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Alternative Gauge Procedure with Fields of Various Ranks

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Making a global phase symmetry local usually involves the introduction of a rank 1, vector field in the definition of the covariant derivative. In this paper we show how a symmetry can be gauged using fields of various ranks in the definition of the covariant derivative. Some of these higher rank gauge theories share similarities with general relativity in that the covariant derivative has terms which are derivatives of some more fundamental quantity. Most of the Lagrangians that we find under these higher rank gauge theories lead to nonrenormalizable quantum theories which is also similar to general relativity.

1 Standard gauge procedure with a rank 1 field

Turning global symmetries into local ones (i.e. gauging a symmetry) is an important feature of most modern field theories. For example, Maxwell's theory can be derived by gauging an Abelian $U(1)$ phase symmetry. In this section the standard gauge procedure is summarized using a complex scalar matter field φ. Throughout the paper the complex scalar field φ will be used to illustrate the higher rank gauge procedures. The same procedure applies equally to other matter field (e.g. a spinor or vector fields). The Lagrange density for φ is

$$\mathcal{L}_{\text{scalar}} = (\partial_\mu \varphi)^* (\partial^\mu \varphi) + \cdots . \tag{1}$$

The ellipses leave off non-derivative terms (e.g. mass $m^2 \varphi^* \varphi$ or self-interaction terms $\lambda(\varphi^*\varphi)^2$). This Lagrange density satisfies the global phase symmetry

$$\varphi(x) \to e^{-ig\Lambda}\varphi(x), \qquad \varphi^*(x) \to e^{ig\Lambda}\varphi^*(x), \tag{2}$$

where g is the coupling and Λ is a constant phase. This phase symmetry can be made local ($g\Lambda \to g\Lambda(x)$) by replacing the ordinary derivative with the covariant derivative

$$\partial_\mu \to D_\mu^{[1]} \equiv \partial_\mu - ig\sigma_{\mu\nu}A^\nu. \tag{3}$$

Throughout the paper the bracketed superscript will indicate the rank of the gauge field. We have introduced a rank-2 operator $\sigma_{\mu\nu}$ since this will allow us to give the general gauging procedure for gauge fields of other ranks. The newly introduced four-vector gauge field A_μ is required to transform as

$$A_\mu \to A_\mu - \Gamma_\mu. \tag{4}$$

In order for the local symmetry version of (2) to be valid Γ_μ, $\sigma_{\mu\nu}$ and $\Lambda(x)$ must satisfy

$$\sigma_{\mu\nu}\Gamma^\nu - \partial_\mu\Lambda = 0. \tag{5}$$

The Maxwell theory case corresponds to choosing

$$\sigma_{\mu\nu} = \eta_{\mu\nu}, \qquad \Gamma_\mu = \partial_\mu\Lambda. \tag{6}$$

However, it is also possible to make the following choice

$$\sigma_{\mu\nu} = \partial_\mu \partial_\nu, \qquad \Lambda = \partial_\mu \Gamma^\mu + f(x), \tag{7}$$

where $f(x)$ is a divergenceless function, $\partial_\mu f(x) = 0$. An interesting difference between the two options is that in (6) Γ_μ is defined in terms of the local phase factor $\Lambda(x)$, while for (7) $\Lambda(x)$ is defined in terms of Γ_μ.

Next one constructs a "kinetic" energy term for A_μ by introducing the field strength tensor

$$F^{[1]}_{\mu\nu} = A_1 \partial_\mu A_\nu + B_1 \partial_\nu A_\mu. \tag{8}$$

This is invariant under (4), (6) if $A_1 + B_1 = 0$, that is to say, if $F^{[1]}_{\mu\nu}$ is anti-symmetric between μ and ν. The reason for writing $F^{[1]}_{\mu\nu}$ as in (8) is that it makes the connection with the field strength tensors of the higher rank cases more transparent. $F^{[1]}_{\mu\nu}$ is invariant in the case of (6), however for the case of (7) it is not since Γ_μ is arbitrary. Because of this arbitrariness one can always make $A_\mu \to 0$ and $F^{[1]}_{\mu\nu} = 0$ by taking $\Gamma_\mu = A_\mu$. If we take case (6) in the construction of $F^{[1]}_{\mu\nu}$ the following Lagrange density

