Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 1, 462–469

Quasigraded Lie Algebras and Matrix Generalization of Landau–Lifshitz Equation

Taras V. SKRYPNYK

Bogoliubov Institute for Theoretical Physics, 14-b Metrologichna Str., 03143 Kyiv-143, Ukraine
E-mail: *tskrypnyk@imath.kiev.ua*

Using special quasigraded Lie algebras we obtain new hierarchies of integrable equations in partial derivatives admitting zero-curvature representations. In particular, we obtain generalization of Landau–Lifshitz equation for the Lie algebras $so(2n)$, $sp(n)$ and $gl(n)$.

1 Introduction

Integrability of equations of $1+1$ field theory and condensed matter physics is based on the possibility to represent them in the form of the so-called zero-curvature equations [1, 2]:

$$\frac{\partial U(x,t,\lambda)}{\partial t} - \frac{\partial V(x,t,\lambda)}{\partial x} + [U(x,t,\lambda), V(x,t,\lambda)] = 0. \tag{1}$$

The most productive interpretation of zero-curvature equation (see [6, 5]) is to consider them as a consistency condition for a set of a commuting Hamiltonian flows on the dual space to some infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ of matrix-valued function of λ written in the Euler–Arnold (generalized Lax) form:

$$\frac{\partial L(\lambda)}{\partial t_l} = \mathrm{ad}^*_{\nabla I_l(L(\lambda))} L(\lambda), \qquad \frac{\partial L(\lambda)}{\partial t_k} = \mathrm{ad}^*_{\nabla I_k(L(\lambda))} L(\lambda), \tag{2}$$

where $L(\lambda) \in \widetilde{\mathfrak{g}}^*$ is the generic element of the dual space, $\nabla I_k(L(\lambda)) \in \widetilde{\mathfrak{g}}$ is algebra-valued gradient of $I_k(L(\lambda))$, and the "Hamiltonians" $I_k(L(\lambda))$, $I_l(L(\lambda))$ belong to the set of mutually commuting with respect to the natural Lie–Poisson bracket functions on $\widetilde{\mathfrak{g}}^*$.

Consistency condition of two commuting flows given by equations (2) yields equation (1) with $U \equiv \nabla I_k$, $V \equiv \nabla I_l$, $t_k \equiv x$, $t_l \equiv t$. Hence in order to construct new integrable hierarchies in the framework of the described approach it is necessary to have some infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ admitting an algorithm of construction of infinite family of commuting Hamiltonians on its dual space. Such the algorithm is famous Kostant–Adler scheme [3, 4].

In our previous paper [9] we proposed a multiparametric family of special quasigraded Lie algebras $\widetilde{\mathfrak{g}}_A$ parametrized by numerical matrices A. The main feature of the constructed Lie algebras is the property to admit Kostant–Adler scheme, i.e. to admit decomposition into the sum of two subalgebras $\widetilde{\mathfrak{g}}_A = \widetilde{\mathfrak{g}}_A^+ + \widetilde{\mathfrak{g}}_A^-$. In the present paper we use them in order to construct new sets of integrable equations in partial derivatives. In the result we obtain an infinite set of equations (1) with $\widetilde{\mathfrak{g}}_A^\pm$-valued $U - V$ pairs constituting so-called integrable hierarchies. We concentrate our attention on one of these hierarchies connected with the subalgebras $\widetilde{\mathfrak{g}}_A^-$.

In the present paper we show that explicit form of the corresponding equations depends on the form of the additional covariant constraint on the matrices entering into the U-operators. We consider the simplest U-operators and the simplest covariant matrix constraint:

$$U = \lambda^{-1} S, \qquad S^2 = \frac{1}{4}\mathbf{1}.$$

Such covariant constraints exist for the cases of underlying simple (reductive) matrix Lie algebras $\mathfrak{g}$ are $so(2n)$, $sp(n)$ and $gl(n)$. In the result we obtain the following equation on the matrix S:

$$\frac{\partial S}{\partial t} = \left[S, \frac{\partial^2 S}{\partial x^2}\right] + \frac{1}{2}\frac{\partial}{\partial x}(AS + SA) + \left[S, \left[S, \frac{\partial S}{\partial x}\right]\right]_A + \frac{1}{2}[S, AS + SA]_A. \tag{3}$$

We show that this equation is the direct matrix generalization of the Landau–Lifshiz equation, where the role of the anisotropy tensor is played by the matrix A. In the case of $\mathfrak{g} = so(4)$ it could be reduced to the standard Landau–Lifshitz equation [15, 5].

