}$ is a complex semisimple Lie algebra, and $\mathfrak{u}$ is its regular reductive subalgebra.

All Lie algebras in this paper assumed to be finite-dimensional, and the ground field is $\mathbb{C}$.

2 General results

In this section we describe a connection between quasi-Poisson homogeneous spaces and classical dynamical r-matrices (see Theorem 2).

First we recall some definitions. Suppose G is a Lie group, $U \subset G$ its connected Lie subgroup. Let $\mathfrak{g}$ and $\mathfrak{u}$ be the corresponding Lie algebras. Choose a basis $x_1, \dots, x_r$ in $\mathfrak{u}$. Denote by D the formal neighborhood of zero in $\mathfrak{u}^*$. By functions from D to a vector space V we mean the elements of the space $V[[x_1, \dots, x_r]]$, where x_i are regarded as coordinates on D. Further, if $\omega \in \Omega^k(D, V)$ is a k-form on D with values in vector space V, then by $\overline{\omega} : D \to \bigwedge^k \mathfrak{u} \otimes V$ we denote the corresponding function.

Definition 1 (see [5]). *Classical dynamical r-matrix for the pair* $(\mathfrak{g}, \mathfrak{u})$ is an $\mathfrak{u}$-equivariant function $r : D \to \mathfrak{g} \otimes \mathfrak{g}$ that satisfies the *classical dynamical Yang–Baxter equation (CDYBE)*:

$$\mathrm{Alt}(\overline{dr}) + \mathrm{CYB}(r) = 0,$$

where $\mathrm{CYB}(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$, and for $x \in \mathfrak{g}^{\otimes 3}$ we set $\mathrm{Alt}(x) = x^{123} + x^{231} + x^{312}$.

We will also require the *quasi-unitarity property*:

$$r + r^{21} = \Omega \in \left(S^2\mathfrak{g}\right)^{\mathfrak{g}}.$$

It is easy to see that if r satisfies the CDYBE and the quasi-unitarity condition, then Ω is constant.

We denote the set of all classical dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$ such that $r + r^{21} = \Omega$ by $\mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$.

Denote by $\mathbf{Map}(D, G)^{\mathfrak{u}}$ the set of all $\mathfrak{u}$-equivariant maps from D to G. Suppose that $r : D \to \mathfrak{g} \otimes \mathfrak{g}$ is an $\mathfrak{u}$-equivariant function. Then for any $g \in \mathbf{Map}(D, G)^{\mathfrak{u}}$ define a function $r^g : D \to \mathfrak{g} \otimes \mathfrak{g}$ by

$$r^g = (\mathrm{Ad}_g \otimes \mathrm{Ad}_g)\left(r - \overline{\eta_g} + \overline{\eta_g}^{21} + \tau_g\right),$$

where $\eta_g = g^{-1}dg$, and $\tau_g(\lambda) = (\lambda \otimes 1 \otimes 1)\left(\left[\overline{\eta_g}^{12}, \overline{\eta_g}^{13}\right](\lambda)\right)$. Then r^g is a classical dynamical r-matrix if and only if r is. The transformation $r \mapsto r^g$ is called a *gauge transformation*. In fact, it is an action of the group $\mathbf{Map}(D, G)^{\mathfrak{u}}$ on $\mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$.

Following [5], we denote the moduli space $\mathbf{Map}_0(D, G)^{\mathfrak{u}} \backslash \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ by $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ (here $\mathbf{Map}_0(D, G)^{\mathfrak{u}} = \{g \in \mathbf{Map}(D, G)^{\mathfrak{u}} : g(0) = e\}$).

Now we recall the definition of quasi-Poisson Lie groups and their quasi-Poisson homogeneous spaces (for details see [7, 1, 2]).