$$\mathcal{L}'_{\text{scalar}} = (D^{[1]}_\mu \varphi)^* (D^{[1]\mu} \varphi) - \tfrac{1}{4} F^{[1]}_{\mu\nu} F^{[1]\mu\nu} + \cdots, \tag{9}$$

is invariant under the combined transformations of equations (4), (6) and the local version of equation (2).

The gravitational interaction follows a similar, although not exactly identical path. One can take the global space-time symmetries of special relativity and make them local [1] to arrive at a theory of the gravitational interaction. Here one again replaces ordinary derivatives with covariant derivatives. For example, the covariant derivative of a vector field V_ν is

$$\partial_\mu V_\nu \to \partial_\mu V_\nu + \Gamma^\alpha_{\mu\nu} V_\alpha \qquad \text{with} \qquad \Gamma^\alpha_{\mu\nu} = \tfrac{1}{2} g^{\alpha\sigma} (\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}) \tag{10}$$

Unlike the gauge field in equation (3), $\Gamma^\alpha_{\mu\nu}$ is not fundamental, but is defined in terms of the first derivatives of the metric tensor $g_{\mu\nu}$.

2 Gauge procedure with a rank 0 field

Here we present the generalized gauge procedure for a rank 0 field. We begin with the Lagrange density of a complex scalar matter field of equation (1) and a local phase transformation (i.e. (2) with $\Lambda \to \Lambda(x)$.) The space-time dependence of $\Lambda(x)$ means that the derivative of φ and φ^* are no longer invariant under (2), but become $\partial_\mu \varphi \to \partial_\mu \varphi - ig(\partial_\mu \Lambda)\varphi$ and $\partial_\mu \varphi^* \to \partial_\mu \varphi^* + ig(\partial_\mu \Lambda)\varphi^*$. As in the case of the gauging procedure with a vector field we want to find a generalization of the derivative operator ∂_μ, which is invariant under the local version of (2). We define this generalized rank 0 derivative operator as

$$D^{[0]}_\mu \equiv \partial_\mu - ig\partial_\mu \Phi. \tag{11}$$

Φ is real, scalar gauge field which is required to undergo the transformation

$$\Phi(x) \to \Phi(x) - \Lambda(x). \tag{12}$$

These transformations of the scalar field Φ are similar to the toy model considered in [4]. With replacement of the ordinary derivative in equation (1) with $D^{[0]}_\mu$ the Lagrangian of equation (1) becomes invariant under the local versions of (2), (12). As in the non-standard rank 1 case of (7) there is no kinetic term, since from the transformation of (12) it is always possible to take $\Phi = 0$ by choosing $\Lambda = \Phi$. Thus the Lagrangian

$$\mathcal{L}_{\text{scalar}} = (D^{[0]}_\mu \varphi)^* (D^{\mu [0]} \varphi) = (\partial_\mu \varphi^* + ig\varphi^* \partial_\mu \Phi)(\partial^\mu \varphi - ig\varphi \partial^\mu \Phi) \tag{13}$$

is invariant under (2), (12). This Lagrangian has no kinetic term for Φ. In contrast to the standard covariant derivative $\partial_\mu - ieA_\mu$, the covariant derivative of (11) involves the derivative of the fundamental gauge field Φ. This can be compared to the covariant derivative of general relativity which involves derivatives of a more fundamental quantity (the metric tensor $g_{\mu\nu}$).