The volume of the paper gives no possibility to consider other equations connected with algebras $\widetilde{\mathfrak{g}}_A$. Interested reader may consult our papers [10, 11] for further examples.

2 Special quasi-graded algebras

In this section we describe a new class of infinite-dimensional Lie algebras $\widetilde{\mathfrak{g}}$ that could be used to generate classical integrable systems. These algebras satisfy the following integrability requirements (IR):

- (IR1) they possess infinite number of algebraically independent invariants of coadjoint representation,
- (IR2) they are decomposable into the direct sum of two subalgebras: $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}^- + \widetilde{\mathfrak{g}}^+$,
- (IR3) subalgebras $\widetilde{\mathfrak{g}}^+$, $\widetilde{\mathfrak{g}}^-$ possess infinite set of embedded ideals $\mathcal{J}_{\pm n}$ of finite co-dimensions.

2.1 General construction

Definition 1. Infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ is called $\mathbb{Z}$-quasigraded of type (p, q) if it admits the decomposition:

$$\widetilde{\mathfrak{g}} = \sum_{j\in\mathbb{Z}} \mathfrak{g}_j, \quad \text{such that} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \sum_{k=-p}^{q} \mathfrak{g}_{i+j+k}.$$

The following proposition holds true [9]:

Proposition 1. *Let $\widetilde{\mathfrak{g}}$ be $\mathbb{Z}$-quasi graded of type $(0, 1)$, or $(1, 0)$. Then $\widetilde{\mathfrak{g}}$ satisfies conditions* (IR2) *and* (IR3)*.*

So our aim in this section will be a construction of $\mathbb{Z}$-quasigraded algebras of type $(0, 1)$. For this purpose we will deform Lie algebraic structure in loop algebras. We will introduce into $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathrm{Pol}\,(\lambda, \lambda^{-1})$ new Lie bracket:

$$[X \otimes p(\lambda), Y \otimes q(\lambda)]_F = [X, Y] \otimes p(\lambda)q(\lambda) - F(X, Y) \otimes \lambda p(\lambda)q(\lambda), \tag{4}$$

where $X, Y \in \mathfrak{g}$, $p(\lambda), q(\lambda) \in \mathrm{Pol}\,(\lambda, \lambda^{-1})$, $[\,,\,]$ in the righthand side of this identity denotes ordinary Lie bracket in $\mathfrak{g}$ and map $F : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is skew. It is evident by the very construction that the Lie algebras with the so defined bracket are $\mathbb{Z}$-quasigraded Lie algebras of type $(0, 1)$ with the quasigrading being defined by degrees of the spectral parameter λ.

The following Propositions answer the question of when bracket (4) satisfy the Jacobi identity:

Proposition 2. *For bracket* (4) *to satisfy the Jacobi identities the cochain F should satisfy the following two requirements:*

$$\text{(J1)} \quad \sum_{\text{c.p. } \{i,j,k\}} (F([X_i, X_j], X_k) + [F(X_i, X_j), X_k]) = 0,$$

$$\text{(J2)} \quad \sum_{\text{c.p. } \{i,j,k\}} F(F(X_i, X_j), X_k) = 0.$$

In the case of classical matrix Lie algebras it is possible to give an explicit construction of large (multiparametric) family of cochains F, that satisfy conditions (J1)–(J2). Let $\mathfrak{g}$ be hereafter a classical matrix Lie algebra of the type $gl(n)$, $so(n)$ and $sp(n)$ over the field of the complex or real numbers. We will realize the algebra $so(n)$ as algebra of skew-symmetric matrices: $so(n) = \{X \in gl(n) | X = -X^\top\}$ and the algebra $sp(n)$ as the following matrix algebra: $sp(n) = \{X \in gl(n) | X = sX^\top s\}$, where n is an even number, $s \in so(n)$ and $s^2 = -1$.

