# Semiclassical Approach to the Geometric Phase Theory for the Hartree Type Equation

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Quasi-energy states and a spectrum of quasi-energies asymptotic in small parameter $\hbar$ ($\hbar \to 0$) are constructed for a multidimensional Hartree type equation with non-local nonlinearity and with an external field cyclic in time. The quasi-energy states are a special case of trajectory coherent solutions of the Hartree type equation, which belong to the class of semiclassically concentrated functions. A function of this class describes a solitary wave localized in a neighborhood of a phase trajectory in the space of moments of the solution. The phase trajectory is closed due to the configuration of the external field. The Aharonov–Anandan geometric phases, which characterize a system "as a whole", are found for the quasi-energy states in a semiclassical approximation accurate to $O(\hbar^{3/2})$, $\hbar \to 0$.

## 1   Introduction

Quantum systems exposed to external periodic fields have nontrivial topological properties and are characterized by the geometric phase (GP) of a wave function [1, 2]. Systems of this type possess a set of quasi-energy states (QES's) and a spectrum of quasi-energies, which were originally introduced in [3, 4]. Mathematically, QES's and GP's are relevant to the properties of a quantum system as a whole. In particular, GP is related to a special kind of gauge symmetry. Details of the GP theory and its applications can be found in the reviews [5, 6].

In constructing quasi-energy states and geometric phases for similar nonlinear systems described by partial differential equations one faces the problem of integrability of nonlinear equations with external fields (variable coefficients) responsible for the nontrivial geometry and topology of the system. The usual symmetry analysis, when directly applied to these systems, fails since ordinary symmetry structures are effectively calculated for nonlinear equations with constant coefficients (see, for example, [7–9]). It should be noted that the above gauge symmetries can be of interest in the symmetry analysis of nonlinear equations of mathematical physics. Gauge symmetries result from the global properties of the geometry of a system and are constructed differently than standard symmetry analysis structures.

In this work, based on the Maslov complex germ method [10] we construct quasi-energy states asymptotic in small parameter $\hbar$, $\hbar \to 0$, in the class of semiclassical concentrated solutions of a multidimensional Hartree type equation (HTE) with a non-local nonlinearity and an external field periodic in time. The solution construction technique developed by the authors [11, 12] relies upon the periodic solutions of the dynamic system of Hamilton–Ehrenfest equations. The existence of periodic phase trajectories is provided by the external field.

The geometric phases for the QES under consideration are obtained in explicit form.

## 2   Problem statement and notations

Consider the Hartree type equation

$$\{-i\hbar\partial_t + \hat{\mathcal{H}}(t) + \varkappa\hat{V}(t, \Psi(t))\}\Psi(\vec{x}, t) = 0, \qquad \Psi \in L_2(\mathbb{R}_x^3). \tag{1}$$

Here, $\partial_t = \partial/\partial t$; $\hbar$ is a "small parameter", $\hbar \in [0,1)$, $\vec{x} \in \mathbb{R}^3$; $|\Psi|^2 = \Psi\Psi^*$, and the function $\Psi^*$ is complex conjugate to $\Psi$. The scalar product of the functions $\Psi$ and $\Phi$ in the space $L_2(\mathbb{R}_x^3)$ is denoted as

$$\langle \Psi | \Phi \rangle = \int_{\mathbb{R}^3} \Psi^*(\vec{x})\Phi(\vec{x})d\vec{x}. \tag{2}$$

Following quantum mechanical terms, we shall refer to the solutions of equation (1) as states.