One can apply this rank 0 procedure starting with matter fields other than a complex scalar field φ. In [5] this was done starting with a vector field, and this could be interpreted as a gauging of the electromagnetic dual symmetry [6]. In [7] a related localizing of the Schwarz–Sen [8] dual symmetry was given. The idea of having a scalar gauge field has also been considered by other authors. In [9] a unified version of the Standard Model was constructed via the introduction of a generalized covariant derivative which involved *both* vector and scalar gauge fields.

3 Classification of Gauge procedures

At this stage we have encountered three different categories of gauging procedures:

• **Trivial Case:** This is illustrated by the rank 0 case of (12). The gauge field transformation of (12) allows one to transform away the gauge field by taking $\Phi = \Lambda$. This case can be seen as a special case of the standard gauge procedure with the association $A_\mu \propto \partial_\mu \Phi$. This is a pure gauge case since for such an A_μ one finds $F^{[1]}_{\mu\nu} = 0$. For this trivial case the phase factor and gauge transformation function are related without the need of a derivative operator.

• **Semi-Trivial Case:** This is illustrated by the rank 1 case of (7). Here the transformation function of the gauge field Γ_μ is arbitrarily given. By choosing $A_\mu = \Gamma_\mu$ it is always possible to transform away the gauge field, making it non-dynamical. Both this case and the trivial case are marked by having covariant derivatives of the form

$$\partial_\mu - ig\partial_\mu (\text{Scalar}), \tag{14}$$

where Scalar is some scalar quantity. In contrast to the trivial case in the semi-trivial case the phase factor and gauge transformation function are related using derivative operator(s).

• **Non-Trivial Case:** This is illustrated by the rank 1 Maxwell Theory case of (6). In contrast to the previous semi-trivial case the phase factor Λ is arbitrary while the gauge transformation function Γ_μ takes a restricted form in terms of the phase function. The gauge field is dynamical and the covariant derivative does not take the form of (14).

In the following sections we show that for the abstract phase symmetries it is possible to formulate a gauge procedure with fields of various ranks as gauge fields. The procedure will employ generalizations of (3), (4) of the form

$$D_\mu = \partial_\mu - ig\sigma_{\mu_1\mu_2\cdots\mu_{n+1}} A^{\mu_1\mu_2\cdots\mu_n}, \qquad A^{\mu_1\mu_2\cdots\mu_n} \to A^{\mu_1\mu_2\cdots\mu_n} - \Gamma^{\mu_1\mu_2\cdots\mu_n}. \tag{15}$$

As with general relativity the covariant derivative of this generalized gauge procedure will have connections, $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}} A^{\mu_1\mu_2\cdots\mu_n}$, which are defined in terms of the derivative of a more fundamental object. Many of the theories obtained from this generalized gauging will have dimensionful coupling constants and lead to nonrenormalizable theories as is also the case with general relativity. Here we do not attempt to give a phenomenological application of the field theories that result from the generalized gauging procedure. We simply demonstrate an alternative, general way in which a global phase symmetry can be made local. This may give some insight into a connection between the way in which one usually gauges abstract phase symmetries via a fundamental vector field and the way in which space-time symmetries are made local. There has been previous work on higher rank (i.e. higher spin) gauge fields [2, 3]. The differences between the present work and [2, 3] are discussed in the concluding section.

4 Gauge procedure with a rank 2 field

In this section we will gauge the local version of the symmetry of equation (2) for the Lagrange density in equation (1) using a rank 2 gauge field. We define a covariant derivative as

$$D^{[2]}_\mu \equiv \partial_\mu - ig\sigma_{\mu\nu\rho}A^{\nu\rho}, \tag{16}$$

where we have introduced a rank 3 operator $\sigma_{\mu\nu\rho}$, and a rank 2 gauge field $A_{\nu\rho}$. Here we consider the case when the gauge field indices are symmetric. The antisymmetric case will be considered in a longer work [10].