The following Proposition holds true:

Proposition 3. *Let $\mathfrak{g}$ be a classical matrix Lie algebra over the field $\mathbb{K}$ of complex or real numbers. Let us define the numerical ($\mathbb{K}$-valued) $n \times n$ matrix A of the following type:*

1) *A is arbitrary for $\mathfrak{g} = gl(n)$,*
2) *$A = A^\top$ for $\mathfrak{g} = so(n)$,*
3) *$A = -sA^\top s$ for $\mathfrak{g} = sp(n)$.*

Then maps $F_A : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of the form $F_A(X, Y) = XAY - YAX$ are correctly defined skew-symmetric maps that satisfy conditions (J1)–(J2).

Remark 1. Cocycle F_A defines a second Lie bracket in the finite-dimensional Lie algebras $\mathfrak{g}$, that is compatible with the standard one. This fact was noticed in the paper [12] and was used to construct compatible Poisson brackets on the finite-dimensional Lie algebras $\mathfrak{g}$. The idea to use the same cocycle to generate infinite-dimensional Lie algebras with Kostant–Adler decomposition was proposed in [9] as a natural generalization of semi-geometric construction of [7, 8] of special quasigraded Lie algebras on the higher genus curves.

Definition 2. We will denote the Lie bracket in $\mathfrak{g}$ defined by the cocycles F_A by $[\,,\,]_A$, the corresponding finite-dimensional Lie algebra by $\mathfrak{g}_A$ and the infinite-dimensional Lie algebra with the Lie bracket given by (4) by $\widetilde{\mathfrak{g}}_A$.

The Lie bracket in the algebra $\widetilde{\mathfrak{g}}_A$ will have the following form:

$$[X(\lambda), (\lambda)]_{F_A} \equiv [X(\lambda), Y(\lambda)]_{A(\lambda)} = [X(\lambda), Y(\lambda)] - \lambda [X(\lambda), Y(\lambda)]_A, \tag{5}$$

where $X(\lambda), Y(\lambda) \in L(\mathfrak{g}) = \mathfrak{g} \otimes \mathrm{Pol}\,(\lambda, \lambda^{-1})$, $A(\lambda) \equiv 1 - \lambda A$, and we extend brackets $[\,,\,]$ and $[\,,\,]_A$ from the Lie algebra $\mathfrak{g}$ to the Lie algebra of $\mathfrak{g}$-valued functions $L(\mathfrak{g})$ in a natural way.

Now we can introduce convenient bases in the algebras $\widetilde{\mathfrak{g}}_A$. Due to the fact that we are dealing with matrix Lie algebras $\mathfrak{g}$, we will denote their basic elements as X_{ij}. For example, for the case $\mathfrak{g} = gl(n)$ we will have that $X_{ij} = I_{ij}$, where $(I_{ij})_{ab} = \delta_{ai}\delta_{bj}$ for the case $\mathfrak{g} = so(n)$ we will have that $X_{ij} = I_{ij} - I_{ji}$ etc. Let

$$X_{ij}^m \equiv X_{ij} \otimes \lambda^m$$

be the natural basis in $\widetilde{\mathfrak{g}}_A$. Commutation relations (5) in this basis have the following form:

$$[X_{ij}^r, X_{kl}^m]_{F_A} = \sum_{p,q} C_{ij,kl}^{pq} X_{pq}^{r+m} - \sum_{p,q} C_{ij,kl}^{pq}(A) X_{pq}^{r+m+1}, \tag{6}$$

where $C_{ij,kl}^{pq}$ and $C_{ij,kl}^{pq}(A)$ are the structure constants of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_A$ respectively.

Remark 2. Algebra $\widetilde{\mathfrak{g}}_A$ could be realized also in the space of special matrix valued functions of λ with an ordinary Lie bracket $[\,,\,]$. Nevertheless we consider realization in the space $\mathfrak{g} \otimes \mathrm{Pol}\,(\lambda, \lambda^{-1})$ with the "deformed" bracket to be the most convenient.

2.2 Coadjoint representation and its invariants

In this subsection we define dual spaces, coadjoint representations and their invariants for the Lie algebras $\widetilde{\mathfrak{g}}_A$. Let us at first explicitly describe the dual space $\widetilde{\mathfrak{g}}_A^*$ of $\widetilde{\mathfrak{g}}_A$. For this purpose we will define the pairing between $\widetilde{\mathfrak{g}}_A$ and $\widetilde{\mathfrak{g}}_A^*$ in the following standard way:

$$\langle X, L\rangle = \operatorname*{res}_{\lambda=0} \mathrm{Tr}\,(X(\lambda)L(\lambda)). \tag{7}$$

