Generalizations of Schouten–Nijenhuis Bracket

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The Schouten–Nijenhuis bracket is generalized for the superspace case and for the Poisson brackets of opposite Grassmann parities.

1 Introduction

Recently a recipe for the construction of new Poisson brackets from the bracket with a definite Grassmann parity was proposed [1]. This recipe is based on the use of exterior differentials of diverse Grassmann parities. It was indicated in [1] that this recipe leads to the generalizations of the Schouten–Nijenhuis bracket [2–8] on both the superspace case and the case of the brackets with diverse Grassmann parities. In the present report we give the details of these generalizations[1].

2 Poisson brackets related with the exterior differentials

Let us recall the recipe for the construction from a given Poisson bracket of a Grassmann parity $\epsilon \equiv 0, 1 \pmod 2$ of another one.

A Poisson bracket, having a Grassmann parity ϵ, written in arbitrary non-canonical phase variables z^a

$$\{A, B\}_\epsilon = A \overleftarrow{\partial}_{z^a} \omega^{ab}_\epsilon(z) \overrightarrow{\partial}_{z^b} B, \tag{1}$$

where $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ are right and left derivatives respectively, has the following main properties:

$$g(\{A, B\}_\epsilon) \equiv g_A + g_B + \epsilon \pmod 2,$$
$$\{A, B\}_\epsilon = -(-1)^{(g_A+\epsilon)(g_B+\epsilon)}\{B, A\}_\epsilon,$$
$$\sum_{(ABC)} (-1)^{(g_A+\epsilon)(g_C+\epsilon)}\{A, \{B, C\}_\epsilon\}_\epsilon = 0,$$

which lead to the corresponding relations for the matrix ω^{ab}_ϵ

$$g\left(\omega^{ab}_\epsilon\right) \equiv g_a + g_b + \epsilon \pmod 2, \tag{2}$$
$$\omega^{ab}_\epsilon = -(-1)^{(g_a+\epsilon)(g_b+\epsilon)}\omega^{ba}_\epsilon, \tag{3}$$
$$\sum_{(abc)} (-1)^{(g_a+\epsilon)(g_c+\epsilon)}\omega^{ad}_\epsilon \partial_{z^d}\omega^{bc}_\epsilon = 0, \tag{4}$$

where $\partial_{z^a} \equiv \partial/\partial z^a$ and $g_a \equiv g(z^a)$, $g_A \equiv g(A)$ are the corresponding Grassmann parities of phase coordinates z^a and a quantity A and a sum with a symbol (abc) under it designates a summation

[1]Concerning the generalizations of the Schouten–Nijenhuis bracket see also [9, 10].

over cyclic permutations of a, b and c. We shall consider the non-degenerate matrix ω^{ab}_ϵ which has an inverse matrix $\omega^\epsilon_{ab}(-1)^{g_b\epsilon}$ (a grading factor is chosen for the convenience)

$$\omega^{ab}_\epsilon \omega^\epsilon_{bc}(-1)^{g_c\epsilon} = \delta^a_c$$

(there is no summation over ϵ in the previous relation) with the properties

$$\begin{aligned}
&g(\omega^\epsilon_{ab}) \equiv g_a + g_b + \epsilon \pmod 2,\\
&\omega^\epsilon_{ab} = (-1)^{(g_a+1)(g_b+1)}\omega^\epsilon_{ba},\\
&\sum_{(abc)}(-1)^{(g_a+1)g_c}\partial_{z^a}\omega^\epsilon_{bc} = 0.
\end{aligned}$$

The Hamilton equations for the phase variables z^a, which correspond to a Hamiltonian H_ϵ $(g(H_\epsilon) = \epsilon)$,

$$\frac{dz^a}{dt} = \{z^a, H_\epsilon\}_\epsilon = \omega^{ab}_\epsilon \partial_{z^b} H_\epsilon \tag{5}$$

can be represented in the form

$$\frac{dz^a}{dt} = \omega^{ab}_\epsilon \partial_{z^b} H_\epsilon \equiv \omega^{ab}_\epsilon \frac{\partial(d_\zeta H_\epsilon)}{\partial(d_\zeta z^b)} \stackrel{\text{def}}{=} (z^a, d_\zeta H_\epsilon)_{\epsilon+\zeta}, \tag{6}$$