Definition 2. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, π_G a bivector field on G, and $\varphi \in \bigwedge^3 \mathfrak{g}$. A triple (G, π_G, φ) is called a *quasi-Poisson Lie group* if

$$\pi_G(gg') = (l_g)_*\pi_G(g') + (r_{g'})_*\pi_G(g),$$
$$\frac{1}{2}[\pi_G, \pi_G] = \overleftarrow{\varphi} - \overrightarrow{\varphi},$$
$$[\pi_G, \overleftarrow{\varphi}] = 0,$$

where l_g (resp. r_g) is left (resp. right) multiplication by g, $\overrightarrow{a}$ (resp. $\overleftarrow{a}$) is the left (resp. right) invariant tensor field on G corresponding to a and $[\cdot, \cdot]$ is the Schouten bracket of multivector fields.

Definition 3. Suppose that (G, π_G, φ) is a quasi-Poisson group, X is a homogeneous G-space equipped with a bivector field π_X. Then (X, π_X) is called a *quasi-Poisson homogeneous (G, π_G, φ)-space* if

$$\pi_X(gx) = (l_g)_*\pi_X(x) + (\rho_x)_*\pi_G(g),$$
$$\frac{1}{2}[\pi_X, \pi_X] = \varphi_X$$

(here l_g denotes the mapping $x \mapsto g \cdot x$, ρ_x is the mapping $g \mapsto g \cdot x$, and φ_X is the trivector field on X induced by φ).

Now take $\rho \in \mathfrak{g}\otimes\mathfrak{g}$ such that $\rho+\rho^{21}=\Omega\in(S^2\mathfrak{g})^{\mathfrak{g}}$. Let $\Lambda=\rho-\frac{\Omega}{2}\in\bigwedge^2\mathfrak{g}$. Define a bivector field on G by $\pi_\rho=\overrightarrow{\rho}-\overleftarrow{\rho}=\overrightarrow{\Lambda}-\overleftarrow{\Lambda}$. Set $\varphi=-\mathrm{CYB}(\rho)$. Then (G,π_ρ,φ) is a quasi-Poisson Lie group (such quasi-Poisson Lie groups are called *quasi-triangular*). Denote by $\mathbf{Homsp}(G,\pi_\rho,\varphi,U)$ the set of all (G,π_ρ,φ)-homogeneous quasi-Poisson structures on G/U. We will see that, under certain conditions, there is a bijection between $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ and $\mathbf{Homsp}(G,\pi_\rho,\varphi,U)$.

Assume that $b\in(\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{u}}$ is such that $b+b^{21}=\Omega$. Let $B=b-\frac{\Omega}{2}$. Define a bivector field on G by $\tilde{\pi}^\rho_b=\overrightarrow{b}-\overleftarrow{\rho}=\overrightarrow{B}-\overleftarrow{\Lambda}$. Then there is a bivector field on G/U defined by $\pi^\rho_b(\underline{g})=p_*(\tilde{\pi}^\rho_b(g))$ (here $p:G\to G/U$ is the canonical projection, and $\underline{g}=p(g)$). It is well defined, since b is $\mathfrak{u}$-invariant.

Proposition 1. *In this setting $(G/U,\pi^\rho_b)$ is a (G,π_ρ,φ)-quasi-Poisson homogeneous space iff* $\mathrm{CYB}(b)=0$ *in* $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$.

Proof. First we check the "multiplicativity" of π^ρ_b. For all $g\in G, u\in U$ we have

$$g\cdot\tilde{\pi}^\rho_b(u)+\pi_\rho(g)\cdot u=gu\cdot b-\rho\cdot gu=\tilde{\pi}^\rho_b(gu).$$

Using p_*, we get the required equality $\pi^\rho_b(\underline{g})=g\cdot\pi^\rho_b(\underline{e})+p_*\pi_\rho(g)$.