The generic element of the dual space $L(\lambda) \in \widetilde{\mathfrak{g}}_A^*$ is written as follows:

$$L(\lambda) = \sum_{k\in\mathbb{Z}} \sum_{i,j=1,n} l_{ij}^{(k)} \lambda^{-(k+1)} X_{ij}^*. \tag{8}$$

The following proposition is true [9]:

Proposition 4. *The coadjoint action of $\widetilde{\mathfrak{g}}_A$ on $\widetilde{\mathfrak{g}}_A^*$ is written as follows:*

$$\mathrm{ad}^*_{X(\lambda)} \circ L(\lambda) = A(\lambda)X(\lambda)L(\lambda) - L(\lambda)X(\lambda)A(\lambda), \tag{9}$$

where $X(\lambda), Y(\lambda) \in \widetilde{\mathfrak{g}}_A$, $L(\lambda) \in \widetilde{\mathfrak{g}}_A^*$.

Remark 3. Note that linear spaces $\widetilde{\mathfrak{g}}_A$ and $\widetilde{\mathfrak{g}}_A^*$ do not coincide as $\widetilde{\mathfrak{g}}_A$-modules. Moreover, rigorously speaking, they also do not coincide as linear spaces, because $\widetilde{\mathfrak{g}}_A^*$ contains formal power series, and $\widetilde{\mathfrak{g}}_A$, by the very definition, consists of the Loran polynomials.

Proposition 4 has the following important corollary:

Corollary 1. *Let $L(\lambda)$ be the generic element of $\widetilde{\mathfrak{g}}_A^*$. Then the functions*

$$I_k^m(L(\lambda)) = \frac{1}{m} \operatorname*{res}_{\lambda=0} \lambda^{-(k+1)} \mathrm{Tr}\left(L(\lambda)A(\lambda)^{-1}\right)^m. \tag{10}$$

are invariants of the coadjoint representation of $\widetilde{\mathfrak{g}}_A$.

Remark 4. The matrix $A(\lambda)^{-1} \equiv (1-\lambda A)^{-1}$ has to be understood as a power series in λ in the neighborhood of 0 or ∞: $A(\lambda)^{-1} = (1+A\lambda+A^2\lambda^2+\cdots)$ or $A(\lambda)^{-1} = -(A^{-1}\lambda^{-1}+A^{-2}\lambda^{-2}+\cdots)$.

2.3 Lie–Poisson structure

Let us define the Poisson structures in the space $\widetilde{\mathfrak{g}}_A^*$ using the pairing $\langle\,,\,\rangle$ defined above. It defines Lie–Poisson (Kirillov–Kostant) bracket on $P(\widetilde{\mathfrak{g}}_{\mathcal{A}}^*)$ in the following standard way:

$$\{F(L(\lambda)), G(L(\lambda))\} = \langle L(\lambda), [\nabla F(L(\lambda)), \nabla G(L(\lambda))]\rangle, \tag{11}$$

where $\nabla F(L(\lambda)) = \sum_{k\in\mathbb{Z}} \sum_{i,j=1}^{n} \dfrac{\partial F}{\partial l_{ij}^{(k)}} X_{ij}^k$, $\nabla G(L) = \sum_{m\in\mathbb{Z}} \sum_{k,l=1}^{n} \dfrac{\partial G(\lambda)}{\partial l_{kl}^{(m)}} X_{kl}^m$.

From the Corollary 1 and standard arguments the next statement follows:

Proposition 5. *Functions $I_\phi^m(L(\lambda))$ are central for the Lie–Poisson bracket* (11)*.*

Let us explicitly calculate Poisson bracket (11). It is easy to show that for the coordinate functions $l_{ij}^{(m)}$ these brackets will have the following form:

$$\{l_{ij}^{(n)}, l_{kl}^{(m)}\} = \sum_{p,q} C_{ij,kl}^{pq} l_{pq}^{(n+m)} - \sum_{p,q} C_{ij,kl}^{pq}(A) l_{pq}^{(n+m+1)}. \tag{12}$$

It is evident, that this bracket determine in the space of linear functions $\{l_{ij}^{(n)}\}$ a structure of the Lie algebra isomorphic to $\widetilde{\mathfrak{g}}_A$. That is why the corresponding Poisson algebra possesses decomposition into the direct sum of two Poisson subalgebras or by other words subspaces $(\widetilde{\mathfrak{g}}_A^{\pm})^*$ are Poisson.