where d_ζ $(\zeta = 0, 1)$ is one of the exterior differentials d_0 or d_1, which have opposite Grassmann parities 0 and 1 respectively and following symmetry properties with respect to the ordinary multiplication

$$\begin{aligned}
&d_0 z^a d_0 z^b = (-1)^{g_a g_b} d_0 z^b d_0 z^a,\\
&d_1 z^a d_1 z^b = (-1)^{(g_a+1)(g_b+1)} d_1 z^b d_1 z^a
\end{aligned} \tag{7}$$

and exterior products

$$\begin{aligned}
&d_0 z^a \wedge d_0 z^b = (-1)^{g_a g_b + 1} d_0 z^b \wedge d_0 z^a,\\
&d_1 z^a \tilde\wedge d_1 z^b = (-1)^{(g_a+1)(g_b+1)} d_1 z^b \tilde\wedge d_1 z^a.
\end{aligned} \tag{8}$$

We use different notations $\wedge$ and $\tilde\wedge$ for the exterior products of $d_0 z^a$ and $d_1 z^a$ respectively.

By taking the exterior differential d_ζ from the Hamilton equations (5), we obtain

$$\frac{d(d_\zeta z^a)}{dt} = (d_\zeta \omega^{ab}_\epsilon)\frac{\partial(d_\zeta H_\epsilon)}{\partial(d_\zeta z^b)} + (-1)^{\zeta(g_a+\epsilon)}\omega^{ab}_\epsilon \partial_{z^b}(d_\zeta H_\epsilon) \stackrel{\text{def}}{=} (d_\zeta z^a, d_\zeta H_\epsilon)_{\epsilon+\zeta}. \tag{9}$$

As a result of equations (6) and (9) we have by definition the following binary composition for functions F and H of the variables z^a and their differentials $d_\zeta z^a \equiv y^a_\zeta$

$$(F, H)_{\epsilon+\zeta} = F\Big[\overleftarrow{\partial}_{z^a}\, \omega^{ab}_\epsilon\, \overrightarrow{\partial}_{y^b_\zeta} + (-1)^{\zeta(g_a+\epsilon)}\, \overleftarrow{\partial}_{y^a_\zeta}\, \omega^{ab}_\epsilon\, \overrightarrow{\partial}_{z^b} + \overleftarrow{\partial}_{y^a_\zeta}\, y^c_\zeta \left(\partial_{z^c}\omega^{ab}_\epsilon\right) \overrightarrow{\partial}_{y^b_\zeta}\Big] H. \tag{10}$$

By using relations (2)–(4) for the matrix ω^{ab}_ϵ, we can establish the following properties for the binary composition (10)

$$\begin{aligned}
&g[(F, H)_{\epsilon+\zeta}] \equiv g_F + g_H + \epsilon + \zeta \pmod 2,\\
&(F, H)_{\epsilon+\zeta} = -(-1)^{(g_F+\epsilon+\zeta)(g_H+\epsilon+\zeta)}(H, F)_{\epsilon+\zeta},\\
&\sum_{(EFH)}(-1)^{(g_E+\epsilon+\zeta)(g_H+\epsilon+\zeta)}(E, (F, H)_{\epsilon+\zeta})_{\epsilon+\zeta} = 0,
\end{aligned}$$

which mean that the composition (10) satisfies all the main properties for the Poisson bracket with the Grassmann parity equal to $\epsilon + \zeta$. Thus, the application of the exterior differentials of opposite Grassmann parities to the given Poisson bracket results in the brackets of the different Grassmann parities.

By transition to the co-differential variables $y_a^{\epsilon+\zeta}$, related with differentials y_ζ^a by means of the matrix ω_ϵ^{ab}

$$y_\zeta^a = y_b^{\epsilon+\zeta}\omega_\epsilon^{ba}, \tag{11}$$

the Poisson bracket (10) takes a canonical form[2]

$$(F,H)_{\epsilon+\zeta} = F\left[\overleftarrow{\partial}_{z^a}\overrightarrow{\partial}_{y_a^{\epsilon+\zeta}} - (-1)^{g_a(g_a+\epsilon+\zeta)}\overleftarrow{\partial}_{y_a^{\epsilon+\zeta}}\overrightarrow{\partial}_{z^a}\right]H, \tag{12}$$

that can be proved with the use of the Jacobi identity (4).