The linear operator $\hat{\mathcal{H}}(t) = \mathcal{H}(\hat{z}, t)$ is a Weyl-ordered function [13] of time $t$ and operators

$$\hat{z} = (\hat{\vec{p}}, \vec{x}), \qquad \hat{\vec{p}} = -i\hbar\nabla = -i\hbar\frac{\partial}{\partial\vec{x}}, \tag{3}$$

for which the following commutation relations are valid:

$$[\hat{z}_k, \hat{z}_j]_- = i\hbar J_{kj}, \qquad k,j = \overline{1,6}, \tag{4}$$

where $J = \|J_{kj}\|_{6\times 6}$ is the unit symplectic matrix

$$J = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}_{6\times 6}, \qquad \mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

In quantum mechanical terms the linear operator $\hat{\mathcal{H}}(t)$ is written as

$$\hat{\mathcal{H}}(t) = \frac{1}{2m}\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}(\vec{x},t)\right)^2 - e\langle \vec{E}(t), \vec{x}\rangle + \frac{k}{2}\vec{x}^2. \tag{6}$$

The external field in the operator (6) is the superposition of a constant magnetic field $\vec{H} = (0,0,H)$ with vector potential $\vec{A} = \frac{1}{2}\vec{H}\times\vec{x}$, an electric field $\vec{E}(t) = (E\cos\omega t, E\sin\omega t, 0)$ periodic in time with frequency $\omega$, and an oscillator field with potential $\frac{k}{2}\vec{x}^2$. The nonlocal operator $\hat{V}(t, \Psi(t))$ in equation (1) has the form

$$\hat{V}(t, \Psi(t))\Psi(\vec{x},t) = \int_{\mathbb{R}^3} V(\vec{x},\vec{y})|\Psi(\vec{y},t)|^2 d\vec{y}\,\Psi(\vec{x},t), \tag{7}$$

$$V(\vec{x},\vec{y}) = V_0 \exp\left[-\frac{(\vec{x}-\vec{y})^2}{2\gamma^2}\right]. \tag{8}$$

The quantities $E$, $H$, $V_0$, $k$, $\omega$, $\gamma$, $\varkappa$, $e$, and $c$ are real parameters; $\langle \vec{x}, \vec{y}\rangle = \sum_{j=1}^{3} x_j y_j$ is the Euclidean scalar product of the vectors $\vec{x}$, $\vec{y}$, $\vec{x}^2 = \langle \vec{x}, \vec{x}\rangle$.

Below we shall use the following notation:

$$\omega_H = \frac{eH}{mc} \quad \text{is the cyclotron frequency,} \tag{9}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{is the oscillator frequency,} \tag{10}$$

$$\omega_a = \omega_0\sqrt{1 + \left(\frac{\omega_H}{2\omega_0}\right)^2}, \tag{11}$$

$$\omega_{\mathrm{nl}} = \sqrt{\frac{|\tilde{\varkappa}V_0|}{m\gamma^2}} \quad \text{is the "nonlinear frequency",} \qquad \tilde{\varkappa} = \varkappa\|\Psi\|^2, \quad \eta = \mathrm{sign}(\tilde{\varkappa}V_0). \tag{12}$$

Equation (1) has many physical applications. In particular, in the Bose–Einstein condensate theory, the solution $\Psi$ has the meaning of wave function, and the nonlocal potential $V(\vec{x}, \vec{y})$ describes the coupling of condensate particles which are in an external field (see, e.g., the reviews [14, 15].

Let us define the geometric phase for the Hartree type equation (1) based on the general concept of geometric phases in quantum mechanics, proposed by Aharonov and Anandan for cyclic states [2] (see also [5, 16]).

Quasi-energy states $\Psi_{\mathcal{E}}(\vec{x}, t, \hbar)$ in quantum mechanics were introduced by Zel'dovich [3] and Ritus [4] by the following condition:

$$\Psi_{\mathcal{E}}(\vec{x}, t, \hbar) = \mathrm{e}^{-i\mathcal{E}t/\hbar}\varphi_{\mathcal{E}}(\vec{x}, t, \hbar), \qquad \text{where} \quad \varphi_{\mathcal{E}}(\vec{x}, t + T, \hbar) = \varphi_{\mathcal{E}}(\vec{x}, t, \hbar). \tag{13}$$

The quantity $\mathcal{E}$ entering into equation (13) is called quasi-energy and defined modulo $\hbar\omega$, ($\omega = 2\pi/T$), i.e. $\mathcal{E}' = \mathcal{E} + m\hbar\omega$, $m \in \mathbb{Z}$.