3 Infinite-dimensional Hamiltonian systems via K-A admissible Lie algebras

In the previous section we have constructed infinite-dimensional Lie algebras $\widetilde{\mathfrak{g}}_A$ that have a decomposition into direct sum of two subalgebras and possess infinite set of invariants of coadjoint representation, i.e. admit so-called Kostant–Adler scheme. In the first subsection of this section we apply Kostant–Adler scheme to $\widetilde{\mathfrak{g}}_A$ in order to construct an infinite set of mutually commuting (with respect to the natural Lie–Poisson bracket) functions on the infinite-dimensional subalgebras $\widetilde{\mathfrak{g}}_A^{\pm}$. In the second subsection we will obtain zero-curvature type equations with $\widetilde{\mathfrak{g}}_A^{\pm}$-valued $U-V$ pairs.

3.1 Integrable Hamiltonian systems connected with algebras $\widetilde{\mathfrak{g}}_A^{\pm}$

Let $L^{\mp}(\lambda) \equiv \sum\limits_{i,j=1,n} L^{\mp}_{ij}(\lambda) X_{ji} = \sum\limits_{k\in\mathbb{Z}_\pm} \sum\limits_{i,j=1,n} l^{(k)}_{ij} \lambda^{-(k+1)} X_{ji}$ be the generic elements of the spaces $(\widetilde{\mathfrak{g}}_A^{\pm})^*$. Let us consider restriction of the invariant functions $\{I^m_k(L(\lambda))\}$ onto these subspaces. Note, that although Poisson subspaces $(\widetilde{\mathfrak{g}}_A^{\pm})^*$ are infinite-dimensional, all functions $\{I^m_k(L^{\pm}(\lambda))\}$ are polynomials, i.e. after restriction to $(\widetilde{\mathfrak{g}}_A^{\pm})^*$ no infinite sums appear in their explicit expressions. Corresponding Hamiltonian equations are written as:

$$\frac{\partial L^{\mp}_{ij}(\lambda)}{\partial t^m_k} = \{L^{\mp}_{ij}(\lambda), I^m_k(L^{\mp}(\lambda))\}. \tag{13}$$

The following important theorem holds true:

Theorem 1. (i) *Time flows defined by Hamiltonian equations* (13) *mutually commute.*

(ii) *Hamiltonian equations* (13) *are written in the "deformed" Lax form:*

$$\frac{\partial L^{\mp}(\lambda)}{\partial t^m_k} = A(\lambda) M^m_k(\lambda) L^{\mp}(\lambda) - L^{\mp}(\lambda) M^m_k(\lambda) A(\lambda), \tag{14}$$

where $M^m_k(\lambda) = \nabla I^m_k(L^{\mp}(\lambda)) \equiv \sum\limits_{s\in\mathbb{Z}_\pm} \sum\limits_{i,j=1}^{n} \dfrac{\partial I^m_k}{\partial l^{(s)}_{ij}} X^s_{ij}$ *is an algebra-valued gradient of* $I^m_k(L^{\mp}(\lambda))$.

(Proof of the theorem follows from the standard framework of the Kostant–Adler scheme [4].)

Remark 5. In this subsection we have obtained Hamiltonian systems of the Euler–Arnold type on the special infinite-dimensional Lie algebras possessing infinite number of the commuting integrals of motion. These Hamiltonian systems are "mechanical" because they are described by ordinary differential equations. Nevertheless we can consider our dynamical variables $l^{(p)}_{ij}$ to be functions of all time variables $t^{m\pm}_k$ and using the commutativity of all time flows obtain differential identities on functions $l^{(p)}_{ij}(t^{m\pm}_k)$ that coincide with the integrable equations in partial derivatives. For this purpose in the next subsection we will derive zero-curvature equations.