For symmetric $A_{\nu\rho}$ ($A_{\nu\rho} = A_{\rho\nu}$) the operator $\sigma_{\mu\nu\rho}$ has the partial symmetry $\sigma_{\mu\nu\rho} = \sigma_{\mu\rho\nu}$. Constructing $\sigma_{\mu\nu\rho}$ from $\eta_{\mu\nu}$ and ∂_μ, we can write a general form as

$$\sigma_{\mu\nu\rho} = \tfrac{1}{2}a_2(\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu) + b_2\eta_{\nu\rho}\partial_\mu + c_2\partial_\mu\partial_\nu\partial_\rho, \tag{17}$$

where a_2, b_2, c_2 are constants with the subscript indicating the rank of $A_{\mu\nu}$. The last term in (17) has a greater symmetry (it is symmetric in all three indices) than required. In conjunction with (2) we require that $A_{\mu\nu}$ transforms as

$$A_{\mu\nu} \to A_{\mu\nu} - \Gamma_{\mu\nu}. \tag{18}$$

If the rank 2 function $\Gamma_{\mu\nu}$ and Λ satisfy

$$\sigma_{\mu\nu\rho}\Gamma^{\nu\rho} - \partial_\mu\Lambda = 0, \tag{19}$$

then the Lagrange density $\mathcal{L}_{\text{scalar}} = (D^{[2]}_\mu\varphi)^*(D^{[2]\,\mu}\varphi) + \cdots$ will be invariant under the combined (2), (18). We will consider three special cases.

1. $[\boldsymbol{a_2 = 1,\ b_2 = c_2 = 0}]$. The covariant derivative and the condition in equation (19) become

$$D^{[2]}_\mu = \partial_\mu - ig\partial_\nu A^\nu_\mu, \qquad \partial_\nu\Gamma^\nu_\mu - \partial_\mu\Lambda = 0. \tag{20}$$

The symmetry of $A_{\mu\nu}$ was used in finding $D^{[2]}_\mu$. A general solution to the last equation in (20) is

$$\Gamma_{\mu\nu} = \eta_{\mu\nu}\Lambda + h_{\mu\nu}, \tag{21}$$

where it is required that $\partial^\nu h_{\mu\nu} = 0$. An example of a solution is

$$\Gamma_{\mu\nu} = \partial_\mu\partial_\nu f(x) + \eta_{\mu\nu}g(x), \qquad \Lambda = \partial_\nu\partial^\nu f(x) + g(x) + h(x), \tag{22}$$

where $f(x)$, $g(x)$ and $h(x)$ are arbitrary scalar functions. The function $h(x)$ must satisfy $\partial_\mu h(x) = 0$, which corresponds to the choice $h_{\mu\nu} = \partial_\mu\partial_\nu f(x) - \eta_{\mu\nu}(\partial_\rho\partial^\rho f(x) + h(x))$ in (21). The explicit gauge field transformation is

$$A_{\mu\nu} \to A_{\mu\nu} - \partial_\mu\partial_\nu f(x) - \eta_{\mu\nu}g(x). \tag{23}$$

2. $[\boldsymbol{b_2 = 1,\ a_2 = c_2 = 0}]$. The covariant derivative and the condition in equation (19) become

$$D^{[2]}_\mu = \partial_\mu - ig\partial_\mu A^\nu_\nu, \qquad \partial_\mu\Gamma^\nu_\nu - \partial_\mu\Lambda = 0. \tag{24}$$

The last equation in (24) can be satisfied by taking

$$\Lambda = \Gamma^\nu_\nu + f(x), \tag{25}$$

with $\partial_\mu f(x) = 0$. The gauge transformation function $\Gamma_{\mu\nu}$, and phase factor Λ are arbitrary, and are related without a derivative operator so this is a trivial case. Since $\Gamma_{\mu\nu}$ is arbitrary it should always be possible to transform the gauge field away via $A_{\mu\nu} \to A_{\mu\nu} - \Gamma_{\mu\nu}$ with $A_{\mu\nu} = \Gamma_{\mu\nu}$.