States of this type play a key role in studying quantum mechanical systems exposed to strong periodic external fields when standard methods of the non-stationary perturbation theory are inapplicable.

Quasi-energy states are special cases of cyclic states introduced by Aharonov and Anandan [2] and have nontrivial GP's.

For the Hartree type equation (1), by a cyclic state on a time interval $[0, T]$, we mean the solution $\Psi(t)$ possessing the property

$$\Psi(t) = \mathrm{e}^{if(t)}\varphi(t), \qquad t \in [0, T], \quad f(T) - f(0) = \phi(\mathrm{mod}\ 2\pi), \quad \varphi(T) = \varphi(0). \tag{14}$$

The complete phase $\phi$ of the function (14) is subdivided into two summands: the dynamic phase

$$\delta = -\frac{1}{\hbar} \int_0^T dt \frac{\langle \Psi(t)|[\hat{\mathcal{H}}(t) + \varkappa\hat{V}(t, \Psi(t))]|\Psi(t)\rangle}{\langle \Psi(t)|\Psi(t)\rangle} \tag{15}$$

and the Aharonov–Anandan geometric phase

$$\gamma = i \int_0^T dt \frac{\langle \varphi(t)|\dot{\varphi}(t)\rangle}{\langle \varphi(t)|\varphi(t)\rangle}. \tag{16}$$

Let us call the quasi-energy state for the Hartree type equation a solution $\Psi_{\mathcal{E}}(\vec{x}, t, \hbar)$ of (1) for which we have

$$\Psi_{\mathcal{E}}(\vec{x}, t + T, \hbar) = \mathrm{e}^{-i\mathcal{E}T/\hbar}\Psi_{\mathcal{E}}(\vec{x}, t, \hbar). \tag{17}$$

It can be easily verified that equations (13) and (17) are equivalent. It is also obvious that quasi-energy states are a special case of the cyclic states (14). The set of the $\mathcal{E}$ values constitutes a spectrum of quasi-energies.

Comparing (14) and (17), we obtain that the function $f(t)$ entering into a quasi-energy state is

$$f(t) = -\mathcal{E}t/\hbar, \tag{18}$$

and for the complete phase $\phi$, in accordance with (14), we have

$$\phi = -\mathcal{E}T/\hbar \quad (\mathrm{mod}\, 2\pi). \tag{19}$$

By virtue of (15)–(19), the Aharonov–Anandan phase $\gamma_{\mathcal{E}}$ corresponding to the quasi-energy state $\Psi_{\mathcal{E}}(\vec{x}, t, \hbar)$ can be determined by the formula

$$\gamma_{\mathcal{E}} = -\frac{\mathcal{E}T}{\hbar} + \frac{1}{\hbar} \int_0^T dt \frac{\langle \Psi_{\mathcal{E}}|[\hat{\mathcal{H}}(t) + \varkappa\hat{V}(t, \Psi(t))]|\Psi_{\mathcal{E}}\rangle}{\langle \Psi_{\mathcal{E}}|\Psi_{\mathcal{E}}\rangle} \quad (\mathrm{mod}\, 2\pi). \tag{20}$$