Now we need to prove that $\frac{1}{2}[\pi^\rho_b,\pi^\rho_b]=\varphi_{G/U}$ iff $\mathrm{CYB}(b)=0$ in $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$. We check it directly:

$$\frac{1}{2}[\tilde{\pi}^\rho_b,\tilde{\pi}^\rho_b]=\frac{1}{2}\left([\overrightarrow{B},\overrightarrow{B}]+[\overleftarrow{\Lambda},\overleftarrow{\Lambda}]\right)=-\overrightarrow{\mathrm{CYB}(B)}+\overleftarrow{\mathrm{CYB}(\Lambda)}=-\overrightarrow{\mathrm{CYB}(b)}+\overleftarrow{\varphi}.$$

Consequently, $\frac{1}{2}[\pi^\rho_b,\pi^\rho_b]=p_*(-\overrightarrow{\mathrm{CYB}(b)}+\overleftarrow{\varphi})=-p_*(\overrightarrow{\mathrm{CYB}(b)})+\varphi_{G/U}$. So we see that $\frac{1}{2}[\pi^\rho_b,\pi^\rho_b]=\varphi_{G/U}$ iff $\mathrm{CYB}(b)=0$ in $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$. ■

Suppose $r\in\mathbf{Dynr}(\mathfrak{g},\mathfrak{u},\Omega)$.

Proposition 2 (see [8]). $\mathrm{CYB}(r(0))=0$ *in* $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$.

Corollary 1. $r\mapsto\pi^\rho_{r(0)}$ *is a map from* $\mathbf{Dynr}(\mathfrak{g},\mathfrak{u},\Omega)$ *to* $\mathbf{Homsp}(G,\pi_\rho,\varphi,U)$.

Proposition 3 (see [6]). *If* $g\in\mathbf{Map}_0(D,G)^{\mathfrak{u}}$, *then* $\pi^\rho_{r(0)}=\pi^\rho_{r^g(0)}$.

Corollary 2. $r\mapsto\pi^\rho_{r(0)}$ *defines a map from* $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ *to* $\mathbf{Homsp}(G,\pi_\rho,\varphi,U)$.

From now on we will assume that the following conditions are satisfied:

$$\mathfrak{u}\text{ has an }\mathfrak{u}\text{-invariant complement }\mathfrak{m}\text{ in }\mathfrak{g}; \tag{2a}$$
$$\Omega\in(\mathfrak{u}\otimes\mathfrak{u})\oplus(\mathfrak{m}\otimes\mathfrak{m}). \tag{2b}$$

Consider the algebraic variety

$$\mathcal{M}_\Omega=\left\{x\in\frac{\Omega}{2}+\left(\textstyle\bigwedge^2\mathfrak{m}\right)^{\mathfrak{u}}\,\middle|\,\mathrm{CYB}(x)=0\text{ in }\textstyle\bigwedge^3(\mathfrak{g}/\mathfrak{u})\right\}.$$

Theorem 1 (Etingof, Schiffman; see [5]). (1) *Any class $\mathcal{C}\in\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ has a representative $r\in\mathcal{C}$ such that $r(0)\in\mathcal{M}_\Omega$. Moreover, this defines an embedding $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)\to\mathcal{M}_\Omega$.*

(2) *Assume that* (2b) *holds. Then the map $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)\to\mathcal{M}_\Omega$ defined above is a bijection.*

Proposition 4. *The mapping $b\mapsto\pi^\rho_b$ from $\mathcal{M}_\Omega$ to* $\mathbf{Homsp}(G,\pi_\rho,\varphi,U)$ *is a bijection.*

Proof. Let us construct the inverse mapping. Assume that π is a bivector field on G/U defining a structure of a (G,π_ρ,φ)-quasi-Poisson homogeneous space. Then $\pi(\underline{e}) \in \bigwedge^2(\mathfrak{g}/\mathfrak{u}) = \bigwedge^2\mathfrak{m}$. Consider $b = \frac{\Omega}{2} + \pi(\underline{e}) + p_*(\Lambda)$. We will prove that $b \in \mathcal{M}_\Omega$ and the mapping $\pi \mapsto b$ is inverse to the mapping $g \mapsto \pi_b^\rho$.

First we prove that $b \in \left(\bigwedge^2\mathfrak{m}\right)^{\mathfrak{u}} + \frac{\Omega}{2}$. For all $u \in U$ we have $\pi(\underline{e}) + p_*(\Lambda) = \pi(u\cdot\underline{e}) + p_*(\Lambda\cdot u) = u\cdot\pi(\underline{e}) + p_*(\pi_\rho(u)) + p_*(\Lambda\cdot u) = u\cdot\pi(\underline{e}) + p_*(u\cdot\rho - u\cdot\frac{\Omega}{2}) = u\cdot(\pi(\underline{e}) + p_*(\Lambda))$. It means that $\pi(\underline{e}) + p_*(\Lambda) \in \left(\bigwedge^2\mathfrak{m}\right)^{\mathfrak{u}}$.