The bracket (10) is given on the functions of the variables z^a, y_ζ^a

$$F = \sum_p \frac{1}{p!} y_\zeta^{a_p}\cdots y_\zeta^{a_1} f_{a_1\ldots a_p}(z), \qquad g(f_{a_1\ldots a_p}) = g_f + g_{a_1} + \cdots + g_{a_p},$$

whereas this bracket, rewritten in the form (12), is given on the functions of variables z^a and $y_a^{\epsilon+\zeta}$

$$F = \sum_p \frac{1}{p!} y_{a_p}^{\epsilon+\zeta}\cdots y_{a_1}^{\epsilon+\zeta} f^{a_1\ldots a_p}(z), \qquad g(f^{a_1\ldots a_p}) = g_f + \epsilon p + g_{a_1} + \cdots + g_{a_p}.$$

We do not exclude a possibility of the own Grassmann parity $g_f \equiv g(f)$ for a quantity f. By taking into account relation (11), we have the following rule for the rising of indices:

$$f^{b_1\ldots b_p} = (-1)^{\sum_{k=1}^{p-1}[g_{b_1}+\cdots+g_{b_k}+k(\epsilon+\zeta)](g_{b_{k+1}}+g_{a_{k+1}}+\epsilon)}\omega_\epsilon^{b_p a_p}\cdots\omega_\epsilon^{b_1 a_1} f_{a_1\ldots a_p}.$$

Note that the quantities $f_{a_1\ldots a_p}$ and $f^{a_1\ldots a_p}$ have in general the different symmetry and parity properties.

In the case $\zeta = 1$, due to relations (7), (8), the terms in the decomposition of a function $F(z^a, y_1^a)$ into degrees p of the variables y_1^a

$$F = \sum_p \frac{1}{p!} y_1^{a_p}\cdots y_1^{a_1} f_{a_1\ldots a_p}(z)$$

can be treated as p-forms and the bracket (10) can be considered as a Poisson bracket on p-forms so that being taken between a p-form and a q-form it results in a $(p+q-1)$-form[3]. Thus, the bracket (10) is a generalization of the bracket introduced in [12] on the superspace case and on the case of the brackets (1) with arbitrary Grassmann parities ϵ ($\epsilon = 0, 1$).

3 Generalizations of the Schouten–Nijenhuis bracket

If we take the bracket in the canonical form (12), then we obtain the generalizations of the Schouten–Nijenhuis bracket [2,3] (see also [4–8,12]) onto the cases of superspace and the brackets

[2]There is no summation over ϵ in relation (11).

[3]Concerning a Poisson bracket between 1-forms and its relation with the Lie bracket of vector fields see in the book [11].

of diverse Grassmann parities. Indeed, let us consider the bracket (12) between monomials F and H having respectively degrees p and q

$$F = \frac{1}{p!} y^{\epsilon+\zeta}_{a_p} \cdots y^{\epsilon+\zeta}_{a_1} f^{a_1\ldots a_p}(z), \qquad g(f^{a_1\ldots a_p}) = g_f + p\epsilon + g_{a_1} + \cdots + g_{a_p},$$
$$H = \frac{1}{q!} y^{\epsilon+\zeta}_{a_q} \cdots y^{\epsilon+\zeta}_{a_1} h^{a_1\ldots a_q}(z), \qquad g(h^{a_1\ldots a_q}) = g_h + q\epsilon + g_{a_1} + \cdots + g_{a_q}.$$

Then as a result we obtain

$$\begin{aligned}(F,H)_{\epsilon+\zeta} = {} & \frac{(-1)^{[g_{b_1}+\cdots+g_{b_{q-1}}+(q-1)(\epsilon+\zeta)](g_f+g_l+p\zeta)}}{p!(q-1)!} \\ & \times y^{\epsilon+\zeta}_{b_{q-1}} \cdots y^{\epsilon+\zeta}_{b_1} y^{\epsilon+\zeta}_{a_p} \cdots y^{\epsilon+\zeta}_{a_1} \left(f^{a_1\ldots a_p} \overleftarrow{\partial}_{z^l} \right) h^{b_1\ldots b_{q-1} l} \\ & - \frac{(-1)^{(g_l+\epsilon+\zeta)(g_f+p\epsilon+g_{a_2}+\cdots+g_{a_p})+[g_{b_1}+\cdots+g_{b_q}+q(\epsilon+\zeta)][g_f+\epsilon+(p-1)\zeta]}}{(p-1)!q!} \\ & \times y^{\epsilon+\zeta}_{b_q} \cdots y^{\epsilon+\zeta}_{b_1} y^{\epsilon+\zeta}_{a_p} \cdots y^{\epsilon+\zeta}_{a_2} f^{l a_2\ldots a_p} \partial_{z^l} h^{b_1\ldots b_q}. \end{aligned} \tag{13}$$

3.1 Particular cases

Let us consider the formula (13) for the particular values of ϵ and ζ.