3. $[\boldsymbol{c_2 = 1,\ a_2 = b_2 = 0}]$. The covariant derivative and the condition in equation (19) become

$$D_\mu^{[2]} = \partial_\mu - ig\partial_\mu\partial_\nu\partial_\rho A^{\nu\rho}, \qquad \partial_\mu\partial_\nu\partial_\rho\Gamma^{\nu\rho} - \partial_\mu\Lambda = 0. \tag{26}$$

The last equation above can be satisfied by taking

$$\Lambda = \partial_\nu\partial_\rho\Gamma^{\nu\rho} + f(x), \tag{27}$$

with $\partial_\mu f(x) = 0$. This is a semi-trivial case since the gauge function is arbitrary, but the relationship between it and the phase factor involves the derivative operator.

For cases 2 and 3 the rank 2 gauge field is arbitrary, and any form for $\Gamma_{\mu\nu}$ works. For case 1 the specific form given in (22) is necessary. This difference can be traced to the different relationships between $\Gamma_{\mu\nu}$ and Λ given in the second equations in (20), (24), (26). Equations (24) and (26) involve the same index on the derivative, while for equation (20) the indices on the derivative operator are different. This is connected with the fact that cases 2 and 3 are trivial and semi-trivial as previously discussed, and have covariant derivatives of the form of equation (14).

Next we want to add a kinetic term involving $A_{\mu\nu}$ alone. Cases 2 and 3 are trivial and semi-trivial, so that $\Gamma_{\mu\nu}$ has a completely arbitrary form. In these cases $A_{\mu\nu}$ is not dynamical since it is possible to transform it away by taking $A_{\mu\nu} = \Gamma_{\mu\nu}$. Thus we only consider case 1 in constructing of an invariant field strength tensor. Also we will work with the special example given in (22) with $g(x) = 0$. Under these conditions the following rank 3 object

$$F^{[2]}_{\mu\nu\rho} = A_2\partial_\mu A_{\nu\rho} + B_2\partial_\nu A_{\mu\rho} + C_2\partial_\rho A_{\mu\nu}, \tag{28}$$

is invariant under the gauge field transformation if the constants obey $A_2 + B_2 + C_2 = 0$. $F^{[2]}_{\mu\nu\rho}$ is neither symmetric nor antisymmetric. Its defining feature is the permutation of the indices which generalizes the form of the rank 2 field strength tensor given in (8). The following Lagrangian is invariant under the local phase and gauge field transformations

$$\mathcal{L}_{\text{scalar}} = (D_\mu^{[2]}\varphi)^*(D^{\mu[2]}\varphi) + KF^{[2]}_{\mu\nu\rho}F^{[2]\mu\nu\rho} + \cdots, \tag{29}$$

where K is a constant. The kinetic energy term involving only the gauge field $A_{\mu\nu}$ is more complex than the rank 1 kinetic term in equation (9).

For all three cases (20), (24) and (26) the coupling g is dimensionful implying that the Lagrangian in equation (29) is nonrenormalizable. The mass dimension of g can be determined from equation (16). The derivative operator has mass dimension $+1$ and in the usual way the gauge field $A_{\mu\nu}$ is taken to have mass dimension $+1$. Thus for case 1 and 2 from (20) and (24) the coupling g has mass dimension -1, while for case 3 from (26) g has mass dimension -3.

5 Gauge procedure with a rank 3 field

In this section we will gauge the local version of the symmetry of equation (2) for the Lagrange density in equation (1) using a rank 3 gauge field. As n becomes large the number of possible terms in the definition of $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}$ and in the construction of a kinetic term for the gauge field becomes larger and more complex. We define a covariant derivative as

$$D_\mu^{[3]} \equiv \partial_\mu - ig\sigma_{\mu\nu\rho\tau}A^{\nu\rho\tau}, \tag{30}$$

where we have introduced a rank 4 operator $\sigma_{\mu\nu\rho\tau}$, and a rank 3 gauge field $A_{\nu\rho\tau}$. In this paper we consider only the rank 3 gauge field with complete symmetry in its indices. Antisymmetric and mixed symmetric cases will be considered in a longer work in preparation [10]. For the totally symmetric gauge field the operator $\sigma_{\mu\nu\rho\tau}$ is symmetric in the last three indices.