Now we prove that $\pi = \pi_b^\rho$. By definition, $\pi_b^\rho(\underline{g}) = p_*(g\cdot\pi(\underline{e}) + g\cdot p_*\Lambda - \Lambda\cdot g) = \pi(\underline{g}) + p_*(g\cdot p_*\Lambda - \Lambda\cdot g - g\cdot\Lambda + \Lambda\cdot g) = \pi(\underline{g})$. So π_b^ρ defines a structure of (G,π_ρ,φ)-quasi-Poisson homogeneous space. By Proposition 1, it means that $b \in \mathcal{M}_\Omega$. ■

Theorem 2. *Suppose* (2a) *and* (2b) *are satisfied. Then the map* $r \mapsto \pi_{r(0)}^\rho$ *from* $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ *to* $\mathbf{Homsp}(G,\pi_\rho,\varphi,U)$ *is a bijection.*

Proof. This theorem follows from Theorem 1 and Proposition 4. ■

Remark 1. If $\varphi = -\mathrm{CYB}(\rho) = 0$, then (G,π_ρ) is a Poisson Lie group. In this case we get a bijection between $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ and the set of all Poisson (G,π_ρ)-homogeneous structures on G/U.

Remark 2. Assume that only (2a) holds. Clearly, in this case the map $r \mapsto \pi_{r(0)}^\rho$ defines an embedding $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega) \hookrightarrow \mathbf{Homsp}(G,\pi_\rho,\varphi,U)$.

Remark 3. If (2a) fails, then the space $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ may be infinite-dimensional (see [9]), while $\mathbf{Homsp}(G,\pi_\rho,\varphi,U)$ is always finite-dimensional.

3 Example: the semisimple case

Assume that $\mathfrak{g}$ is a semisimple Lie algebra. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by $\mathbf{R}$ the corresponding root system. Suppose $\langle\cdot,\cdot\rangle$ is a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$, and $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$ is the corresponding tensor. We will describe $\mathcal{M}_\Omega$ for a reductive Lie subalgebra $\mathfrak{u} \subset \mathfrak{g}$ containing $\mathfrak{h}$.

Precisely, consider a set $\mathbf{U} \subset \mathbf{R}$ such that $\mathfrak{u} = \mathfrak{h} \oplus \sum_{\alpha\in\mathbf{U}} \mathfrak{g}_\alpha$ is a reductive Lie subalgebra. In this case we will call $\mathbf{U}$ *reductive* (in other words, a set $\mathbf{U} \subset \mathbf{R}$ is reductive iff $(\mathbf{U}+\mathbf{U})\cap\mathbf{R} \subset \mathbf{U}$ and $-\mathbf{U} = \mathbf{U}$). Note that in this situation condition (2a) is satisfied, since $\mathfrak{m} = \sum_{\alpha\in\mathbf{R}\setminus\mathbf{U}} \mathfrak{g}_\alpha$ is an $\mathfrak{u}$-invariant complement to $\mathfrak{u}$ in $\mathfrak{g}$.

Fix $E_\alpha \in \mathfrak{g}_\alpha$ such that $\langle E_\alpha, E_{-\alpha}\rangle = 1$ for all $\alpha \in \mathbf{R}$. Then $\Omega = \Omega_{\mathfrak{h}} + \sum_{\alpha\in\mathbf{R}} E_\alpha \otimes E_{-\alpha}$, where $\Omega_{\mathfrak{h}} \in S^2\mathfrak{h}$. Notice that (2b) is also satisfied.