1. We start from the case which leads to the usual Schouten–Nijenhuis bracket for the skew-symmetric contravariant tensors. In this case, when $\epsilon = 0$, $\zeta = 1$ and the matrix $\omega^{ab}_0(x) = -\omega^{ba}_0(x)$ corresponds to the usual Poisson bracket for the commuting coordinates $z^a = x^a$, we have

$$\begin{aligned}(F,H)_1 = {} & \frac{(-1)^{(q-1)(g_f+p)}}{p!(q-1)!} \Theta_{b_{q-1}} \cdots \Theta_{b_1} \Theta_{a_p} \cdots \Theta_{a_1} \left(f^{a_1\ldots a_p} \overleftarrow{\partial}_{x^l} \right) h^{b_1\ldots b_{q-1} l} \\ & - \frac{(-1)^{g_f(q+1)+q(p-1)}}{(p-1)!q!} \Theta_{b_q} \cdots \Theta_{b_1} \Theta_{a_p} \cdots \Theta_{a_2} f^{l a_2\ldots a_p} \partial_{x^l} h^{b_1\ldots b_q}, \end{aligned} \tag{14}$$

where $\Theta_a \equiv y^1_a$ are Grassmann co-differential variables related owing to (11) with the Grassmann differential variables $\Theta^a \equiv d_1 x^a$

$$\Theta^a = \Theta_b \omega^{ba}_0.$$

When Grassmann parities of the quantities f and h are equal to zero $g_f = g_h = 0$, we obtain from (14)

$$(F,H)_1 \stackrel{\text{def}}{=} (-1)^{(p+1)q+1} \Theta_{a_{p+q}} \cdots \Theta_{a_2} [F,H]^{a_2\ldots a_{p+q}},$$

where $[F,H]^{a_2\ldots a_{p+q}}$ are components of the usual Schouten–Nijenhuis bracket (see, for example, [7]) for the contravariant antisymmetric tensors[4]. This bracket has the following symmetry property

$$[F,H] = (-1)^{pq}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(FHE)} (-1)^{ps}[[F,H],E] = 0,$$

where s is a degree of a monomial E.

[4]Here and below we use the same notation $[F,H]$ for the different brackets. We hope that this will not lead to the confusion.

2. In the case $\epsilon = \zeta = 0$ and $\omega_0^{ab}(x) = -\omega_0^{ba}(x)$ we obtain the bracket for symmetric contravariant tensors (see, for example, [6])

$$(F,H)_0 = \frac{1}{p!(q-1)!} y^0_{b_{q-1}} \cdots y^0_{b_1} y^0_{a_p} \cdots y^0_{a_1} \left(\partial_{x^l} f^{a_1\ldots a_p}\right) h^{b_1\ldots b_{q-1} l} - \frac{1}{(p-1)!q!} y^0_{b_q} \cdots y^0_{b_1} y^0_{a_p} \cdots y^0_{a_2} f^{l a_2\ldots a_p} \partial_{x^l} h^{b_1\ldots b_q} \stackrel{\mathrm{def}}{=} y^0_{a_{p+q}} \cdots y^0_{a_2} [F,H]^{a_2\ldots a_{p+q}},$$

where commuting co-differentials y^0_a connected with commuting differentials $y^a_0 \equiv d_0 x^a$ in accordance with (11)

$$y^a_0 = y^0_b \omega_0^{ba}$$

and the bracket $[F,H]^{a_2\ldots a_{p+q}}$ has the following symmetry property

$$[F,H] = -(-1)^{g_f g_h}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{g_e g_h} [E,[F,H]] = 0.$$

