Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 1, 142–148

Invariants for Evolution Equations

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In the spirit of the recent work of Ibragimov [1] who adopted the infinitesimal method for calculating invariants of families of differential equations using the equivalence groups, we apply the method to evolution type equations of the form $u_t = f(x,u)u_{xx} + g(x,u,u_x)$. We show that the equivalence Lie algebra admitted by this equation has two functionally independent differential invariants of the second order.

1 Introduction

We consider evolution equations of the form

$$u_t = f(x,u)u_{xx} + g(x,u,u_x). \tag{1}$$

A number of many special cases in this class of equations have been successfully used to model physical problems. Such example is the nonlinear diffusion equation $u_t = [D(u)u_x]_x$. Group properties of this equation were studied by Ovsiannikov [2]. Other examples of such equations that appear in the literature are $u_t = [g(x)D(u)u_x]_x$, $u_t = [g(x)D(u)u_x]_x - K(u)u_x$, $u_t = (u^n)_{xx} + g(x)u^m + f(x)u^s u_x$, etc.

It can be shown that equations (1) admit equivalence transformation

$$x' = P(x), \qquad t' = c_1 t + c_2, \qquad u' = R(x,u) \tag{2}$$

with

$$f' = \frac{P_x^2 f}{c_1}, \qquad g' = \frac{P_x R_u g + (P_{xx}R_u u_x + P_{xx}R_x - 2P_xR_{ux}u_x - P_xR_{uu}u_x^2 - P_xR_{xx})f}{c_1 P_x}.$$

If we set $P(x) = x + \epsilon\phi(x)$ and $R(x,u) = u + \epsilon\psi(x,u)$, we can write the above transformations in infinitesimal form. That is, in the form

$$Y = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta\frac{\partial}{\partial u_x} + \mu\frac{\partial}{\partial f} + \nu\frac{\partial}{\partial g}, \tag{3}$$

where ξ^1, ξ^2 and η depend on t, x and u, while μ and ν depend on t, x, u, u_x, f and g, and ζ is given by $\zeta = D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2)$. The operator D_x is the total derivative with respect to x.

We deduce that the class of equations (1) has an infinite continuous group of equivalence transformations generated by the infinite-dimensional Lie algebra which is spanned by the operators:

$$
\begin{aligned}
&Y_1 = \frac{\partial}{\partial t}, \qquad Y_2 = t\frac{\partial}{\partial t} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g},\\
&Y_\phi = \phi(x)\frac{\partial}{\partial x} - \phi' u_x \frac{\partial}{\partial u_x} + 2\phi' f\frac{\partial}{\partial f} + \phi'' f u_x \frac{\partial}{\partial g},\\
&Y_\psi = \psi(x,u)\frac{\partial}{\partial u} + (\psi_x + \psi_u u_x)\frac{\partial}{\partial u_x} + \left[\psi_u g - (\psi_{uu} u_x^2 + 2\psi_{xu} u_x + \psi_{xx})f\right]\frac{\partial}{\partial g}.
\end{aligned} \tag{4}
$$

In this paper we calculate differential invariants of equivalence transformations of equations (1) by using the infinitesimal method for calculations of invariants of families of equations developed in [3]. In the following three sections we consider the problem of classifying differential invariants of equations (1) of zero, first and second order.

2 Differential invariants of order zero

We search for invariants of order zero. That is, invariants of the form

$$
J = J(t, x, u, u_x, f, g).
$$

We apply the invariant test $Y(J) = 0$ to the operators Y_1, Y_2, Y_ϕ and Y_ψ and using the fact that $\phi(x)$ and $\psi(x,u)$ are arbitrary functions, we obtain $J = \text{const}$. Hence, equations (1) do not admit differential invariants of order zero.