Proposition 5. *Suppose that* $x = \sum_{\alpha\in\mathbf{R}} x_\alpha E_\alpha \otimes E_{-\alpha}$. *Then* $x + \frac{\Omega}{2} \in \mathcal{M}_\Omega$ *iff*

$$x_\alpha = 0 \textit{ for } \alpha \in \mathbf{U}; \tag{3a}$$

$$x_{-\alpha} = -x_\alpha \textit{ for } \alpha \in \mathbf{R}; \tag{3b}$$

$$\textit{if } \alpha,\beta \in \mathbf{R}\setminus\mathbf{U}, \gamma \in \mathbf{U}, \alpha+\beta+\gamma = 0, \textit{then } x_\alpha + x_\beta = 0; \tag{3c}$$

$$\textit{if } \alpha,\beta,\gamma \in \mathbf{R}\setminus\mathbf{U}, \alpha+\beta+\gamma = 0, \textit{then } x_\alpha x_\beta + x_\beta x_\gamma + x_\gamma x_\alpha = -1/4. \tag{3d}$$

Note that (3c) is equivalent to the following condition:

$$\text{if } \alpha \in \mathbf{R}\setminus\mathbf{U}, \beta \in \mathbf{U}, \text{then } x_{\alpha+\beta} = x_\alpha.$$

Proof. It is easy to see that $x \in \left(\bigwedge^2 \mathfrak{m}\right)^{\mathfrak{h}}$ iff (3a) and (3b) are satisfied.

Suppose that $c_{\alpha\beta}$ are defined by $[E_\alpha, E_\beta] = c_{\alpha\beta}E_{\alpha+\beta}$.

For any $\gamma \in \mathbf{U}$ we have

$$\begin{aligned}
[E_\gamma, x] &= \sum_{\alpha\in\mathbf{R}\setminus\mathbf{U}} x_\alpha([E_\gamma, E_\alpha]\otimes E_{-\alpha} + E_\alpha \otimes [E_\gamma, E_{-\alpha}]) = \\
&= \sum_{\alpha,\beta\in\mathbf{R}\setminus\mathbf{U},\alpha+\beta+\gamma=0} (x_\alpha c_{\gamma\alpha}E_{-\beta}\otimes E_{-\alpha} - x_\alpha c_{\gamma\alpha}E_{-\alpha}\otimes E_{-\beta}) \\
&= \sum_{\alpha,\beta\in\mathbf{R}\setminus\mathbf{U},\alpha+\beta+\gamma=0} (x_\beta c_{\gamma\alpha} - x_\alpha c_{\gamma\beta})\, E_{-\alpha}\otimes E_{-\beta} \\
&= \sum_{\alpha,\beta\in\mathbf{R}\setminus\mathbf{U},\alpha+\beta+\gamma=0} (x_\alpha + x_\beta)c_{\gamma\alpha}E_{-\alpha}\otimes E_{-\beta}.
\end{aligned}$$

Thus x is $\mathfrak{u}$-invariant if and only if $x_\alpha + x_\beta = 0$ for all $\alpha, \beta \in \mathbf{R}\setminus\mathbf{U}$ such that $\alpha + \beta \in \mathbf{U}$.

Finally, we calculate $\mathrm{CYB}\left(x + \frac{\Omega}{2}\right) = \mathrm{CYB}(x) + \mathrm{CYB}\left(\frac{\Omega}{2}\right)$ (see [1]):

