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Invariance of Maxwell's Equations under Nonlinear Representations of Poincaré Algebra

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It is shown that the Maxwell equations are conditionally invariant with respect to two nonlinear representations of Poincaré algebra. The corresponding groups of finite transformations are constructed. The nonlinear generalization of Maxwell equations invariant under the above-mentioned nonlinear representations of Poincaré algebra is found.

1 Conditional invariance

Let us consider the Maxwell equations in vacuum written in the complex form:

$$\vec{\nabla}\vec{\Sigma} = 0, \qquad \partial_0\vec{\Sigma} + i\vec{\nabla}\times\vec{\Sigma} = 0, \tag{1}$$

where $\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$, $\partial_0 = \frac{\partial}{\partial t}$, $\vec{\Sigma} = \vec{E} + i\vec{H}$ is a complex-valued vector of electromagnetic field. As it is known [1, 2], the system (1) is invariant under the linear representation of the Poincaré algebra $AP(1,3)$ with basis operators

$$\begin{aligned}
&AP^{\rm lin}(1,3) = \langle P_\mu, J^{\rm lin}_{ab}, J^{\rm lin}_{0a}\rangle, \\
&P_\mu = \partial_\mu, \qquad J^{\rm lin}_{ab} = x_a\partial_b - x_b\partial_a + \Sigma_a\partial_{\Sigma_b} - \Sigma_b\partial_{\Sigma_a}, \qquad J^{\rm lin}_{0a} = x_0\partial_a + x_a\partial_0 + N^{\rm lin}_a, \\
&\vec{N}^{\rm lin} = (N^{\rm lin}_1, N^{\rm lin}_2, N^{\rm lin}_3) = i\vec{\Sigma}\times\vec{\nabla}_\Sigma, \qquad \mu = \overline{0,3},
\end{aligned} \tag{2}$$

where $\vec{\nabla}_\Sigma = \left(\frac{\partial}{\partial\Sigma_1}, \frac{\partial}{\partial\Sigma_2}, \frac{\partial}{\partial\Sigma_3}\right)$.

A nonlinear representation for the algebra $AP(1,3)$ and for the electromagnetic field $\vec{E}$, $\vec{H}$ that has the following form for complex-valued vectors was found in [3]:

$$\begin{aligned}
&AP^{\rm nli}(1,3) = \langle P_\mu, J^{\rm nli}_{ab}, J^{\rm nli}_{0a}\rangle, \\
&P_\mu = \partial_\mu, \qquad J^{\rm nli}_{ab} = J^{\rm lin}_{ab}, \qquad J^{\rm nli}_{0a} = x_0\partial_a + x_a\partial_0 + N^{\rm nli}_a, \\
&\vec{N}^{\rm nli} = \left(N^{\rm nli}_1, N^{\rm nli}_2, N^{\rm nli}_3\right) = \vec{\nabla}_\Sigma - \vec{\Sigma}(\Sigma_k\partial_{\Sigma_k}), \qquad k = 1,2,3
\end{aligned} \tag{3}$$

(summation by k is implied). Representations (2) and (3) are not equivalent as there is only one independent invariant for the operators (3) and two functionally independent invariants for the operators (2). Let us consider invariance properties of the system (1) with respect to the operators (3).

Theorem 1. *The Maxwell equations* (1) *are invariant with respect to the representation* (3) *of* $AP(1,3)$ *iff the following conditions are satisfied:*

$$\begin{aligned}
&\left[\partial_0 + \vec{\Sigma}\vec{\nabla}\right]\Sigma^k = 0, \\
&\left[\vec{\nabla} + \vec{\Sigma}\partial_0 - i\vec{\Sigma}\times\vec{\nabla}\right]\Sigma^k = 0, \qquad k = 1,2,3.
\end{aligned} \tag{4}$$

Besides, the system (1), (4) *is invariant with respect to the linear representation* $AP^{\rm lin}(1,3)$ (2).