We now state for equation (1) the problem of finding, in a semiclassical approximation, of the quasi-energy states and the quasi-energy spectrum in the class of trajectory concentrated functions [11, 12]

$$
\begin{aligned}
\mathcal{P}_\hbar^t &= \mathcal{P}_\hbar^t\big(Z(t,\hbar), S(t,\hbar)\big) \\
&= \left\{ \Phi : \Phi(\vec{x},t,\hbar) = \varphi\left(\frac{\Delta\vec{x}}{\sqrt{\hbar}}, t, \hbar\right) \exp\left[\frac{i}{\hbar}\big(S(t,\hbar) + \langle \vec{P}(t,\hbar), \Delta\vec{x}\rangle\big)\right] \right\},
\end{aligned}
\tag{21}
$$

where the function $\varphi(\vec{\xi}, t, \hbar)$ belongs to the Schwartz space $\mathbb{S}$ with respect to the variable $\vec{\xi} \in \mathbb{R}^3$, smoothly depends on $t$, and is regular in $\sqrt{\hbar}$, $\hbar \to 0$. Here, $\Delta\vec{x} = \vec{x} - \vec{X}(t,\hbar)$; the real function $S(t,\hbar)$ and the 6-component vector function $Z(t,\hbar) = (\vec{P}(t,\hbar), \vec{X}(t,\hbar))$, which specify the class $\mathcal{P}_\hbar^t(Z(t,\hbar), S(t,\hbar))$, regularly depend on $\sqrt{\hbar}$ in the neighborhood of $\hbar = 0$ and *are to be determined*. In the cases where this does not lead to ambiguity, we use the shorthand symbol $\mathcal{P}_\hbar^t$ for $\mathcal{P}_\hbar^t(Z(t,\hbar), S(t,\hbar))$.

The functions of the class $\mathcal{P}_\hbar^t$ are normalizable $\|\Phi(t)\|^2 = \langle \Phi(t)|\Phi(t)\rangle$ in the space $L_2(\mathbb{R}_x^3)$ with respect to the scalar product (2). If $\Phi(t)$ is a solution of equation (1), then $\|\Phi(t)\|^2 = \|\Phi(0)\|^2$; therefore, in what follows the argument $t$ in $\|\Phi(t)\|^2$ will be omitted.

Let us define for a linear operator $\hat{A} : \mathcal{P}_\hbar^t \to \mathcal{P}_\hbar^t$ and $\Psi \in \mathcal{P}_\hbar^t$ its mean value as

$$
\langle \hat{A} \rangle = \frac{1}{\|\Psi\|^2} \langle \Psi | \hat{A} | \Psi \rangle.
\tag{22}
$$

For a solution $\Psi$ of equation (1), we have

$$
\frac{d\langle \hat{A}(t)\rangle}{dt} = \left\langle \frac{\partial \hat{A}(t)}{\partial t} \right\rangle + \frac{i}{\hbar} \langle [\hat{\mathcal{H}}(t) + \varkappa \hat{V}(t, \Psi(t)), \hat{A}(t)]\rangle,
\tag{23}
$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator of the linear operators $\hat{A}$ and $\hat{B}$.

The vector function $Z(t,\hbar) = (\vec{P}(t,\hbar), \vec{X}(t,\hbar))$ is a parameter of the class $\mathcal{P}_\hbar^t$ of the form (21), which is to be determined.

When constructing a solution $\Psi \in \mathcal{P}_\hbar^t$ of equation (1), the vector function $Z(t,\hbar)$ will be chosen to satisfy the condition

$$
\vec{X}(t,\hbar) = \langle \vec{x} \rangle, \qquad \vec{P}(t,\hbar) = \langle \hat{\vec{p}} \rangle.
\tag{24}
$$

Let us construct asymptotic solutions for equation (1) with an accuracy of $O(\hbar^{3/2})$, $\hbar \to 0$, which correspond to the leading term of the asymptotic expansion. To do this, we expand the function $V(\vec{x}, \vec{y})$ of the form (8) in the operator (7) in a Taylor power series of $\Delta\vec{x} = \vec{x} - \vec{X}(t,\hbar)$, $\Delta\vec{y} = \vec{y} - \vec{X}(t,\hbar)$ and restrict ourselves to the terms of the order two inclusive in $\Delta\vec{x}$ and $\Delta\vec{y}$. Then equation (1) can be written

$$
\left\{ -i\hbar\partial_t + \tilde{\mathcal{H}}(t) + \langle \mathcal{H}_z(t), \Delta\hat{z}\rangle + \frac{1}{2}\langle \Delta\hat{z}, \tilde{\mathcal{H}}_{zz}(t)\Delta\hat{z}\rangle \right\} \Psi(\vec{x}, t) = O\big(\hbar^{3/2}\big).
\tag{25}
$$

