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Application of the Orbits Method to Integration of Linear Differential Equations with Non-Commutative Symmetries

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Linear PDE's are considered in the article. The equation to be solved is reduced to the equation on Lagrange submanifold to the coadjoint orbit. As a result we obtained necessary and sufficient conditions of integrability for invariant linear differential equations.

1 Introduction

The most widespread integration method the linear partial differential equations

$$H(z,\partial_z)\varphi(z) = E\varphi(z), \qquad z \in \mathbb{Z} \subset \mathbb{R}^N, \quad \varphi(z) \in C^\infty(\mathbb{Z}) \tag{1}$$

is separation of variables [1] that uses commutative algebra of symmetries of equation (1). The main goal in separation of variables is to solve eigenvalue problem

$$Y_\mu\psi(z) = \lambda_\mu\psi(z), \qquad Y_1 = H, \qquad \lambda_1 = E, \qquad [Y_\mu, Y_\nu] = 0. \tag{2}$$

Commutative algebra of symmetries is not a sufficient condition for the separation of variables. It is necessary for the solvability of system (2) that operators Y_μ must meet additional requirements [2].

Let linear differential equation (1) allow a non-Abelian symmetry group G. We shall now investigate the most efficient way how the non-Abelian symmetry group can be used for the integration of that equation.

2 Main theorem

Let $M = G/H$ be the regular orbit of group action G on Z; x are local coordinates on homogeneous space M, $X_A(x,\partial_x) \equiv X_A^a(x)\partial_{x^a}$ are generators of the transformation group which form Lie algebra $\mathcal{G}$: $[X_A, X_B] = C_{AB}^C X_C$, y are the invariants of the transformation group ($X_A y = 0$), $z \leftrightarrow (x, y)$.

Theorem 1. *Equation* (1) *with N variables that allows a Lie symmetry group G can be reduced to linear differential equations with N' variables:*

$$N' = N - \dim M + d(M), \tag{3}$$

where

$$d(M) = \frac{1}{2}\dim \mathcal{G}/\mathcal{G}^\lambda - \dim \mathcal{H}/\mathcal{H}^\lambda. \tag{4}$$

Here $\mathcal{H}$ is Lie algebra of isotropy subgroup H, λ is regular element of the space $\mathcal{H}^\perp = \{f \in \mathcal{G}^* \mid \langle f, \mathcal{H}\rangle = 0\} \subset \mathcal{G}^*$, $\mathcal{G}^\lambda$ is annihilator of the covector $\lambda \in \mathcal{H}^\perp$, $\mathcal{H}^\lambda = \mathcal{G}^\lambda \cap \mathcal{H}$.

Non-negative integer $d(M)$, which is called *defect* of homogeneous space M can be easily calculated using structural constants of Lie algebra $\mathcal{G}$ and its subalgebra $\mathcal{H}$ [3]:

$$d(M) = \frac{1}{2}\operatorname{rank}\langle\lambda, [\mathcal{G},\mathcal{G}]\rangle - \operatorname{rank}\langle\lambda, [\mathcal{G},\mathcal{H}]\rangle.$$

In the case when N' from (3) equals 1 we obtain ordinary differential equation and we call initial equation *integrable*.

Homogeneous spaces for which $d(M) = 0$ are called *commutative*. The class of commutative spaces includes, in particular, all of the symmetric and weakly symmetries spaces [4].

Let us show the main constructions, sufficient for the proof of the Theorem 1 [5].

Let us consider associative algebra D, generated by the finite set of elements $\{E_\mu\}$. Let $\mathcal{F}$ be a linear space with the basis $\{E_\mu\}$, $S(\mathcal{F})$ is a symmetric algebra, then $S(\mathcal{F}) \simeq D$.

We define a skew-symmetric bilinear form (commutator)

$$[A, B] \in S(\mathcal{F}), \qquad \forall\, A, B \in \mathcal{F},$$

such that Jacobi identities are satisfied

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B] = 0, \qquad \forall\, A, B, C \in \mathcal{F}.$$

(We assume here that the form acts on $S(\mathcal{F})$ according to the Leibnitz rule).