$$\begin{aligned}
\mathrm{CYB}(x) &= \sum_{\alpha,\beta\in\mathbf{R}} x_\alpha x_\beta\big([E_\alpha, E_\beta]\otimes E_{-\alpha}\otimes E_{-\beta} + E_\alpha\otimes[E_{-\alpha}, E_\beta]\otimes E_{-\beta} \\
&\quad + E_\alpha\otimes E_\beta\otimes[E_{-\alpha}, E_{-\beta}]\big) \\
&= \sum_{\alpha,\beta,\gamma\in\mathbf{R},\alpha+\beta+\gamma=0} \big(x_\alpha x_\beta c_{\alpha\beta}E_{-\gamma}\otimes E_{-\alpha}\otimes E_{-\beta} \\
&\quad - x_\alpha x_\beta c_{\alpha\beta}E_{-\alpha}\otimes E_{-\gamma}\otimes E_{-\beta} + x_\alpha x_\beta c_{\alpha\beta}E_{-\alpha}\otimes E_{-\beta}\otimes E_{-\gamma}\big) \\
&= \sum_{\alpha,\beta,\gamma\in\mathbf{R},\alpha+\beta+\gamma=0} c_{\alpha\beta}(x_\alpha x_\beta + x_\alpha x_\gamma + x_\beta x_\gamma)E_{-\alpha}\otimes E_{-\beta}\otimes E_{-\gamma}, \\
\mathrm{CYB}\left(\frac{\Omega}{2}\right) &\equiv \frac{1}{4}\sum_{\alpha,\beta,\gamma\in\mathbf{R}\setminus\mathbf{U},\alpha+\beta+\gamma=0} c_{\alpha\beta}E_{-\alpha}\otimes E_{-\beta}\otimes E_{-\gamma} \\
&(\mathrm{mod}\ \mathfrak{u}\otimes\mathfrak{g}\otimes\mathfrak{g} + \mathfrak{g}\otimes\mathfrak{u}\otimes\mathfrak{g} + \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{u}).
\end{aligned}$$

So the image of $\mathrm{CYB}\left(x + \frac{\Omega}{2}\right)$ in $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$ vanishes if and only if the condition (3d) is satisfied. ■

Proposition 6. *Suppose* $\Pi \subset \mathbf{R}$ *is a set of simple roots,* $\mathbf{R}_+$ *is the corresponding set of positive roots. Choose a subset* $\Delta \subset \Pi$ *such that* $\mathbf{N} = (\mathrm{span}\Delta) \cap \mathbf{R}$ *contains* $\mathbf{U}$*. Find* $h \in \mathfrak{h}$ *such that* $\alpha(h) \notin \pi i\mathbb{Z}$ *for* $\alpha \in \mathbf{N}\setminus\mathbf{U}$ *and* $\alpha(h) \in \pi i\mathbb{Z}$ *for* $\alpha \in \mathbf{U}$*. Then* x_α *defined by*

$$x_\alpha = \begin{cases} 0, & \alpha \in \mathbf{U}, \\ \frac{1}{2}\coth\alpha(h), & \alpha \in \mathbf{N}\setminus\mathbf{U}, \\ \pm 1/2, & \alpha \in \pm\mathbf{R}_+\setminus\mathbf{N} \end{cases}$$

satisfies (3a)–(3d)*. Moreover, any function satisfying* (3a)–(3d) *is of this form.*

First, we prove the second part of the proposition. Set

$$\mathbf{P} = \{\alpha \,|\, x_\alpha \neq -1/2\}.$$

It is obvious that $\mathbf{U} \subset \mathbf{P}$.

Lemma 1. $\mathbf{P}$ *is parabolic.*

Proof. Obviously, $\mathbf{P} \cup (-\mathbf{P}) = \mathbf{R}$.

We have to prove that if $\alpha, \beta \in \mathbf{P}$ and $\alpha + \beta \in \mathbf{R}$, then $\alpha + \beta \in \mathbf{P}$. We do it by considering several cases. If $\alpha, \beta \in \mathbf{U}$, then $\alpha+\beta \in \mathbf{U} \subset \mathbf{P}$. If $\alpha \in \mathbf{P}\backslash\mathbf{U}$ and $\beta \in \mathbf{U}$, then $x_{\alpha+\beta} = x_\alpha \neq -1/2$ by (3c) and $\alpha + \beta \in \mathbf{P}$. If $\alpha, \beta \in \mathbf{P}\backslash\mathbf{U}$, there are two possibilities. If $\alpha + \beta \in \mathbf{U}$, then there is nothing to prove. If $\alpha + \beta \notin \mathbf{U}$, then, by (3d), $x_\alpha x_\beta - x_{\alpha+\beta}(x_\alpha + x_\beta) = -1/4$. If $x_{\alpha+\beta} = -1/2$, then from this equation it follows that $x_\alpha = -1/2$. Consequently, $\alpha + \beta \in \mathbf{P}$. ■

Since $\mathbf{P}$ is parabolic, there exists a set of positive roots $\Pi \subset \mathbf{R}$ and a subset $\Delta \subset \Pi$ such that $\mathbf{P} = \mathbf{R}_+ \cup \mathbf{N}$ (see [4], chapter VI, § 1, proposition 20); here $\mathbf{R}_+$ is the set of positive roots corresponding to Π, and $\mathbf{N} = (\operatorname{span}\Delta) \cap \mathbf{R}$ is the Levi subset corresponding to Δ.