Here, we take into account (24) and use the following notation: $\Delta\hat{z} = (\Delta\hat{\vec{p}}, \Delta\vec{x}) = (\hat{\vec{p}} - \vec{P}(t,\hbar), \vec{x} - \vec{X}(t,\hbar))$. The central moments of the function $\Psi(\vec{x}, t)$ are written as a $(6 \times 6)$-matrix $\Delta_2(t)$ of the form

$$
\Delta_2(t) = \begin{pmatrix} \sigma_{pp}(t) & \sigma_{px}(t) \\ \sigma_{xp}(t) & \sigma_{xx}(t) \end{pmatrix},
\tag{26}
$$

where $\sigma_{pp}(t) = (\sigma_{p_k p_l}(t)) = (\langle \Delta \hat{p}_k \Delta \hat{p}_l \rangle)$, $\sigma_{xp}(t) = (\sigma_{x_k p_l}(t)) = \left( \frac{1}{2} \langle \Delta x_k \Delta \hat{p}_l + \Delta \hat{p}_l \Delta x_k \rangle \right)$, $\sigma_{xx}(t) = (\sigma_{x_k x_l}) = (\langle \Delta x_k \Delta x_l \rangle)$;

$$\tilde{\mathcal{H}}(t) = \mathcal{H}(t) + \tilde{\varkappa} V_0 - \eta \frac{m \omega_{\rm nl}^2}{2} \sum_{j=1}^{3} \sigma_{x_j x_j}, \tag{27}$$

$$\mathcal{H}(t) = \frac{1}{2m} \vec{P}^2 + \frac{m \omega_a^2}{2} \left( X_1^2 + X_2^2 \right) + \frac{m \omega_0^2}{2} X_3^2$$
$$+ \frac{\omega_H}{2} (P_1 X_2 - P_2 X_1) - eE(X_1 \cos \omega t + X_2 \sin \omega t), \tag{28}$$

$$\mathcal{H}_z(t) = \begin{pmatrix} \mathcal{H}_p(t) \\ \mathcal{H}_x(t) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{m} P_1 + \dfrac{\omega_H}{2} X_2 \\ \dfrac{1}{m} P_2 - \dfrac{\omega_H}{2} X_1 \\ \dfrac{1}{m} P_3 \\ -\dfrac{\omega_H}{2} P_2 + m \omega_a^2 X_1 - eE \cos \omega t \\ \dfrac{\omega_H}{2} P_1 + m \omega_a^2 X_2 - eE \sin \omega t \\ m \omega_0^2 X_3 \end{pmatrix}, \tag{29}$$

where $\vec{P} = \vec{P}(\hbar, t)$, $\vec{X} = \vec{X}(\hbar, t)$ and the notation (9)–(12) is used;

$$\tilde{\mathcal{H}}_{zz}(t) = \begin{pmatrix} \tilde{\mathcal{H}}_{pp}(t) & \tilde{\mathcal{H}}_{px}(t) \\ \tilde{\mathcal{H}}_{xp}(t) & \tilde{\mathcal{H}}_{xx}(t) \end{pmatrix}, \tag{30}$$

$$\tilde{\mathcal{H}}_{pp}(t) = (\tilde{\mathcal{H}}_{p_k p_l}(t)) = \mathrm{diag} \left( \frac{1}{m}, \frac{1}{m}, \frac{1}{m} \right), \tag{31}$$

$$\tilde{\mathcal{H}}_{xx}(t) = (\tilde{\mathcal{H}}_{x_k x_l}(t)) = \mathrm{diag} \left( m(\omega_a^2 - \eta \omega_{\rm nl}^2), m(\omega_a^2 - \eta \omega_{\rm nl}^2), m(\omega_0^2 - \eta \omega_{\rm nl}^2) \right), \tag{32}$$

$$\tilde{\mathcal{H}}_{px}(t) = (\tilde{\mathcal{H}}_{p_k x_l}(t)) = \begin{pmatrix} 0 & \dfrac{\omega_H}{2} & 0 \\ -\dfrac{\omega_H}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