For basis elements we have:

$$[E_\mu, E_\nu] = \Omega_{\mu\nu}(E) \in S(\mathcal{F}), \qquad \mu,\nu = 1,\ldots,\dim\mathcal{F}.$$

Algebra $S(\mathcal{F})$ is called *functional algebra* ($\mathcal{F}$-algebra). In particular, if $\Omega_{\mu\nu}$ in formula (4) appears to be a quadratic polynomial, then we call it a quadratic algebra, and so on.

The space of smooth functions on $\mathcal{F}^*$ is Poisson algebra with the Poisson bracket:

$$\{\varphi,\psi\}^{\mathcal{F}}(g) = \Omega_{\mu\nu}(g)\frac{\partial\varphi(g)}{\partial g_\mu}\frac{\partial\psi(g)}{\partial g_\nu}, \qquad g = g_\mu E^\mu \in \mathcal{F}^*, \qquad \varphi,\psi \in C^\infty(\mathcal{F}^*).$$

The Poisson–Lie bracket is defined on the space of functions on $\mathcal{G}^*$:

$$\{\varphi,\psi\}(f) = C^C_{AB} f_C \frac{\partial\varphi(f)}{\partial f_A}\frac{\partial\psi(f)}{\partial f_B}, \qquad f = f_A e^A \in \mathcal{G}^*, \qquad \varphi,\psi \in C^\infty(\mathcal{G}^*). \tag{5}$$

Every Lie group G, acting on homogeneous space M, corresponds to $\mathcal{F}$-algebra of invariant linear operators on $C^\infty(M)$. Let us note as $L_\mu(x,\partial_x)$ independent operators, forming the $\mathcal{F}$-algebra

$$[L_\mu(x,\partial_x), X_A(x,\partial_x)] = 0, \qquad \mu = 1,\ldots,\dim\mathcal{F}, \qquad A = 1,\ldots,\dim\mathcal{G},$$
$$[L_\mu, L_\nu] = \Omega_{\mu\nu}(L) \in S(\mathcal{F}). \tag{6}$$

The dimension of the $\mathcal{F}$-algebra invariant operators is given by the following formula [3]

$$\dim\mathcal{F} = \dim\mathcal{G} + \dim\mathcal{H}^\lambda - 2\dim\mathcal{H}, \qquad \lambda \in \mathcal{H}^\perp. \tag{7}$$

We introduce symbols of operators $X_A(x,p) \equiv X^a_A(x)p_a$, $L_\mu(x,p) \in C^\infty(T^*M)$:

$$\{X_A(x,p), X_B(x,p)\} = C^C_{AB} X_C(x,p), \qquad \{L_\mu(x,p), L_\nu(x,p)\} = \Omega_{\mu\nu}(L), \tag{8}$$
$$\{X_A(x,p), L_\mu(x,p)\} = 0. \tag{9}$$

Momentum mappings

$$\mu:\ T^*X \to \mathcal{G}^*,\quad X(x,p) = f \in \mathcal{G}^*,\qquad \tilde{\mu}:\ T^*X \to \mathcal{F}^*,\quad L(x,p) = g \in \mathcal{F}^*$$

are Poisson mappings on Poisson algebra of functions on T^*M in Poisson algebra on $\mathcal{G}^*$ and $\mathcal{F}^*$. Also symplectic sheets $\Omega \subset \mathcal{G}^*$ and $\tilde{\Omega} \subset \mathcal{F}^*$ mutually correspond [6]:

$$\Omega = \mu\big(\tilde{\mu}^{-1}(\tilde{\Omega})\big),\qquad \tilde{\Omega} = \tilde{\mu}\big(\mu^{-1}(\Omega)\big),\qquad \operatorname{codim}\Omega = \operatorname{codim}\tilde{\Omega}.$$

Let Q be a Lagrange submanifold on the symplectic sheet Ω (coadjoint orbit) in $\mathcal{G}^*$. Let us represent the algebra $\mathcal{G}$ by differential operators $l(q, \partial_q, J)$, which form an irreducible representation of algebra $\mathcal{G}$ in the space of functions on Q [7]:

$$[l_A(q,\partial_q;J), l_B(q,\partial_q;J)] = C^C_{AB} l_C(q,\partial_q;J). \tag{10}$$

Here q are local coordinates on Q, J are parameters, “numerating” orbits in $\mathcal{G}^*$ and satisfying the condition for orbits to be integer.