3 Differential invariants of first order

In order to determine differential invariants of the first order,

$$
J = J(t, x, u, u_x, f, g, f_x, f_u, g_x, g_u, g_{u_x})
$$

we need to consider the first prolongation of Y,

$$
Y^{(1)} = Y + \mu^x \frac{\partial}{\partial f_x} + \mu^u \frac{\partial}{\partial f_u} + \nu^x \frac{\partial}{\partial g_x} + \nu^u \frac{\partial}{\partial g_u} + \nu^{u_x}\frac{\partial}{\partial g_{u_x}},
$$

where

$$
\begin{aligned}
&\mu^k = \tilde{D}_k(\mu) - f_x \tilde{D}_k(\xi^2) - f_u \tilde{D}_k(\eta), \qquad k = x,\ u,\\
&\nu^k = \tilde{D}_k(\nu) - g_x \tilde{D}_k(\xi^2) - g_u \tilde{D}_k(\eta) - g_{u_x}\tilde{D}_k(\zeta), \qquad k = x,\ u,\ u_x,
\end{aligned} \tag{5}
$$

where $\tilde{D}_x$, $\tilde{D}_u$ and $\tilde{D}_{u_x}$ denote the total derivatives with respect to x, u and u_x:

$$
\begin{aligned}
\tilde{D}_x &= \frac{\partial}{\partial x} + f_x\frac{\partial}{\partial f} + g_x\frac{\partial}{\partial g} + f_{xx}\frac{\partial}{\partial f_x} + f_{xu}\frac{\partial}{\partial f_u} + g_{xx}\frac{\partial}{\partial g_x} + g_{xu}\frac{\partial}{\partial g_u} + g_{xu_x}\frac{\partial}{\partial g_{u_x}} + \cdots,\\
\tilde{D}_u &= \frac{\partial}{\partial u} + f_u\frac{\partial}{\partial f} + g_u\frac{\partial}{\partial g} + f_{xu}\frac{\partial}{\partial f_x} + f_{uu}\frac{\partial}{\partial f_u} + g_{xu}\frac{\partial}{\partial g_x} + g_{uu}\frac{\partial}{\partial g_u} + g_{uu_x}\frac{\partial}{\partial g_{u_x}} + \cdots,\\
\tilde{D}_{u_x} &= \frac{\partial}{\partial u_x} + f_{u_x}\frac{\partial}{\partial f} + g_{u_x}\frac{\partial}{\partial g} + f_{xu_x}\frac{\partial}{\partial f_x} + f_{uu_x}\frac{\partial}{\partial f_u}\\
&\quad + g_{xu_x}\frac{\partial}{\partial g_x} + g_{uu_x}\frac{\partial}{\partial g_u} + g_{u_xu_x}\frac{\partial}{\partial g_{u_x}} + \cdots.
\end{aligned} \tag{6}
$$

Using the formulae (5) and (6) we obtain the first extension of the generators Y_1, Y_2, Y_ϕ, Y_ψ given by equations (4):

$$\begin{aligned}
Y_1^{(1)} &= Y_1, \qquad Y_2^{(1)} = Y_2 - f_x\frac{\partial}{\partial f_x} - f_u\frac{\partial}{\partial f_u} - g_x\frac{\partial}{\partial g_x} - g_u\frac{\partial}{\partial g_u} - g_{u_x}\frac{\partial}{\partial g_{u_x}},\\
Y_\phi^{(1)} &= Y_\phi + (2\phi'' f + \phi' f_x)\frac{\partial}{\partial f_x} + 2\phi' f_u\frac{\partial}{\partial f_u}\\
&\quad + (u_x\phi''' f + u_x\phi'' f_x - g_x\phi' + \phi'' u_x g_{u_x})\frac{\partial}{\partial g_x} + u_x\phi'' f_u\frac{\partial}{\partial g_u} + (\phi'' f + \phi' g_{u_x})\frac{\partial}{\partial g_{u_x}},\\
Y_\psi^{(1)} &= Y_\psi - f_u\psi_x\frac{\partial}{\partial f_x} - f_u\psi_u\frac{\partial}{\partial f_u} + [\psi_{xu} - (\psi_{xuu}u_x^2 + 2\psi_{xxu}u_x + \psi_{xxx})f\\
&\quad - (\psi_{uu}u_x^2 + 2\psi_{xu}u_x + \psi_{xx})f_x + \psi_u g_x - \psi_x g_u - (\psi_{xx} + \psi_{xu}u_x)g_{u_x}]\frac{\partial}{\partial g_x}\\
&\quad + [\psi_{uu}g - (\psi_{uuu}u_x^2 + 2\psi_{xuu}u_x + \psi_{xxu}) - (\psi_{uu}u_x^2 + 2\psi x u u_x + \psi_{xx})f_u\\
&\quad - (\psi_{uu}u_x + \psi_{xu})g_{u_x}]\frac{\partial}{\partial g_u} - 2(\psi_{uu}u_x + \psi_{xu})f\frac{\partial}{\partial g_{u_x}}.
\end{aligned} \tag{7}$$