3.2 "Deformed" zero curvature equations

In this section we will obtain zero curvature-type equations as compatibility conditions for the set of the commutative Hamiltonian flows constructed in the previous section. The following theorem holds true:

Theorem 2. *Let infinite-dimensional Lie algebras $\widetilde{\mathfrak{g}}_A$, $\widetilde{\mathfrak{g}}_A^{\pm}$, their dual spaces and polynomial Hamiltonians $I_k^m(L^{\pm}(\lambda))$, $I_l^n(L^{\pm}(\lambda))$ on them be defined as in previous sections. Then algebra-valued gradients of these functions satisfy the "deformed" zero-curvature equations:*

$$\frac{\partial \nabla I_k^m(L^{\pm}(\lambda))}{\partial t_l^{n\pm}} - \frac{\partial \nabla I_l^n(L^{\pm}(\lambda))}{\partial t_k^{m\pm}} + [\nabla I_k^m(L^{\pm}(\lambda)), \nabla I_l^n(L^{\pm}(\lambda))]_{A(\lambda)} = 0. \tag{15}$$

(Proof follows from the commutativity of two time flows defined by equations (14).)

Remark 6. By means of other realizations of $\widetilde{\mathfrak{g}}_A$ deformed zero-curvature equations can be rewritten in the form of the standard zero-curvature equations, but in this case corresponding $U - V$ pairs will be more complicated as functions of the spectral parameter λ.

Theorem 2 provides us with an infinite number of $\widetilde{\mathfrak{g}}_A^{\pm}$-valued $U - V$ pairs that satisfy zero curvature-type equations. The latter are non-linear equations in the partial derivatives on the dynamical variables – matrix elements of the matrices $L^{\pm}(\lambda)$. In the terminology of [2] equations generated by the infinite set of $U-V$ pairs are called "integrable in the kinematic sense". In the next subsections we will consider the simplest examples of such integrable equations and their hierarchies.

Now let us explain the technique of obtaining integrable equation in partial derivatives starting from zero-curvature equations. Let us at first note that in the described approach no "space" variable x is *a priori* singled out: all times $t_k^{m\pm}$ are equivalent. Fixation of the "space" flow is equivalent to the fixation of integrable hierarchy. For this purpose one should fix a Hamiltonian that generate x-flow. For the case of integrable systems, connected with algebras $\widetilde{\mathfrak{g}}_A^{\pm}$, this choice yields fixation of dynamical variables. In more details, for the dynamical variables in this case serve the matrix elements of $\nabla I_k^m(L^{\mp}(\lambda))$, where the Hamiltonian $I_k^m(L^{\mp}(\lambda))$ is chosen to generate an x-flow. Using zero-curvature conditions one can express matrix elements of all other matrix gradients $\nabla I_l^n(L^{\mp}(\lambda))$ via these dynamical variables and their derivatives with respect to the "space" coordinate. Substituting these expressions back to zero-curvature condition we obtain the desired equation in partial derivatives on the matrix elements of $\nabla I_k^m(L^{\mp}(\lambda))$.

4 Matrix generalization of L-L hierarchy

In this section we will obtain integrable hierarchies of differential equations in partial derivatives, admitting a zero curvature type representation (15) with the values in $\widetilde{\mathfrak{g}}_A^{-}$.

Let us consider dual space $(\widetilde{\mathfrak{g}}_A^{-})^*$. Its generic element has the following form:

$$L^{+}(\lambda) = \sum_{k<0} \sum_{i,j=1,n} l_{ij}^{(k)} \lambda^{-(k+1)} X_{ji} = L^{(-1)} + \lambda L^{(-2)} + \lambda^2 L^{(-3)} + \lambda^3 L^{(-4)} + \cdots, \tag{16}$$

where $L^{(-k)} \equiv \sum\limits_{i,j=1,n} l_{ij}^{(-k)} X_{ji}$. Let us now calculate the Hamiltonians $I_k^m(L^{+}(\lambda))$. In order for the Hamiltonians $I_k^m(L^{+}(\lambda))$ to be polynomials we have to expand expression $A(\lambda)^{-1}$ in the power series in the neighborhood of zero:

$$A(\lambda)^{-1} = 1 + A\lambda + A^2\lambda^2 + \cdots .$$

From the results of the of the previous section it follows that the matrix gradients of Hamiltonians $I_k^m(L^{+}(\lambda))$ satisfy "deformed" zero-curvature equation (15). We will be interested in the two simplest Hamiltonians of the set $I_k^2(L^{+}(\lambda))$. By the direct calculations we obtain for them the following expressions:

$$I_0^2(L^{+}(\lambda)) = 1/2\,\mathrm{Tr}\,\big(L^{(-1)}\big)^2, \qquad I_1^2(L^{+}(\lambda)) = \mathrm{Tr}\,\big(A\big(L^{(-1)}\big)^2\big) + \mathrm{Tr}\,\big(L^{(-1)}L^{(-2)}\big). \tag{17}$$