Corollary. *There exists another representation of the Poincaré algebra that is an invariance algebra of the system under consideration.*

Let us consider the set of operators

$$J^3_{0a} = x_0\partial_a + x_a\partial_0 + \alpha N^{\rm lin}_a + \beta N^{\rm nli}_a, \qquad a = 1, 2, 3 \tag{5}$$

and its completion by means of commutation. As a result a Lie algebra is generated (generally speaking it is infinitely dimensional). It is not difficult to prove the following

Statement. *The completion of the set* (5) *will be the Poincaré algebra iff either* $(\alpha, \beta) = \left(\frac{1}{2}, \frac{1}{2}\right)$, *or* $(\alpha, \beta) = \left(\frac{1}{2}, -\frac{1}{2}\right)$. *These representations are equivalent with respect to the substitution* $\vec{\Sigma} \to -\vec{\Sigma}$.

The explicit form of the operators for $\alpha = \beta = \frac{1}{2}$ is as follows:

$$AP^3(1,3) = \langle P_\mu, J^3_{ab}, J^3_{0a}\rangle, \tag{6}$$
$$J^3_{ab} = x_a\partial_b - x_b\partial_a - \frac{i}{2}\varepsilon_{abc}\left(N^{\rm lin}_c + N^{\rm nli}_c\right), \qquad J^3_{0a} = x_0\partial_a + x_a\partial_0 + N^3_a = \frac{1}{2}\left(J^{\rm lin}_{0a} + J^{\rm nli}_{0a}\right).$$

As $J^{\rm lin}_{0a}$ and $J^{\rm nli}_{0a}$ are symmetry operators of the system (1), (4), this system is invariant with respect to the operators J^3_{0a}, and whence with respect to the representation (6) of $AP^3(1,3)$. Let us point out that operators of space rotations J^3_{ab}, as distinct from representations (2), (3), are nonlinear with respect to $\vec{\Sigma}$.

It is not difficult to show that the following relation is true on the solutions of the system (4):

$$\vec{\Sigma}^2 = \Sigma^a\Sigma_a = 1, \tag{7}$$

or, in the notations of electromagnetic field $\vec{E}$, $\vec{H}$,

$$\vec{E}\cdot\vec{H} = 0, \qquad \vec{E}^2 - \vec{H}^2 = 1.$$

It is interesting to note that if Σ_a satisfy the conditions (7), then the representations (2), (3) and (6) are connected by the following expressions:

$$\vec{N}^{\rm nli} = \left[i\vec{\Sigma}\times\vec{N}^{\rm lin}\right], \qquad \vec{N}^{\rm lin} = \left[i\vec{\Sigma}\times\vec{N}^{\rm nli}\right], \qquad \vec{N}^3_a = \left[i\vec{\Sigma}\times\vec{N}^3_a\right].$$

The system of equations (4) is much overdetermined: there are 12 equations with respect to 3 unknown functions. However, we can go on to 4 equations for 2 unknown functions. Actually it is not difficult to check that the system (4) is equivalent to its first equation (for Σ_k), first "component" of the second equation

$$[\partial_1 + \Sigma_1\partial_0 - i\Sigma_2\partial_3 + i\Sigma_3\partial_2]\Sigma_k = 0,$$

and the condition (7). Whence making the substitution $\Sigma_3 = \pm\sqrt{1-\Sigma_1^2-\Sigma_2^2}$, we go on to the system of 4 equations with respect to Σ_1, Σ_2.

The system (1), (4) is compatible, and it has, for example, the following solution

$$\vec{\Sigma} = (1, i\varphi(t,x), \varphi(t,x)),$$
$$\partial_2\varphi + i\partial_3\varphi = 0, \qquad \partial_0\varphi + \partial_1\varphi = 0, \qquad \varphi\in\mathbb{C}. \tag{8}$$

Among solutions of (8) there are functions of the following form

$$\vec{\Sigma} = (1, if(t-x_1), f(t-x_1)), \tag{9}$$

where f is an arbitrary differentiable function.

Thus, at the certain subset of solutions of the Maxwell equations that include flat wave solutions (9), the electromagnetic field $\vec{E}$, $\vec{H}$ can be transformed by transformations in the 4-dimensional space, not only as an electromagnetic tensor, but also in a nonlinear way.