We note that $Y_1^{(n)} = Y_1$. Hence for any order of differential invariants $J_t = 0$.

Now from the differential invariant test $Y^{(1)}(J) = 0$, we get three identities

$$E_2 = Y_2^{(1)}(J) = 0, \qquad E_\phi = Y_\phi^{(1)}(J) = 0, \qquad E_\psi = Y_\psi^{(1)}(J) = 0. \tag{8}$$

Since $\phi(x)$ and $\psi(x,u)$ are arbitrary functions, coefficients of ϕ in $E_\phi = 0$ and ψ in $E_\psi = 0$ give $J_x = J_u = 0$. Now coefficients of ψ_{uuu}, ψ_{xuu}, ψ_{uu} and ψ_{xu} give $J_{g_u} = J_{g_x} = J_g = J_{g_{u_x}} = 0$. Coefficient of ϕ'' in $E_\phi = 0$ gives $J_{f_x} = 0$ and coefficient of ψ_x in $E_\psi = 0$ gives $J_{u_x} = 0$. Hence, $J = J(f, f_u)$ and equations (8) read

$$E_1 = -\left(f\frac{\partial J}{\partial f} + f_u\frac{\partial J}{\partial f_u}\right) = 0, \qquad E_\phi = 2\left(f\frac{\partial J}{\partial f} + f_u\frac{\partial J}{\partial f_u}\right)\phi' = 0, \qquad E_\psi = f_u\frac{\partial J}{\partial f_u}\psi_u = 0.$$

If $f_u \neq 0$ from the above relations we deduce that $J_{f_u} = J_f = 0$ and therefore equations (1) do not admit differential invariant of the first order. However the equation

$$f_u = 0 \tag{9}$$

is invariant under the group which is spanned by (7). That is,

$$Y_1^{(1)}(f_u)|_{f_u=0} = 0, \qquad Y_2^{(1)}(f_u)|_{f_u=0} = 0, \qquad Y_\phi^{(1)}(f_u)|_{f_u=0} = 0, \qquad Y_\psi^{(1)}(f_u)|_{f_u=0} = 0.$$

4 Differential invariants of second order

Now we determine differential invariants that depend on the second derivatives of f and g. Here we need to calculate the second prolongation of (4). As in the previous case it is straightforward to deduce that $J_t = J_x = J_u = 0$. Hence,

$$J = J(u_x, f, g, f_x, f_u, g_x, g_u, g_{u_x}, f_{xx}, f_{xu}, f_{uu}, g_{xx}, g_{xu}, g_{xu_x}, g_{uu}, g_{uu_x}, g_{u_xu_x}).$$