In [11, 12], equation (25) is called an associated linear Schrödinger equation. Its Hamiltonian is square in $\Delta \hat{z}$:

$$\hat{\tilde{\mathcal{H}}}(t) = \tilde{\mathcal{H}}(t) + \langle \mathcal{H}_z(t), \Delta \hat{z} \rangle + \frac{1}{2} \langle \Delta \hat{z}, \tilde{\mathcal{H}}_{zz}(t) \Delta \hat{z} \rangle. \tag{34}$$

The solutions of the linear equation (25) enable us to find approximate solutions of the nonlinear Hartree type equation (1).

## 3   The Hamilton–Ehrenfest system

The solution $\Psi$ of equation (1) is constructed in the class of trajectory concentrated functions (21) with the help of the solutions of the dynamic Hamilton–Ehrenfest equations (HEE's) for the centered moments of the field $\Psi$ [11,12]. To obtain a solution $\Psi$ exactly up to $O(\hbar^{3/2})$, it suffices to take into account the moments of the second order. It should also be noted that to construct the quasi-energy states (17) for equation (1), one must take the periodic solutions of the HEE's.

Let us substitute the operators $\hat{\vec{p}}$, $\vec{x}$, and $\Delta_2(t)$ into (23) instead of the operator $\hat{A}$. In view of (24) and the notation (26)–(33), we obtain the following HEE's:

$$\dot{Z}(t,\hbar) = J\mathcal{H}_z(t), \tag{35}$$

$$\dot{\Delta}_2(t) = J\tilde{\mathcal{H}}_{zz}(t)\Delta_2 - \Delta_2\tilde{\mathcal{H}}_{zz}(t)J. \tag{36}$$

Obviously, equations (35), which describe the motion of the centroid of the field $\Psi$, are integrable independently of equations (36) for the second moments $\Delta_2(t)$. Therefore, we first find the periodic solution of the system (35) and then integrate equations (36). The elementary particular periodic solution of the system (35) can be taken in the form

$$Z_0(t,\hbar) = (\vec{P}_0(t,\hbar), \vec{X}_0(t,\hbar)),$$

$$\vec{P}_0(t,\hbar) = \left(-m\left(\frac{\omega_H}{2}+\omega\right)\xi\sin\omega t, m\left(\frac{\omega_H}{2}+\omega\right)\xi\cos\omega t, 0\right), \tag{37}$$

$$\vec{X}_0(t,\hbar) = (\xi\cos\omega t, \xi\sin\omega t, 0), \tag{38}$$

$$\xi = \frac{eE}{m(\omega_0^2 - \omega_H\omega - \omega^2)}. \tag{39}$$

For the solution (37), the Hamiltonian (34) of equation (25) is periodic with the period $T = 2\pi/\omega$. In what follows we assume that $\xi^{-1} \neq 0$ which means the absence of a resonance.

In [11, 12], the system (36) is shown to be equivalent to the system of equations in variations

$$\dot{A}(t) = J\tilde{\mathcal{H}}_{zz}(t)A(t), \tag{40}$$

where $A(t)$ is a $6 \times 6$ nonsingular matrix and $\tilde{\mathcal{H}}_{zz}(t)$ has the form of (30).

Solving the system (40), one can find the trajectory-coherent states (TCS's) for the associated linear Schrödinger equation (25) in explicit form using well-known methods (see, e.g., [17]). These TCS's, in turn, lead to approximate TCS's for the nonlinear Hartree type equation (1) in the class of trajectory concentrated functions (21) [11, 12].