Using the operators $\eta_{\mu\nu}$ and ∂_μ we can write a general form for $\sigma_{\mu\nu\rho\tau}$

$$\begin{aligned}\sigma_{\mu\nu\rho\tau} &= \tfrac{1}{3}a_3(\eta_{\mu\nu}\eta_{\rho\tau}+\eta_{\mu\rho}\eta_{\nu\tau}+\eta_{\mu\tau}\eta_{\rho\nu})+\tfrac{1}{3}b_3(\eta_{\mu\nu}\partial_\rho\partial_\tau+\eta_{\mu\rho}\partial_\nu\partial_\tau+\eta_{\mu\tau}\partial_\rho\partial_\nu)\\ &\quad+\tfrac{1}{3}c_3(\eta_{\rho\nu}\partial_\mu\partial_\tau+\eta_{\tau\rho}\partial_\mu\partial_\nu+\eta_{\nu\tau}\partial_\mu\partial_\rho)+d_3\partial_\mu\partial_\nu\partial_\rho\partial_\tau,\end{aligned}\tag{31}$$

where a_3, b_3, c_3, d_3 are constants with the subscript indicating the rank of $A_{\mu\nu\tau}$. The last term in (5) has a greater index symmetry than required: it is symmetric in all four indices. In conjunction with the transformation of equation (2) we require that $A_{\mu\nu\tau}$ transforms as

$$A_{\mu\nu\tau}\to A_{\mu\nu\tau}-\Gamma_{\mu\nu\tau}.\tag{32}$$

If the rank 3 gauge function $\Gamma_{\mu\nu\tau}$ and Λ satisfy

$$\sigma_{\mu\nu\rho\tau}\Gamma^{\nu\rho\tau}-\partial_\mu\Lambda=0,\tag{33}$$

then the Lagrange density $\mathcal{L}_{\text{scalar}}=(D^{[3]}_\mu\varphi)^*(D^{[3]\,\mu}\varphi)+\cdots$ will be invariant under the combined transformation (2), (32). As in the previous section we will consider four special cases.

1. $[\boldsymbol{a_3=1,\ b_3=c_3=d_3=0}]$. The covariant derivative and the condition in equation (33) become

$$D^{[3]}_\mu=\partial_\mu-igA_{\mu\rho}{}^{\rho},\qquad \Gamma_{\mu\rho}{}^{\rho}-\partial_\mu\Lambda=0,\tag{34}$$

where the symmetry of $A_{\mu\nu\rho}$ was used in determining $D^{[3]}_\mu$. An example of a solution to the last equation of (34) is

$$\Gamma_{\mu\nu\rho}=\partial_\mu\partial_\nu\partial_\rho f(x)+\tfrac{1}{6}(\eta_{\mu\nu}\partial_\rho+\eta_{\nu\rho}\partial_\mu+\eta_{\rho\mu}\partial_\nu)g(x),\qquad \Lambda=\partial_\rho\partial^\rho f(x)+g(x)+h(x),\tag{35}$$

where $f(x)$, $g(x)$, $h(x)$ are arbitrary functions with $\partial_\mu h(x)=0$. Equation (35) appears as a generalization of the rank 2 example solution of (22). The explicit gauge field transformation is

$$A_{\mu\nu\rho}\to A_{\mu\nu\rho}-\partial_\mu\partial_\nu\partial_\rho f(x)-\tfrac{1}{6}(\eta_{\mu\nu}\partial_\rho+\eta_{\nu\rho}\partial_\mu+\eta_{\rho\mu}\partial_\nu)g(x).\tag{36}$$

This gauge field transformation appears as a generalized version of the rank 2 gauge field case given in (23). This case is closer to the standard vector gauge procedure of (6) than any of the previous rank 2 cases, since the covariant derivative in (34) only involves the gauge field and not derivatives of the gauge field.