Now the second prolongation of (4) reads

$$
\begin{aligned}
Y_2^{(2)} &= Y_2^{(1)} - f_{xx}\frac{\partial}{\partial f_{xx}} - f_{xu}\frac{\partial}{\partial f_{xu}} - f_{uu}\frac{\partial}{\partial f_{uu}} - g_{xx}\frac{\partial}{\partial g_{xx}} - g_{xu}\frac{\partial}{\partial g_{xu}} - g_{xu_x}\frac{\partial}{\partial g_{xu_x}} \\
&\quad - g_{uu}\frac{\partial}{\partial g_{uu}} - g_{uu_x}\frac{\partial}{\partial g_{uu_x}} - g_{u_xu_x}\frac{\partial}{\partial g_{u_xu_x}}, \\
Y_\phi^{(2)} &= Y_\phi^{(1)} + (2\phi''' f + 3\phi'' f_x)\frac{\partial}{\partial f_{xx}} + (2\phi'' f_u + \phi' f_{xu})\frac{\partial}{\partial f_{xu}} + 2\phi' f_{uu}\frac{\partial}{\partial f_{uu}} \\
&\quad + (u_x\phi^{(iv)} f + \cdots)\frac{\partial}{\partial g_{xx}} + (u_x\phi''' f_u + \cdots)\frac{\partial}{\partial g_{xu}} \\
&\quad + (\phi''' f + \phi'' g_{u_x} + \phi'' f_x + u_x\phi'' g_{u_xu_x})\frac{\partial}{\partial g_{xu_x}} + u_x\phi'' f_{uu}\frac{\partial}{\partial g_{uu}} \\
&\quad + (\phi'' f_u + \phi' g_{uu_x})\frac{\partial}{\partial g_{uu_x}} + 2\phi' g_{u_xu_x}\frac{\partial}{\partial g_{u_xu_x}}, \\
Y_\psi^{(2)} &= Y_\psi^{(1)} - (\psi_{xu} f_u + 2\psi_x f_{xu})\frac{\partial}{\partial f_{xx}} - (\psi_{xu} f_u + \psi_u f_{xu} + \psi_x f_{uu})\frac{\partial}{\partial f_{xu}} \\
&\quad - (f_u\psi_{uu} + 2f_{uu}\psi_u)\frac{\partial}{\partial f_{uu}} + (-\psi_{xxxx} f + \cdots)\frac{\partial}{\partial g_{xx}} + (-\psi_{xxxu} f + \cdots)\frac{\partial}{\partial g_{xu}} \\
&\quad + (-\psi_{uuuu} f u_x^2 + \cdots)\frac{\partial}{\partial g_{uu}} - [2\psi_{xuu}u_x f + 2\psi_{xxu} f + 2\psi_{uu} f_x u_x + 2\psi_{xu} f_x + \psi_x g_{uu_x} \\
&\quad + (\psi_{xx} + \psi_{xu}u_x)g_{u_xu_x}]\frac{\partial}{\partial g_{xu_x}} - [2\psi_{uuu}u_x f + 2\psi_{xuu} f + 2\psi_{uu} f_u u_x + 2\psi_{xu} f_u \\
&\quad + \psi_u g_{uu_x} + (\psi_{xu} + \psi_{uu}u_x)g_{u_xu_x}]\frac{\partial}{\partial g_{uu_x}} - (2\psi_{uu} f + \psi_u g_{u_xu_x})\frac{\partial}{\partial g_{u_xu_x}}.
\end{aligned}
\tag{10}
$$

The invariant test produces three identities

$$
E_2 = Y_2^{(2)}(J) = 0, \qquad E_\phi = Y_\phi^{(2)}(J) = 0, \qquad E_\psi = Y_\psi^{(2)}(J) = 0. \tag{11}
$$

Coefficients of ψ_{xxxx}, ψ_{xxxu}, ψ_{uuuu} and ψ_{xxx} in $E_\psi = 0$ give $J_{g_{xx}} = J_{g_{xu}} = J_{g_{uu}} = J_{g_x} = 0$. Hence,

$$
J = J(u_x, f, g, f_x, f_u, g_u, g_{u_x}, f_{xx}, f_{xu}, f_{uu}, g_{xu_x}, g_{uu_x}, g_{u_xu_x}).
$$

Equation $E_2 = 0$ now reads

$$
\begin{aligned}
&f J_f + g J_g + f_x J_{f_x} + f_u J_{f_u} + g_u J_{g_u} + g_{u_x} J_{g_{u_x}} + f_{xx} J_{f_{xx}} + f_{xu} J_{f_{xu}} + f_{uu} J_{f_{uu}} \\
&\qquad + g_{xu_x} J_{g_{xu_x}} + g_{uu_x} J_{g_{uu_x}} + g_{u_xu_x} J_{g_{u_xu_x}} = 0.
\end{aligned}
$$