The corresponding matrix gradients are:

$$\nabla I_0^2(L^+(\lambda)) = L^{(-1)}\lambda^{-1},$$
$$\nabla I_1^2(L^+(\lambda)) = L^{(-1)}\lambda^{-2} + \big(\big(AL^{(-1)} + L^{(-1)}A\big) + L^{(-2)}\big)\lambda^{-1}. \tag{18}$$

As it follows from the said above, in order to fix an integrable hierarchy we should choose the Hamiltonian that generates an x-flow. In order to choose the simplest hierarchy we have to choose the Hamiltonian with the simplest matrix gradient. That is why we will take for such Hamiltonian the function $I_0^2(L^+(\lambda))$, putting $t_0^{2+} \equiv x$; $\nabla I_0^2(L^+(\lambda)) \equiv U(x,\lambda)$, $\nabla I_k^m(L^+(\lambda)) \equiv V_k^m(x,\lambda)$, $m,k>0$ as the basic $U-V$ pairs that generate this hierarchy. In this case the role of the dynamical variables is played by the matrix elements of the matrix $L^{(-1)}$.

Let us obtain an explicit form of the simplest equation of the hierarchy described above. For this purpose we have to choose a Hamiltonian that generates the "time" flow in the simplest possible way. We take for such Hamiltonian the function $I_1^2(L^+(\lambda))$, i.e. $t \equiv t_1^{2+}$. In the result we obtain that zero-curvature equation (15) is equivalent the following λ-independent equations:

$$\frac{\partial S}{\partial t} - \frac{\partial M}{\partial x} = [S,M]_A, \tag{19a}$$
$$\frac{\partial S}{\partial x} = [S,M], \tag{19b}$$

where $S \equiv L^{(-1)}$, $M \equiv L^{(-2)} + \big(AL^{(-1)} + L^{(-1)}A\big)$.

In order to obtain equations in partial derivatives for the dynamical variables – matrix elements of the matrix S, it is necessary to solve equation (19b), i.e. to express M via S and S_x and then substitute this expression into equation (19a). We will illustrate this procedure by the simplest, but most interesting example.

Let us now consider the case of the higher-rank algebras. In this case in order to solve equation (19b), i.e. in order to obtain one matrix equation in partial derivatives instead of two equations (19) it is necessary to impose additional constraints on the matrix S. Let $\mathfrak{g} = gl(n)$, $so(2n)$ or $sp(n)$ and for the matrix S to satisfy G-invariant constraint:

$$S^2 = \frac{\alpha}{4}\mathbf{1}. \tag{20}$$

This constraint means that S belongs to the degenerated coadjoint orbits of G of the following type: $Gl(n)/GL(p)\times GL(q)$, $SO(2n)/Gl(n)$ or $SP(n)/GL(n)$. On this orbits we may solve equation (19b) in the following way:

$$M = \frac{1}{\alpha}\left[S, \frac{\partial S}{\partial x}\right] + M', \quad \text{where} \quad M' \in \ker \mathrm{ad}_L.$$

Ambiguity connected with the existence of $\ker \mathrm{ad}_S$ is removed by the requirement that the constraint (20) is consistent with equations (19). It is easy to show, that for $M' = 1/2(AS+SA)$ the constraint (20) is consistent with equation (19), i.e. $\left(\frac{\partial S^2}{\partial x}\right)|_{S^2=\frac{\alpha}{4}\mathbf{1}} = 0$ and $\left(\frac{\partial S^2}{\partial t}\right)_{S^2=\frac{\alpha}{4}\mathbf{1}} = 0$. Resulting matrix equation acquires the following form:

$$\frac{\partial S}{\partial t} = \frac{1}{\alpha}\left[S, \frac{\partial^2 S}{\partial x^2}\right] + \frac{1}{2}\frac{\partial}{\partial x}(AS+SA) + \frac{1}{\alpha}\left[S, \left[S, \frac{\partial S}{\partial x}\right]\right]_A + \frac{1}{2}[S, AS+SA]_A, \tag{21}$$

where $S^2 = \frac{\alpha}{4}\mathbf{1}$. In the case $\alpha = 1$ we obtain equation (3). In the limit $A \to 0$ this equation goes to standard higher-rank generalization of Heisenberg magnet equation [13, 14].