3. By taking the Martin bracket [13] $\omega_0^{ab}(\theta) = \omega_0^{ba}(\theta)$ with Grassmann coordinates $z^a = \theta^a$ ($g_a = 1$) as an initial bracket (1), we have in the case $\zeta = 0$ for antisymmetric contravariant tensors on the Grassmann algebra

$$(F,H)_0 = \frac{(-1)^{(q-1)(g_f+1)}}{p!(q-1)!} \Theta_{b_{q-1}} \cdots \Theta_{b_1} \Theta_{a_p} \cdots \Theta_{a_1} (f^{a_1\ldots a_p} \overleftarrow{\partial}_{\theta^l}) h^{b_1\ldots b_{q-1} l} + \frac{(-1)^{(q-1)g_f + p}}{(p-1)!q!} \Theta_{b_q} \cdots \Theta_{b_1} \Theta_{a_p} \cdots \Theta_{a_2} f^{l a_2\ldots a_p} \partial_{\theta^l} h^{b_1\ldots b_q} \stackrel{\mathrm{def}}{=} \Theta_{a_{p+q}} \cdots \Theta_{a_2} [F,H]^{a_2\ldots a_{p+q}},$$

where the Grassmann co-differentials Θ_a related with the Grassmann differentials Θ^a as

$$d_0 \theta^a \equiv \Theta^a = \Theta_b \omega_0^{ba}.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H] = -(-1)^{g_f g_h}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{g_e g_h} [E,[F,H]] = 0.$$

4. By taking the Martin bracket again, in the case $\zeta = 1$

$$d_1 \theta^a \equiv x^a = x_b \omega_0^{ba}$$

we obtain for the symmetric tensors on Grassmann algebra

$$(F,H)_1 = \frac{1}{p!(q-1)!} x_{b_{q-1}} \cdots x_{b_1} x_{a_p} \cdots x_{a_1} (f^{a_1\ldots a_p} \overleftarrow{\partial}_{\theta^l}) h^{b_1\ldots b_{q-1} l} - \frac{1}{(p-1)!q!} x_{b_q} \cdots x_{b_1} x_{a_p} \cdots x_{a_2} f^{l a_2\ldots a_p} \partial_{\theta^l} h^{b_1\ldots b_q} \stackrel{\mathrm{def}}{=} x_{a_{p+q}} \cdots x_{a_2} [F,H]^{a_2\ldots a_{p+q}}.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H] = -(-1)^{(g_f+p+1)(g_h+q+1)}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{(g_e+s+1)(g_h+q+1)}[E,[F,H]] = 0.$$

5. In general, if we take the even bracket in superspace with coordinates $z^a = (x,\theta)$, where x and θ are respectively commuting and anticommuting (Grassmann) variables, then in the case $\zeta = 1$ we have

$$\begin{aligned}(F,H)_1 = {} & \frac{(-1)^{(g_{b_1}+\cdots+g_{b_{q-1}}+q-1)(g_f+g_l+p)}}{p!(q-1)!} y^1_{b_{q-1}}\cdots y^1_{b_1} y^1_{a_p}\cdots y^1_{a_1}(f^{a_1\ldots a_p}\overleftarrow{\partial}_{z^l})h^{b_1\ldots b_{q-1}l} \\ & - \frac{(-1)^{(g_l+1)(g_f+g_{a_2}+\cdots+g_{a_p})+(g_{b_1}+\cdots+g_{b_q}+q)(g_f+p-1)}}{(p-1)!q!} y^1_{b_q}\cdots y^1_{b_1} y^1_{a_p}\cdots y^1_{a_2} f^{la_2\ldots a_p}\partial_{z^l}h^{b_1\ldots b_q} \\ & \stackrel{\text{def}}{=} y^1_{a_{p+q}}\cdots y^1_{a_2}[F,H]^{a_2\ldots a_{p+q}},\end{aligned}$$

where

$$d_1 z^a \equiv y^a_1 = y^1_b \omega^{ba}_0.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H] = -(-1)^{(g_f+p+1)(g_h+q+1)}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{(g_e+s+1)(g_h+q+1)}[E,[F,H]] = 0.$$