We note as U a Lagrange submanifold to symplectic sheet $\tilde{\Omega}$. The defect $d(M)$ of homogeneous space M is defined as dimension of Lagrange submanifold to symplectic sheet on the coalgebra of invariant operators [3]:

$$d(M) = \dim U = \frac{1}{2}\dim\tilde{\Omega}. \tag{11}$$

We will note *index* of $\mathcal{F}$-algebra ($\operatorname{ind}\mathcal{F}$) the number of functionally independent elements that generate center of the enveloping field of algebra $\mathcal{F}$.

Obviously the equality

$$\dim\mathcal{F} - \operatorname{ind}\mathcal{F} = \dim\tilde{\Omega} = 2d(M) \tag{12}$$

takes place.

We now construct irreducible representation of algebra $\mathcal{F}$ on Lagrange submanifold U of symplectic sheet $\tilde{\Omega}$ in $\mathcal{F}^*$ with differential operators $\zeta(u, \partial_u; J)$:

$$[\zeta_\mu(u,\partial_u;J), \zeta_\nu(u,\partial_u;J)] = -\Omega_{\mu\nu}(\zeta(u,\partial_u;J)). \tag{13}$$

Set of generalized functions $D^J_{qu}(x)$, is defined from the equations

$$(X_A(x,\partial_x) + l_A(q,\partial_q;J))D^J_{qu}(x) = 0,\qquad (L_\mu(x,\partial_x) - \zeta_\mu(u,\partial_u;J))D^J_{qu}(x) = 0. \tag{14}$$

It is full and orthogonal on $C^\infty(M)$ [5]:

$$\int_M D^{\tilde{J}}_{\tilde{q}\tilde{u}}(x)\overline{D^J_{qu}(x)}\, d\mu(x) = \delta(J,\tilde{J})\delta(q,\tilde{q})\delta(u,\tilde{u}), \tag{15}$$

$$\int D^J_{qu}(x)\overline{D^J_{qu}(\tilde{x})}\, d\mu(J) d\mu(q) d\mu(u) = \delta(x,\tilde{x}). \tag{16}$$

Since operator $H(y,\partial_y,x,\partial_x)$ of equation (1) is G-invariant, so it may be expressed as follows:

$$H(y,\partial_y,x,\partial_x) = F(y,\partial_y,L(x,\partial_x)).$$

Because of that solution basis of equation (1), marked by the parameters (J,q) may be constructed following way:

$$\varphi^J_q(y,x) = \int_U \tilde{\varphi}^J(y,u) D^J_{qu}(x)\, d\mu(u), \tag{17}$$

where function $\tilde{\varphi}^J(y,u)$ is a solution of the equation

$$\tilde{H}(y,\partial_y,u,\partial_u;J)\tilde{\varphi}^J(y,u)=0. \tag{18}$$

Here $H(y,\partial_y,u,\partial_u;J)=F(y,\partial_y,\zeta^+(u,\partial_u;J))$.

Since the expression is correct (11), the number of independent variables N' in equation (18) is defined by formula (3).

Finally from (11) we have the number of independent variables N' in equation (18) to be given by formula (3).