From this first order partial differential equation we get 12 integrals

$$
\begin{aligned}
&p_1 = \frac{f}{g}, \quad p_2 = \frac{f_x}{f}, \quad p_3 = \frac{f_u}{f}, \quad p_4 = \frac{g_u}{g}, \quad p_5 = \frac{g_{u_x}}{g}, \quad p_6 = \frac{f_{xx}}{f}, \\
&p_7 = \frac{f_{xu}}{f}, \quad p_8 = \frac{f_{uu}}{f}, \quad p_9 = \frac{g_{xu_x}}{g}, \quad p_{10} = \frac{g_{uu_x}}{g}, \quad p_{11} = \frac{g_{u_xu_x}}{g}, \quad p_{12} = u_x.
\end{aligned}
\tag{12}
$$

Coefficient of ψ_{xxu} in $E_\psi = 0$ gives

$$
J_{p_4} + 2J_{p_9} = 0,
$$

where we have used the new variables p_i. The above relation reduces the integrals by one:

$$
p_1, \quad p_2, \quad p_3, \quad p_5, \quad p_6, \quad p_7, \quad p_8, \quad p_{10}, \quad p_{11}, \quad p_{12}, \quad q_4 = 2p_4 - p_9. \tag{13}
$$

From the coefficient of ψ_{xuu} in $E_\psi = 0$ we get

$$J_{p_{10}} + p_{12}J_{q_4} = 0$$

and therefore we have the following 10 integrals

$$p_1, \quad p_2, \quad p_3, \quad p_5, \quad p_6, \quad p_7, \quad p_8, \quad p_{11}, \quad p_{12}, \quad r_4 = q_4 - p_{12}p_{10}. \tag{14}$$

Coefficient of ψ_x in $E_\psi = 0$ gives

$$J_{p_{12}} - p_3J_{p_2} - 2p_7J_{p_6} - p_8J_{p_7} = 0$$

which produces the integrals

$$p_1, \quad p_3, \quad p_5, \quad p_8, \quad p_{11}, \quad r_4, \quad q_2 = p_2 + p_3p_{12}, \quad q_6 = p_7^2 - p_6p_8, \quad q_7 = p_7 + p_8p_{12}. \tag{15}$$

Coefficient of ψ_{xx} in $E_\psi = 0$ produces

$$p_1^2J_{p_1} + p_1p_{11}J_{p_{11}} + p_1p_5J_{p_5} + p_3p_8J_{q_6} + (p_{11} + p_1r_4 - 2p_1p_3)J_{r_4} = 0$$

which implies the integrals

$$p_3, \quad p_8, \quad q_2, \quad q_7, \quad q_5 = \frac{p_5}{p_1}, \quad q_{11} = \frac{p_{11}}{p_1}, \quad r_6 = q_6 + \frac{p_3p_8}{p_1}, \quad \mu_4 = \frac{r_4}{p_1} + \frac{p_{11}}{p_1^2} - 2\frac{p_3}{p_1}. \tag{16}$$

We take the coefficient of ψ_{xu} in $E_\psi = 0$,

$$2J_{q_5} + p_3J_{q_7} + 2p_3q_7J_{r_6} + 2(q_5 - q_2)J_{\mu_4} = 0.$$

We obtain the integrals

$$p_3, \quad p_8, \quad q_2, \quad q_{11}, \quad r_5 = q_5^2 - 2q_2q_5 - 2\mu_4, \quad r_7 = 2q_7 - p_3q_5, \quad \mu_6 = q_7^2 - r_6. \tag{17}$$

Coefficient of ψ_{uu} in $E_\psi = 0$ gives

$$p_3\mu_6J_{\mu_6} + 2p_8J_{q_{11}} + p_3p_8J_{p_8} = 0$$

which produces the integrals

$$p_3, \quad q_2, \quad r_5, \quad r_7, \quad r_{11} = p_3q_{11} - 2p_8, \quad \lambda_6 = \frac{\mu_6}{p_8}. \tag{18}$$