Let us now show, that this equation is direct matrix generalization of the well-known Landau–Lifshitz equation. For this purpose we consider the following example:

Example. Let $\mathfrak{g} = so(4)$. and matrix S satisfy G-invariant constraint (20) with $\alpha = -1$. In this case it is easy to show that matrix S can be written in the form:

$$S = \begin{pmatrix} 0 & -s_3 & s_2 & s_1 \\ s_3 & 0 & -s_1 & s_2 \\ -s_2 & s_1 & 0 & s_3 \\ -s_1 & -s_2 & -s_3 & 0 \end{pmatrix},$$

where three-component vector $\vec{s}$ belongs to $S^2 = SO(3)/SO(2) = SO(4)/GL(2)$:

$$\langle \vec{s}, \vec{s} \rangle = 1/4.$$

As a result of the special form of matrix S the second and third item of the right-hand side of equation (21) are eliminated, and it could be rewritten as equation for the vector $\vec{s}$ as follows:

$$\frac{\partial \vec{s}}{\partial t} = \left[\vec{s} \times \frac{\partial^2 \vec{s}}{\partial x^2}\right] + \left[\vec{s} \times J(\vec{s})\right].$$

It coincides with the well-known Landau–Lifshiz equation (here $J = 1/4\big(\widehat{A}^2 - 2a_4\widehat{A}\big)$, $\widehat{A} = \operatorname{diag}(a_1, a_2, a_3)$).

Acknowledgements

The author is grateful to P.I. Holod for many discussions. The research was partially supported by INTAS YS Fellowship 03-55-2233.

[1] Zaharov V. and Shabat A., Integration of nonlinear equations of mathematical physics using the method of inverse scattering problem, *Funct. Anal. and Appl.*, 1979, V.13, N 3, 13–21.

[2] Tahtadjan L. and Faddejev L., Hamiltonian approach in the theory of solitons, Moscow, Nauka, 1986.

[3] Kostant B., The solution of the generalized Toda lattice and the representation theory, *Adv. Math.*, 1979, V.34, 195–338.

[4] Reyman A.G. and Semenov-Tian-Shansky M.A., Group theoretical methods in the theory of finite-dimensional integrable systems, *Current Problems in Mathematics. Fundamental Directions*, Vol. 6, Moscow, VINITI, 1989, 145–147.

[5] Holod P.I., Hamiltonian systems connected with the anisotropic affine Lie algebras and higher Landau–Lifschits equations, *Doklady Acad. Nauk Ukr. SSR, Ser. A*, 1984, N 5, 6–8.

[6] Flaschka H., Newell A. and Ratiu T., Kac–Moody Lie algebras and soliton equations, *Physica D*, 1983, V.9, 303–323, 324–332.

[7] Skrypnyk T., Lie algebras on hyperelliptic curves and finite-dimensional integrable system, in Proceedings of the XXIII International Colloquium "Group Theoretical Methods in Physics" (1–5 August, 2000, Dubna, Russia), *Yadernaya Physica*, 2002, V.65, N 6, 1141–1145.

[8] Skrypnyk T., Quasi-graded Lie algebras on hyperelliptic curves and classical integrable systems, *J. Math. Phys.*, 2001, V.48, N 9, 4570–4582.

[9] Skrypnyk T., Quasigraded deformations of loop algebras and classical integrable systems, *Czechoslovak J. Phys.*, 2002, V.52, N 11, 1283–1288.

[10] Skrypnyk T., Quasigraded Lie algebras and hierarchies of integrable equations, *Czechoslovak J. Phys.*, 2002, V.52, N 11, 1283–1288.

[11] Skrypnyk T., Deformations of loop algebras and integrable systems II: hierarchies of integrable equations, *J. Math. Phys.*, submitted.

[12] Cantor I.L. and Persits D.E., Closed stacks of Poisson brackets, in Proceedings of the IX USSR Conference in Geometry, Kishinev, Shtinitsa, 1988, 141.

[13] Oh P. and Park Q.H., More on generalized Heizenberg ferromagnet models, hep-th/9604031.

[14] Golubchik I. and Sokolov V., Generalized Heisenberg equations on $\mathbb{Z}$-graded Lie algebras, *Theor. Math. Phys.*, 1999, V.120, N 2, 248–255.

[15] Sklyanin E., On complete integrability of Landau–Lifshiz equation, Preprint LOMI E-3, 1979.