Let $\mathbf{N}_+ = \mathbf{N} \cap \mathbf{R}_+$ be the set of positive roots in $\mathbf{N}$ corresponding to Δ. For all $\alpha \in \Delta\backslash\mathbf{U}$ let $y_\alpha = \operatorname{arccoth} 2x_\alpha$, for $\alpha \in \Delta \cap \mathbf{U}$ let $y_\alpha = 0$. Find $h \in \mathfrak{h}$ such that $y_\alpha = \alpha(h)$. Now we prove that h satisfies Proposition 6.

Lemma 2. *$\alpha(h) \notin \pi i\mathbb{Z}$ and $x_\alpha = \frac{1}{2}\coth\alpha(h)$ for all $\alpha \in \mathbf{N}\backslash\mathbf{U}$; $\alpha(h) \in \pi i\mathbb{Z}$ for $\alpha \in \mathbf{U}$.*

Proof. It is enough to prove this for α positive, so that we can use the induction on the length $l(\alpha)$. The case $l(\alpha) = 1$ is trivial. Suppose that $l(\alpha) = k$. Then we can find $\alpha' \in \mathbf{N}_+$ and $\alpha_k \in \Delta$ such that $l(\alpha') = k - 1$ and $\alpha = \alpha' + \alpha_k$. Consider two cases.

First, suppose that $\alpha \in \mathbf{U}$.

If $\alpha_k \in \mathbf{U}$, then $\alpha' \in \mathbf{U}$. By induction, $\alpha(h) = \alpha'(h) \in \pi i\mathbb{Z}$.

If $\alpha_k \notin \mathbf{U}$, then $\alpha' \notin \mathbf{U}$. By induction assumption, $x_{\alpha'} = \frac{1}{2}\coth\alpha'(h)$. From (3c) it follows that $0 = x_{\alpha'} + x_{\alpha_k} = \frac{1}{2}(\coth\alpha'(h) + \coth\alpha_k(h))$ and, consequently, $\alpha(h) \in \pi i\mathbb{Z}$.

Now suppose that $\alpha \notin \mathbf{U}$.

If $\alpha_k \in \mathbf{U}$, then $\alpha' \notin \mathbf{U}$. Since $\alpha_k(h) = 0$, by (3c) we have $x_\alpha = x_{\alpha'+\alpha_k} = x_{\alpha'} = \frac{1}{2}\coth\alpha'(h) = \frac{1}{2}\coth\alpha(h)$.

When $\alpha_k \notin \mathbf{U}$, then there are two possibilities again. If $\alpha' \in \mathbf{U}$, then by induction $\alpha'(h) \in \pi i\mathbb{Z}$. By (3c), $0 = x_\alpha + x_{-\alpha_k}$. Consequently, $x_\alpha = x_{\alpha_k} = \frac{1}{2}\coth\alpha_k(h) = \frac{1}{2}\coth\alpha(h)$. If $\alpha' \notin \mathbf{U}$, then, by (3d), $x_\alpha x_{-\alpha'} + x_{-\alpha'}x_{-\alpha_k} + x_{-\alpha_k}x_\alpha = -1/4$. This equation can be rewritten as

$$x_\alpha = \frac{1/4 + x_{\alpha'}x_{\alpha_k}}{x_{\alpha'} + x_{\alpha_k}} = \frac{1}{2}\,\frac{1 + \coth\alpha'(h)\coth\alpha_k(h)}{\coth\alpha'(h) + \coth\alpha_k(h)} = \frac{1}{2}\coth\alpha(h),$$

and the lemma is proved. ■

To prove the first part of the proposition we need the following root theory lemma.