3 Example

Let us illustrate the Theorem presented in this article by a non-trivial example. We consider an equation of type (1), where H is the Laplace operator

$$\Delta\varphi(x)\equiv\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}g^{ij}(x)\sqrt{g}\frac{\partial}{\partial x^j}\varphi(x)=E\varphi(x), \qquad g\equiv\det g_{ij} \tag{19}$$

on the Riemannian manifold with metric (below $c_i=\mathrm{const}$)

$$g^{ij}(x)=\begin{pmatrix} c_1e^{2x_4}x_3^2 & c_1e^{2x_4}x_3 & -c_4x_3e^{-x_4}/2 & c_3x_3e^{x_4}/2+c_4e^{-x_4}/2\\ c_1e^{2x_4}x_3 & c_1e^{2x_4} & -c_4e^{-x_4}/2 & c_3e^{x_4}/2\\ -c_4x_3e^{-x_4}/2 & -c_4e^{-x_4}/2 & 0 & 0\\ c_3x_3e^{x_4}/2+c_4e^{-x_4}/2 & c_3e^{x_4}/2 & 0 & c_2\end{pmatrix}. \tag{20}$$

That metric is not Stakkel which means that Hamilton–Jacobi equation as well as Klein–Gordon and Dirac equations cannot be solved using separation of variables. From the point of existing methods the corresponding equation is not integrable.

Metric (20) allows five-dimensional motion group formed by operators

$$\begin{gathered} X_1=\partial_1, \qquad X_2=\partial_2, \qquad X_3=x_2\partial_1+\partial_3,\\ X_4=-x_1\partial_1+x_2\partial_2-2x_3\partial_3+\partial_4, \qquad X_5=x_1\partial_2-x_3^2\partial_3+x_3\partial_4, \end{gathered} \tag{21}$$

that form a basis of the Lie algebra $\mathcal{G}=\{e_A\}$ with the following nonzero commutation rules:

$$\begin{gathered} [e_1,e_4]=-e_1, \qquad [e_1,e_5]=e_2, \qquad [e_2,e_3]=e_1, \qquad [e_2,e_4]=e_2,\\ [e_3,e_4]=-2e_3, \qquad [e_3,e_5]=e_4, \qquad [e_4,e_5]=-2e_5. \end{gathered}$$

Here this group acts on a four-dimensional homogeneous Riemannian space with one dimensional isotropy subgroup. Substituting $x=0$ in operators X_A (21), we find isotropy subalgebra $\mathcal{H}=\{e_5\}$. Algebra $\mathcal{G}$ is of an odd index 1, i.e. it has only one Casimir function $K(f)=f_1f_2f_4+f_1^2f_5-f_2^2f_3$, generating center of the Poisson algebra with the Poisson–Lie bracket defined by (5). Regular element of space $\mathcal{H}^\perp$ is non-degenerate and looks like: $\lambda=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,0)$. Annihilator of that covector λ is one-dimensional and, because it is degenerate, is the gradient of Casimir function:

$$\mathcal{G}^\lambda=\bigl\{\nabla K(f)|_{f=\lambda}=\lambda_2\lambda_4e_1+(\lambda_1\lambda_4-2\lambda_2\lambda_3)e_2-\lambda_2^2e_3+\lambda_1\lambda_2e_4+\lambda_1^2e_5\bigr\}.$$

So we get: $\mathcal{H}^\lambda=\{0\}$. According to (4) our Riemannian space is non-commutative and its defect is $d(M)=1$. Making a substitution in (3) $N=\dim M=4$, $d(M)=1$, we get $N'=1$, which means according to the Theorem that equation (19) may be reduced to a linear first-order differential equation. Also formulae (7), (12) give $\dim\mathcal{F}=3$, $\operatorname{ind}\mathcal{F}=1$, i.e. the algebra of invariant operators is three-dimensional and center of enveloping field is generated by one operator.

Let us find the algebra of invariant operators for this case. Symmetry operators X_A are projections of left-invariant vector fields ξ_A: $X_A = \pi_* \xi_A$, where mapping $\pi : G \to M = G/H$ is a projection of group G on the right coset space. Here we note as η_A the algebra of right-invariant vector fields:

$$[\xi_A, \xi_B] = C^C_{AB}\xi_C, \qquad [\eta_A, \eta_B] = C^C_{AB}\eta_C, \qquad [\xi_A, \eta_B] = 0.$$