Coefficient of ψ_u in $E_\psi = 0$ gives the equation

$$p_3J_{p_3} + 2r_{11}J_{r_{11}} + r_7J_{r_7} = 0$$

from which we get the solutions

$$q_2, \quad r_5, \quad \lambda_6, \quad \mu_7 = \frac{r_7}{p_3}, \quad \mu_{11} = \frac{r_{11}}{p_3^2}. \tag{19}$$

Solutions (19) satisfy $E_\psi = 0$ for any arbitrary function $\psi(x,u)$. Now we use the identity $E_\phi = 0$. Coefficient of ϕ''' gives

$$J_{r_5} + J_{\lambda_6} = 0$$

and therefore we have

$$q_2, \quad \mu_7, \quad \mu_{11}, \quad \mu_5 = r_5 - \lambda_6. \tag{20}$$

Coefficient of ϕ'' in $E_\phi = 0$ produces the equation

$$2J_{q_2} + 3J_{\mu_7} - 3q_2 J_{\mu_5} = 0$$

which gives the integrals

$$\mu_{11}, \quad \lambda_7 = 2\mu_7 - 3q_2, \quad \lambda_5 = 4\mu_5 + 3q_2^2. \tag{21}$$

Finally equation $E_\phi = 0$ reads

$$(2\lambda_5 J_{\lambda_5} + \lambda_7 J_{\lambda_7})\phi' = 0.$$

Hence we obtain the solutions

$$J_1 = \mu_{11}, \qquad J_2 = \frac{\lambda_5}{\lambda_7^2}. \tag{22}$$

Now, using the sequence of integrals (12)–(22), we can write the forms of J_1 and J_2 in terms of the original variables, u_x, f, g, f_x, f_u, g_u, g_{u_x}, f_{xx}, f_{xu}, f_{uu}, g_{xu_x}, g_{uu_x}, $g_{u_xu_x}$. We therefore conclude that equation (1) has two invariants of the second order:

$$J_1 = \frac{f_u g_{u_xu_x} - 2ff_{uu}}{f_u^2}, \tag{23}$$

$$\begin{aligned} J_2 = {} & f_u^2(-4u_x^2 ff_{uu} - 8u_x ff_{xu} - 4ff_{xx} - 16fg_u \\ & + 8u_x fg_{uu_x} + 8fg_{xu_x} + 3u_x^2 f_u^2 + 6u_x f_x f_u + 20gf_u - 8u_x f_u g_{u_x} + 3f_x^2 - 8f_x g_{u_x} \\ & - 8gg_{u_xu_x} + 4g_{u_x}^2)/(4u_x ff_{uu} + 4ff_{xu} - 3u_x f_u^2 - 3f_x f_u - 2f_u g_{u_x})^2. \end{aligned} \tag{24}$$

In addition to the invariant equation $f_u = 0$ (equation (9)) that we found in the previous section, here we have also the following three invariant equations:

$$f_u g_{u_xu_x} - 2ff_{uu} = 0, \tag{25}$$

$$\begin{aligned} & -4u_x^2 ff_{uu} - 8u_x ff_{xu} - 4ff_{xx} - 16fg_u + 8u_x fg_{uu_x} + 8fg_{xu_x} + 3u_x^2 f_u^2 \\ & \quad + 6u_x f_x f_u + 20gf_u - 8u_x f_u g_{u_x} + 3f_x^2 - 8f_x g_{u_x} - 8gg_{u_xu_x} + 4g_{u_x}^2 = 0, \end{aligned} \tag{26}$$

$$4u_x ff_{uu} + 4ff_{xu} - 3u_x f_u^2 - 3f_x f_u - 2f_u g_{u_x} = 0. \tag{27}$$

To show this we need to apply the second prolongation (10) of (4) to these equations. That is, we have to show that

$$Y_2^{(2)}(\phi)|_{\phi=0} = 0, \qquad Y_\phi^{(2)}(\phi)|_{\phi=0} = 0, \qquad Y_\psi^{(2)}(\phi)|_{\phi=0} = 0,$$

where ϕ is the left hand side of equations (25), (26) and (27).