2. $[\boldsymbol{b_3=1,\ a_3=c_3=d_3=0}]$. The covariant derivative and the condition in equation (33) become

$$D^{[3]}_\mu=\partial_\mu-ig\partial^\rho\partial^\tau A_{\mu\rho\tau},\qquad \partial^\nu\partial^\rho\Gamma_{\mu\nu\rho}-\partial_\mu\Lambda=0.\tag{37}$$

The last equation in (37) can be satisfied by taking the same $\Gamma_{\mu\nu\rho}$ as in (35), and a phase factor of the form

$$\Lambda=\partial_\rho\partial^\rho\partial_\tau\partial^\tau f(x)+\tfrac{1}{2}\partial_\tau\partial^\tau g(x)+h(x),\tag{38}$$

where $f(x)$, $g(x)$, $h(x)$ are arbitrary functions with $\partial_\mu h(x)=0$. The gauge transformation of $A_{\mu\nu\rho}$ is identical to (36).

3. $[c_3 = 1,\ a_3 = b_3 = d_3 = 0]$. The covariant derivative and the condition in equation (33) become

$$D^{[3]}_\mu = \partial_\mu - ig\partial_\mu\partial^\tau A^\rho_{\rho\tau}, \qquad \partial_\mu\partial^\tau\Gamma^\rho_{\rho\tau} - \partial_\mu\Lambda = 0. \tag{39}$$

The last equation above can be satisfied by taking

$$\Lambda = \partial^\tau\Gamma^\rho_{\rho\tau} + f, \tag{40}$$

where f is gradient-less. Since the covariant derivative is of the form (14) and the gauge function and phase factor are related using derivatives this case is characterized as semi-trivial. $\mathcal{L}_{\rm scalar} = (D^{[3]}_\mu\varphi)^*(D^{[3]\ \mu}\varphi) + \cdots$ is invariant under the phase transformation of $\varphi(x)$ and an arbitrary gauge transformation $A_{\mu\nu\rho} \to A_{\mu\nu\rho} - \Gamma_{\mu\nu\rho}$.

4. $[d_3 = 1,\ a_3 = b_3 = c_3 = 0]$. The covariant derivative and the condition in equation (33) become

$$D^{[3]}_\mu = \partial_\mu - ig\partial_\mu\partial_\nu\partial_\rho\partial_\tau A^{\nu\rho\tau}, \qquad \partial_\mu\partial_\nu\partial_\rho\partial_\tau\Gamma^{\nu\rho\tau} - \partial_\mu\Lambda = 0. \tag{41}$$

This last equation above can be satisfied by taking

$$\Lambda = \partial_\nu\partial_\rho\partial_\tau\Gamma^{\nu\rho\tau} + f, \tag{42}$$

where f is gradient-less. As for the preceding case this is also semi-trivial.

As before, we want to add a kinetic term involving $A_{\mu\nu\rho}$ alone. Cases 1 and 2 are both non-trivial and have the same transformation of $A_{\mu\nu\rho}$ given by (36). Thus both cases will have the same pure gauge invariant field strength tensors. Cases 3 and 4 are both semi-trivial and have arbitrary forms for the gauge transformation. Thus we only consider cases 1 and 2 in constructing the field strength tensor. Furthermore, as in the rank 2 fields of the previous section, we will work with the special case $g(x) = 0$ in (36) and (38). Thus, the rank 4 object

$$F^{[3]}_{\mu\nu\rho\tau} = A_3\partial_\mu A_{\nu\rho\tau} + B_3\partial_\nu A_{\rho\tau\mu} + C_3\partial_\rho A_{\tau\mu\nu} + D_3\partial_\tau A_{\mu\nu\rho}, \tag{43}$$

is invariant under (36) and (38) (given $g(x) = 0$) if the constants obey $A_3 + B_3 + C_3 + D_3 = 0$. The common feature between this invariant field strength tensor and the rank 1 and 2 cases of (8) and (28) is the permutation of indices. Thus the following Lagrangian is invariant under the local phase transformation and the gauge field transformation equation (36).