In canonical coordinates of type II: $g = g_5 g_4 g_3 g_2 g_1$ on the Lie group, where $g_k = \exp(x_k e_k)$, left- and right-invariant vector fields are:

$$\begin{aligned}
&\xi_1 = \partial_1, \qquad \xi_2 = \partial_2, \qquad \xi_3 = x_2\partial_1 + \partial_3,\\
&\xi_4 = -x_1\partial_1 + x_2\partial_2 - 2x_3\partial_3 + \partial_4, \qquad \xi_5 = x_1\partial_2 - x_3^2\partial_3 + x_3\partial_4 + e^{-2x_4}\partial_5,\\
&\eta_1 = -\big(e^{-x_4} + x_3x_5e^{x_4}\big)\partial_1 - e^{x_4}x_5\partial_2, \qquad \eta_2 = -e^{x_4}x_3\partial_1 - e^{x_4}\partial_2,\\
&\eta_3 = -e^{-2x_4}\partial_3 - x_5\partial_4 + x_5^2\partial_5, \qquad \eta_4 = -\partial_4 + 2x_5\partial_5, \quad \eta_5 = -\partial_5.
\end{aligned}$$

(It is obvious that symmetry operators X_A are left-invariant vector fields restricted on the space of functions, which are independent of coordinate x_5). Invariant operators $L(x, \partial_x)$ are operator functions of right-invariant field restricted on the class of functions which are constant on every right coset (in our case functions, which are independent of coordinate x_5) and commuting with every field η_α, which form isotropy algebra $\mathcal{H} = \{\eta_\alpha\}$ (in our case, commuting with η_5). Invariant functions $L(x,p)$, which are symbols of invariant operators are easily obtained:

$$L_\mu(x,p) = a_\mu(f)|_{f=\eta^{cl}}, \qquad \eta^{cl}_A \equiv \eta^i_A(x)p_i,$$

where functions $a_\mu(f) \in C^\infty(\mathcal{G}^*)$ are solutions of the equations

$$\left.\left(C^B_{\alpha A} f_B \frac{\partial a(f)}{\partial f_A}\right)\right|_{f=\lambda} = 0, \qquad \lambda \in \mathcal{H}^\perp, \qquad \alpha = 1, \ldots, \dim \mathcal{H}. \tag{22}$$

In our case equation (22) transforms in

$$f_2\frac{\partial a(f)}{\partial f_1} + f_4\frac{\partial a(f)}{\partial f_3} = 0,$$

As we solve that equation, we get: $a_1 = f_2$, $a_2 = f_4$, $a_3 = f_1f_4 - f_2f_3$. So, invariant functions are:

$$\begin{aligned}
&L_1(x,p) = \eta^{cl}_2(x,p)|_{p_5=0} = -e^{x_4}(x_3p_1 + p_2), \qquad L_2(x,p) = \eta^{cl}_4(x,p)|_{p_5=0} = -p_4,\\
&L_3(x,p) = (\eta^{cl}_1(x,p)\eta^{cl}_4(x,p) - \eta^{cl}_2(x,p)\eta^{cl}_3(x,p))|_{p_5=0} = e^{-x_4}(p_1p_4 - p_2p_3 - x_3p_1p_3).
\end{aligned}$$

Invariant operators look as follows $L(x, \partial_x)$

$$L_1 = ie^{x_4}(x_3\partial_1 + \partial_2), \qquad L_2 = i(\partial_4 + 1), \qquad L_3 = -e^{-x_4}(\partial_{14} - x_3\partial_{13} - \partial_{23}),$$

are self-adjoint in respect to Riemannian measure $\sqrt{g}\,dx = C\exp(2x_4)\,dx$ and form solvable three-dimensional Lie algebra: $i[L_1, L_2] = L_1$, $[L_1, L_3] = 0$, $i[L_2, L_3] = L_3$. Center of the enveloping $\mathcal{F}$-algebra is generated by one element $Z = L_1L_3$, which coincides with Casimir operator $K(-iX)$ for algebra $\mathcal{G}$: $Z(x, \partial_x) = L_1(x, \partial_x)L_3(x, \partial_x) = K(-iX(x, \partial_x))$:

$$L_1L_3 = -i\{X_1, X_2, X_4\} - i\{X_1, X_1, X_5\} + i\{X_2, X_2, X_3\}.$$

Here and below figure brackets note the symmetrized product of operators. The Laplace operator belongs to the enveloping field of the $\mathcal{F}$-algebra since it is invariant operator:

$$-\Delta = c_1 L_1^2(x, \partial_x) + c_2 L_2^2(x, \partial_x) + c_3\{L_1(x, \partial_x), L_2(x, \partial_x)\} + c_4 L_3(x, \partial_x) + c_2. \tag{23}$$