We make the following remarks: If equation (1) is such that

1. All four equations (9), (25)–(27) hold, then it has no invariants. We note that if (9) holds, then equations (25) and (27) are satisfied.
2. Equation (9) holds, then it has one invariant, $J_2 = 0$.
3. Equations (26) and (27) hold, then it has one invariant, J_1.
4. Equation (25) holds (but not (9)), then it has two invariants, $J_1 = 0$, J_2.
5. Equation (26) holds, then it has two invariants, J_1, $J_2 = 0$.
6. Equation (27) holds, then it has two invariants, J_1, $J_2' = \frac{1}{J_2} = 0$.

Finally, we make a comment on the invariant equation $f_u = 0$. From this relation we deduce that when two equations of the form (1) are connected by a point transformation, the corresponding functions $f(x,u)$ must both depend on u, or both do not depend on u. From this we can deduce that there exists **no** point transformation that maps an equation of the form (1) with $f_u \neq 0$ to the linear heat equation $u_t = u_{xx}$. In general, an equation of the form (1) with $f_u \neq 0$ cannot be linearised by a point transformation.

Example. We consider the integrable equation

$$u_t = u^2 u_{xx} \tag{28}$$

and the class of equations

$$u_t = u^n u_{xx} + g(x,u,u_x). \tag{29}$$

Both of these equations are special forms of (1). Setting $f = u^2$ and $g = 0$ into equations (23) and (24) we find that equation (28) has invariants $J_1 = -1$ and $J_2 = 1$. From (23) and (24) we deduce that equation (29) has invariants $J_1 = -1$ and $J_2 = 1$ if it is of the form

$$u_t = u^n u_{xx} + \frac{1}{2}(n-2)u^{n-1}u_x^2 + k(x)u^n u_x + \frac{2}{n}\frac{\mathrm{d}k}{\mathrm{d}x}u^{n+1} + h(x)u^{\frac{3n+4}{4}}. \tag{30}$$

Now if we consider transformation (2), it can be shown that the most general form of (30) (and consequently of (29)) that can be linked with (28) is

$$u_t = u^n u_{xx} + \frac{1}{2}(n-2)u^{n-1}u_x^2 + k(x)u^n u_x + \frac{2}{n}\frac{\mathrm{d}k}{\mathrm{d}x}u^{n+1}. \tag{31}$$

In fact, it can be shown that the transformation

$$x \mapsto \int \mathrm{e}^{\int k(x)\mathrm{d}x}\mathrm{d}x, \qquad t \mapsto t, \qquad u \mapsto \mathrm{e}^{\int k(x)\mathrm{d}x}u^{\frac{n}{2}}$$

maps (28) into (31).

5 Remarks

We have shown that the class of equations (1) has no differential invariants of order zero and order one. We have determined two functionally independent differential invariants of second order. In order to produce higher order invariants, we need to follow the procedure as above by considering higher order prolongations, or alternatively we can introduce the idea of invariant differentiation. Details about invariant differentiation can be found in the book of Ibragimov [3].

We note that for the invariants (23) and (24) we need to have $f_u \neq 0$. Hence in the case where $f_u = 0$, that is equation $u_t = f(x)u_{xx} + g(x,u,u_x)$, needs to be considered separately. However, by introducing a new space variable $\xi = \int \frac{1}{f(x)}\mathrm{d}x$, this latter equation takes the form $u_t = u_{\xi\xi} + h(\xi,u,u_\xi)$. The problem of classification of differential invariants for the class of equations $u_t = u_{xx} + g(x,u,u_x)$ will be considered in a separate article in the near future.

[1] Ibragimov N.H., Laplace type invariants for parabolic equations, *Nonlinear Dynamics*, 2002, V.28, 125–133.

[2] Ovsiannikov L.V., Group relations of the equation of non-linear heat conductivity, *Dokl. Akad. Nauk SSSR*, 1959, V.125, 492–495 (in Russian).

[3] Ibragimov N.H., Elementary Lie group analysis and ordinary differential equations, New York, Wiley, 1999.