6. In the case of the even bracket in superspace as initial one with $\zeta = 0$ we obtain

$$\begin{aligned}(F,H)_0 = {} & \frac{(-1)^{(g_{b_1}+\cdots+g_{b_{q-1}})(g_f+g_l)}}{p!(q-1)!} y^0_{b_{q-1}}\cdots y^0_{b_1} y^0_{a_p}\cdots y^0_{a_1}(f^{a_1\ldots a_p}\overleftarrow{\partial}_{z^l})h^{b_1\ldots b_{q-1}l} \\ & - \frac{(-1)^{g_l(g_f+g_{a_2}+\cdots+g_{a_p})+g_f(g_{b_1}+\cdots+g_{b_q})}}{(p-1)!q!} y^0_{b_q}\cdots y^0_{b_1} y^0_{a_p}\cdots y^0_{a_2} f^{la_2\ldots a_p}\partial_{z^l}h^{b_1\ldots b_q} \\ & \stackrel{\text{def}}{=} y^0_{a_{p+q}}\cdots y^0_{a_2}[F,H]^{a_2\ldots a_{p+q}},\end{aligned}$$

where

$$d_0 z^a \equiv y^a_0 = y^0_b \omega^{ba}_0.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H] = -(-1)^{g_f g_h}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{g_e g_h}[E,[F,H]] = 0.$$

7. Taking as an initial bracket the odd Poisson bracket in superspace with coordinates z^a, for the case $\zeta = 0$ we have

$$(F,H)_1 = \frac{(-1)^{(g_{b_1}+\cdots+g_{b_{q-1}}+q-1)(g_f+g_l)}}{p!(q-1)!} y^1_{b_{q-1}} \cdots y^1_{b_1} y^1_{a_p} \cdots y^1_{a_1} (f^{a_1\ldots a_p} \overleftarrow{\partial}_{z^l}) h^{b_1\ldots b_{q-1}l}$$
$$- \frac{(-1)^{(g_l+1)(g_f+p+g_{a_2}+\cdots+g_{a_p})+(g_f-1)(g_{b_1}+\cdots+g_{b_q}+q)}}{(p-1)!q!}$$
$$\times\, y^1_{b_q} \cdots y^1_{b_1} y^1_{a_p} \cdots y^1_{a_2} f^{la_2\ldots a_p} \partial_{z^l} h^{b_1\ldots b_q} \overset{\text{def}}{=} y^1_{a_{p+q}} \cdots y^1_{a_2} [F,H]^{a_2\ldots a_{p+q}},$$

where

$$d_0 z^a \equiv y^a_0 = y^1_b \omega^{ba}_1.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H] = -(-1)^{(g_f+1)(g_h+1)}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{(g_e+1)(g_h+1)}[E,[F,H]] = 0.$$

8. At last for the odd Poisson bracket in superspace, taking as an initial one, we obtain in the case $\zeta = 1$

$$(F,H)_0 = (-1)^{(g_{b_1}+\cdots+g_{b_{q-1}})(g_f+p)} \left[\frac{1}{p!(q-1)!} y^0_{b_{q-1}} \cdots y^0_{b_1} y^0_{a_p} \cdots y^0_{a_1} (f^{a_1\ldots a_p} \overleftarrow{\partial}_{z^l}) h^{lb_1\ldots b_{q-1}} \right.$$
$$\left. - \frac{(-1)^{(g_f+p)(g_l+g_{b_q})}}{(p-1)!q!} y^0_{b_q} \cdots y^0_{b_1} y^0_{a_p} \cdots y^0_{a_2} f^{a_2\ldots a_p l} \partial_{z^l} h^{b_1\ldots b_q} \right]$$
$$\overset{\text{def}}{=} y^0_{a_{p+q}} \cdots y^0_{a_2} [F,H]^{a_2\ldots a_{p+q}},$$

where

$$d_1 z^a \equiv y^a_1 = y^0_b \omega^{ba}_1.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H] = -(-1)^{(g_f+p)(g_h+q)}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{(g_e+s)(g_h+q)}[E,[F,H]] = 0.$$

Thus, we see that the formula (13) contains as particular cases quite a number of the Schouten–Nijenhuis type brackets.

4 Conclusion

We give the recipe for the construction from a given Poisson bracket of the definite Grassmann parity another bracket. For this construction we use the exterior differentials with different Grassmann parities. We proved that the resulting Poisson bracket essentially depends on the parity of the exterior differential in spite of these differentials give the same exterior calculus [1]. The recipe leads to the set of different generalizations for the Schouten–Nijenhuis bracket. Thus, we see that the Schouten–Nijenhuis bracket and its possible generalizations are particular cases of the usual Poisson brackets of different Grassmann parities (12). We hope that these generalizations will find their own application for the deformation quantization (see, for example, [7,14]) as well as the usual Schouten–Nijenhuis bracket.

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