$$\mathcal{L}_{\rm scalar} = (D^{[3]}_\mu\varphi)^*(D^{\mu[3]}\varphi) + KF^{[3]}_{\mu\nu\rho\tau}F^{[3]\mu\nu\rho\tau} + \cdots, \qquad K = \text{const.} \tag{44}$$

For the symmetric rank 3 gauge field the dimension of the coupling, and therefore whether the theory is renormalizable or not is different for each of the four cases above. For case 1, there are no derivative terms which appear in the second term in the covariant derivative in (34). Taking $A_{\mu\nu\rho}$ to have the usual mass dimension of +1, then g is dimensionless. This case may lead to a renormalizable theory. For cases 2 and 3 two derivatives appear in the second term of the covariant derivative in (37), (39), which implies that g should have mass dimension -2. Finally, for case 4 four derivatives appear in the second term of the covariant derivative in (41), which gives g a mass dimension of -4.

6 Summary and conclusion

Here we summarize the overall structure of the various gauging procedures presented here.

1. Starting from some initial matter field a local version of the phase symmetry in equation (2) is imposed. A covariant derivative of the form of the first equation in (15) and a rank n gauge field which transforms like the second equation in (15) are introduced.

2. The definition of the covariant derivative in (15) involves the introduction of a rank $n+1$ operator $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}$, which is constructed from ∂_μ and $\eta_{\mu\nu}$. The freedom in the construction of $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}$ can be seen from (6), (7), (17), and (5).

3. There were three different categories of gauge procedures: trivial, semi-trivial, or non-trivial. The trivial and semi-trivial cases involve introduction of an arbitrary gauge function $\Gamma_{\mu_1\mu_2\cdots}$ in terms of which the phase factor Λ was defined either without (trivial case) or with (semi-trivial) use of the derivative operator. The non-trivial case involves the introduction of an arbitrary phase factor Λ in terms of which the gauge function was defined. For the trivial and semi-trivial cases the gauge field could always be transformed away. For the non-trivial case one can construct an invariant field strength tensor and a kinetic energy term.

4. For the rank n symmetric gauge field one can define a rank $n+1$ field strength tensor (as in equations (8), (28), (43)) which was invariant under just the transformation of the gauge field, equations (23) or (36). This allows the construction of kinetic terms for the gauge field in the Lagrangian. The rank $n+1$ field strength tensor has a permutation symmetry among its indices.

5. The coupling constant g in general will have a non-zero mass dimension which can be determined by the covariant derivative, the structure of the $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}$ and taking the gauge field $A_{\mu_1\mu_2\cdots\mu_m}$ to have the standard mass dimension +1. The mass dimension of g is then the inverse of $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}$. For example, (20) and (24) imply g has a mass dimension of -1; equations (7) and (37) imply g has a mass dimension of -2. These theories having a dimensionful coupling are nonrenormalizable. There are cases (e.g. the standard rank 1 case of (6) or the rank 3 case of (34)), where g has mass dimension 0 and should therefore be renormalizable.

There have been other studies of higher rank (i.e. higher spin) gauge fields. In particular, the work of Fronsdal sought to extend a gauge procedure to higher rank fields of both integer [2] and half-integer spin [3]. In these works the gauge transformation of the gauge fields is different from that in the present work. In [2] the transformation of the rank n gauge field involves one derivative operator acting on rank $n-1$ gauge parameters. Here the transformation of the rank n gauge field involves up to n derivative operators acting on a scalar function (for example see equations (32) and (38)). In both the present work and in [2] the rank $n \geq 2$ gauge fields are totally symmetric under exchange of indices. However, in [2] the gauge fields must satisfy a traceless condition that is not necessary in the present work. A current review of these higher spin gauge theories, and connections to supersymmetry and string theory can be found in [11].

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