2 Finite transformations

Having solved the respective system of the Lie equations, we can find the finite group transformations corresponding to operators of Lorentz boosts $J^3_{\mu\nu}$ (6) (for the representation $AP^{\rm nli}(1,3)$ these transformations were found in [4]):

$$\begin{aligned}
J^3_{ab}:\ & x_a \to \tilde{x}_a = x_a\cos(2\theta) - x_b\sin(2\theta),\\
& x_b \to \tilde{x}_b = x_b\cos(2\theta) + x_a\sin(2\theta),\\
& x_c \to \tilde{x}_c = x_c, \quad c \neq a, b,\\
& \Sigma_k \to \tilde{\Sigma} = \frac{\Sigma_k\cos\theta + (\delta_{kb}\Sigma_a - \delta_{ka}\Sigma_b - i\varepsilon_{abk})\sin\theta}{\cos\theta - i\varepsilon_{abc}\Sigma_c\sin\theta}; \qquad (10)\\
J^3_{0a}:\ & x_0 \to \tilde{x}_0 = x_0\,{\rm ch}\,(2\theta) + x_a\,{\rm sh}\,(2\theta),\\
& x_a \to \tilde{x}_a = x_a\,{\rm ch}\,(2\theta) + x_0\,{\rm sh}\,(2\theta),\\
& x_k \to \tilde{x}_k = x_k, \quad k \neq a,\\
& \Sigma_k \to \tilde{\Sigma} = \frac{\Sigma_k\,{\rm ch}\,\theta + (\delta_{ka} - i\varepsilon_{akl}\Sigma_l)\,{\rm sh}\,\theta}{{\rm ch}\,\theta + \Sigma_a\,{\rm sh}\,\theta}. \qquad (11)
\end{aligned}$$

The transformations (10), (11) can be used for group generations of solutions for the Maxwell equations by means of e.g. initial solution (8). The newly obtained solutions can be further multiplied by the symmetry groups of the Maxwell equations.

3 Generalizations of the Maxwell equations

In the paper [4] the following nonlinear generalizations of the Maxwell equations is considered:

$$\begin{aligned}
&\vec{\nabla}\vec{\Sigma} = A(\omega)\partial_0\omega,\\
&\partial_0\vec{\Sigma} + i\vec{\nabla}\times\vec{\Sigma} = A(\omega)\vec{\nabla}(\omega), \qquad (12)
\end{aligned}$$

where ω are invariants of the representation (2) of the algebra $AP^{\rm lin}(1,3)$. Let us consider the following system of equations

$$\begin{aligned}
&\vec{\nabla}\vec{\Sigma} = A(\Omega)\partial_0\Omega + B(\Omega)\vec{\Sigma}\vec{\nabla}\Omega,\\
&\partial_0\vec{\Sigma} + i\vec{\nabla}\times\vec{\Sigma} = A(\Omega)\vec{\nabla}(\Omega) + B(\Omega)\big(\vec{\Sigma}\partial_0\Omega - i\vec{\Sigma}\times\vec{\nabla}\Omega\big), \qquad (13)
\end{aligned}$$

where $\Omega = f\big(\Sigma_a^2\big)$. Without losing generality we can assume that $\Omega = \ln\sqrt{\Sigma_a^2 - 1}$. If $B = 0$ then the system (13) has the form (12). The following theorem is true.

Theorem 2. *The system of equations* (13) *is invariant with respect to the linear representation of the Poincaré algebra* (2) *with any* $A(\Omega)$*,* $B(\Omega)$*, and with respect to the nonlinear representation* (3) *if and only if* $A = B = 1$*. In this case the right-hand parts of the equations* (13) *coincides with left-hand parts of the equations* (4)*, if the substitution* $\Sigma^k \to \Omega$ *is made.*

It follows from the Theorem that the system (13) with $A = B = 1$ is invariant also with respect to the representation of the Poincaré algebra $AP^3(1,3)$ (6).

Let us write the equations (13) in the covariant form. Let us introduce the following notations for this purpose. $G^{\mu\nu}$ is an antisymmetric tensor:

$$G^{0a} = \Sigma_a, \qquad G^{ab} = -i\varepsilon_{abc}\Sigma_c.$$

In terms of the tensor of the electromagnetic field $F^{\mu\nu}$, $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\delta}F^{\rho\delta}$ we can write $G^{\mu\nu} = F^{\mu\nu} + i\tilde{F}^{\mu\nu}$. Then (13) can be written in the same notations as:

$$\partial_\mu G^{\mu\nu} = G^{\mu\nu}\partial_\mu\varphi - \partial^\nu\varphi, \qquad \varphi = \ln\sqrt{-\frac{G^{\mu\nu}G_{\mu\nu}}{4} - 1}.$$

For the Maxwell equation (1) $\varphi \equiv 0$.

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