Lemma 3. *Suppose $\mathbf{P} \subset \mathbf{R}$ is parabolic. Then $\mathbf{Y} = \mathbf{R}\backslash\mathbf{P}$ has the following properties:*

$$(-\mathbf{Y}) \cap \mathbf{Y} = \varnothing; \tag{4a}$$

$$(\mathbf{Y} + \mathbf{Y}) \cap \mathbf{R} \subset \mathbf{Y}; \tag{4b}$$

$$\text{if } \alpha \in \mathbf{Y}, \beta \in \mathbf{R}\backslash\mathbf{Y} \text{ and } \alpha - \beta \in \mathbf{R}, \text{ then } \alpha - \beta \in \mathbf{Y}. \tag{4c}$$

Proof. Since (4a) is obvious and (4b) follows from (4a) and (4c), we prove only the latter property: if $\alpha \in \mathbf{Y}$ and $\beta \in \mathbf{P}$ are such that $\alpha - \beta \in \mathbf{P}$, then, since $\mathbf{P}$ is parabolic, we would have $\alpha = (\alpha - \beta) + \beta \in \mathbf{P}$. So $\alpha - \beta \in \mathbf{Y}$. ■

Now we just check (3a)–(3d) directly. Suppose that $\mathbf{N}$ is defined as in the proposition. Let $\mathbf{Y} = \mathbf{R}_+\backslash\mathbf{N}$. Then $\mathbf{P} = \mathbf{R}\backslash\mathbf{Y} = -\mathbf{R}_+ \cup \mathbf{N}$ is a parabolic set, and $\mathbf{Y}$ satisfies (4a)–(4c).

Lemma 4. *Suppose x_α is as defined in Proposition 6. Then x_α satisfies* (3a)–(3d).

Proof. (3a) we have already, (3b) is trivial.

To prove (3c), consider the following cases. First, take $\alpha, \beta \in \mathbf{N}\backslash\mathbf{U}$, $\gamma \in \mathbf{U}$, $\alpha+\beta+\gamma=0$. Then $x_\alpha + x_\beta = \frac{1}{2}(\coth\alpha(h) + \coth\beta(h)) = 0$ as $\alpha+\beta=-\gamma\in\mathbf{U}$. The case $\alpha\in\mathbf{N}\backslash\mathbf{U}$, $\beta\in\mathbf{R}\backslash\mathbf{N}$, $\gamma\in\mathbf{U}$ is impossible, because then we would have $\beta=-\alpha-\gamma\in\mathbf{N}$. The case $\alpha,\beta\in\pm\mathbf{Y}$, $\gamma\in\mathbf{U}$ is also impossible, because $-\gamma=\alpha+\beta\in\pm\mathbf{Y}$. Finally, if $\alpha\in\pm\mathbf{Y}$, $\beta\in\mp\mathbf{Y}$, $\gamma\in\mathbf{U}$, then $x_\alpha+x_\beta=\pm\frac{1}{2}\mp\frac{1}{2}=0$.

Condition (3d) can be proved in a similar way. ∎

Now to summarize:

Theorem 3. *Suppose $U\subset G$ is the connected Lie subgroup corresponding to $\mathfrak{u}\subset\mathfrak{g}$. Take $\rho\in\mathfrak{g}\otimes\mathfrak{g}$ such that $\rho+\rho^{21}=\Omega$ and set $\varphi=-\mathrm{CYB}(\rho)$. Then any (G,π_ρ,φ)-homogeneous quasi-Poisson space structure on G/U is exactly of the form $\pi=\pi^\rho_{x+\Omega/2}$ for some $x=\sum\limits_{\alpha\in R} x_\alpha E_\alpha\otimes E_{-\alpha}$, where x_α is defined in Proposition 6.*

Remark 4. Let ρ be any solution of the classical Yang–Baxter equation such that $\rho+\rho^{21}=\Omega$ (see [3]). Then (G,π_ρ) is a Poisson Lie group and therefore Theorem 3 provides the list of all (G,π_ρ)-homogeneous Poisson space structures on G/U.

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