Let us construct an irreducible representation of algebra $\mathcal{G}$ in the space of function on the Lagrange submanifold Q of coadjoint orbit Ω (symplectic sheet) (10):

$$l_1 = iq_1, \qquad l_2 = -iq_2, \qquad l_3 = q_1\partial_{q_2}, \qquad l_4 = q_1\partial_{q_1} - q_2\partial_{q_2}, \qquad l_5 = q_2\partial_{q_1} + iJ/q_1^2, \tag{24}$$

here $(q_1, q_2) \in Q = \mathbb{R}^2$. Because of irreducibility $K(-il(q, \partial_q; J)) = J$. Let us note that operators l_A (24) are skew-Hermitian in respect to measure $d\mu(q) = dq_1dq_2$ on Q. Parameter J that numerates coadjoint orbits is not quantized and may be of any real value.

Let us construct in the same way irreducible representation of $\mathcal{F}$-algebra in the space of function on Lagrange submanifold U of the symplectic sheet $\tilde{\Omega}$ (13) corresponding with representation (24):

$$\zeta_1 = u, \qquad \zeta_2 = -iu\partial_u, \qquad \zeta_3 = J/u, \qquad \left(u \in U = \mathbb{R}^1\backslash\{0\}\right). \tag{25}$$

These operator are self-adjoint in respect to measure $d\mu(u) = du/u$ on U.

We find set of functions $D^J_{qu}(x)$, numerated by variables (J, q, u) from solution of overdetermined system (14):

$$D^J_{qu}(x) = \exp\left(i(q_2x_2 - q_1x_1) - iJe^{x_4}/q_1u\right)\ \delta\left(e^{x_4}(x_3q_1 - q_2) - u\right). \tag{26}$$

That set of functions satisfies ortogonality and completeness conditions on M (15), (16) (in the right sides of these formulae delta-functions are presented with respect to corresponding measures $d\mu(J) = dJ/(2\pi)^3$).

Let us present solution $\varphi^J_q(x)$ of equation (19) as (17)

$$\varphi^J_q(x) = \int_U \tilde{\varphi}^J(u) D^J_{qu}(x)\, d\mu(u), \tag{27}$$

then, using expressions (14), (23), (25) for function $\tilde{\varphi}^J(u)$ we obtain ordinary differential equation

$$\left(c_2u^2\partial_u^2 + u(c_2 - ic_3u)\partial_u + ic_3u/2 - c_1u^2 - c_4J/u - c_2\right)\tilde{\varphi}^J(u) = E\tilde{\varphi}^J(u).$$

[1] Miller W., Symmetry and separation of variables, Addison Wesley, 1977.

[2] Bagrov V.G. and Gitman D.M., Exact solutions of relativistic wave equations, Dordrecht – Boston – London, Kluwer Academic Press, 1990.

[3] Shirokov I.V., Identities and invariant operators on homogeneous spaces, *Theor. Math. Phys.*, 2001, V.126, 326–338.

[4] Akhiezer D.N. and Vinberg E.B., Weakly symmetric spaces and spherical varieties, *Transformation Groups*, V.4, N 1, 1999, 3–24.

[5] Shirokov I.V., K-orbits, harmonic analysis on homogeneous spaces, and integration of differential equations, Preprint, Omsk, Omsk State Univ., 1998 (in Russian).

[6] Karasev M.V. and Maslov V.P., Nonlinear Poisson brackets. Geometry and quantization, Moscow, Nauka, 1990 (in Russian).

[7] Shirokov I.V., Darboux coordinates on K-orbits and the spectra of Casimir operators on Lie groups, *Theor. Math. Phys.*, 2000, V.123, 754–767.