Taking into account the periodic behavior of the Hamiltonian (34), (37), we now state the problem of constructing quasi-energy states in the class of TCS's (quasi-energy TCS's).

To do this, we use the solutions of the system (40) that satisfy the Floquet condition

$$a(t+T) = \exp(i\Omega T)a(t), \tag{41}$$

where $a(t)$ is a column of the matrix $A(t)$ and $\Omega$ is a real constant. The fundamental matrix of the system (40), (41) is written as

$$A(t) = (a_1(t), a_2(t), a_3(t), a_1^*(t), a_2^*(t), a_3^*(t)),$$

where the columns of the matrix $A(t)$ are denoted by $a_j(t)$, $j = \overline{1,3}$ and $a_j^*(t)$ is complex conjugate to $a_j(t)$ normalized by the condition $\langle a_j, J^T a_l^* \rangle = 2i\delta_{jl}$, $\langle a_j, J^T a_l \rangle = 0$. The columns are given by the following expressions:

$$a_1(t) = \frac{e^{i\omega_+ t}}{\sqrt{2}}\left(g_0, ig_0, 0, -\frac{i}{g_0}, \frac{1}{g_0}, 0\right)^T, \qquad a_2(t) = \frac{e^{i\omega_- t}}{\sqrt{2}}\left(g_0, -ig_0, 0, -\frac{i}{g_0}, -\frac{1}{g_0}, 0\right)^T,$$

$$a_3(t) = e^{i\omega_s t}\left(0, 0, g_s, 0, 0, \frac{-i}{g_s}\right)^T. \tag{42}$$

Here

$$\omega_+ = \sqrt{\omega_a^2 - \eta\omega_{\text{nl}}^2} + \frac{\omega_H}{2}, \tag{43}$$

$$\omega_- = \sqrt{\omega_a^2 - \eta\omega_{\text{nl}}^2} - \frac{\omega_H}{2} \tag{44}$$

are combined frequencies, $\omega_s = \sqrt{\omega_0^2 - \eta\omega_{nl}^2}$, $g_0 = \sqrt{\dfrac{m}{2}(\omega_+ + \omega_-)}$, $g_s = \sqrt{m\omega_s}$,

$$A(t) = \begin{pmatrix} B(t) & B^*(t) \\ C(t) & C^*(t) \end{pmatrix}, \tag{45}$$

where

$$B(t) = \begin{pmatrix} \dfrac{g_0 e^{i\omega_+ t}}{\sqrt{2}} & \dfrac{g_0 e^{i\omega_- t}}{\sqrt{2}} & 0 \\ \dfrac{ig_0 e^{i\omega_+ t}}{\sqrt{2}} & \dfrac{-ig_0 e^{i\omega_- t}}{\sqrt{2}} & 0 \\ 0 & 0 & g_s e^{i\omega_s t} \end{pmatrix}, \qquad C(t) = \begin{pmatrix} \dfrac{-ie^{i\omega_+ t}}{\sqrt{2}g_0} & \dfrac{-ie^{i\omega_- t}}{\sqrt{2}g_0} & 0 \\ \dfrac{e^{i\omega_+ t}}{\sqrt{2}g_0} & \dfrac{-e^{i\omega_- t}}{\sqrt{2}g_0} & 0 \\ 0 & 0 & \dfrac{-ie^{i\omega_s t}}{g_s} \end{pmatrix}.$$

The periodic motion (37), (38) is stable in the linear approximation provided $\omega_0^2 > \eta\omega_{nl}^2$. This is assumed to be fulfilled.

The solution of the system (26), (36) is presented via the blocks $B(t)$ and $C(t)$ of the matrix $A(t)$ as

$$\sigma_{xx}(t) = \frac{\hbar}{4}(C(t)D_\nu C^+(t) + C^*(t)D_\nu C^T(t)),$$

$$\sigma_{pp}(t) = \frac{\hbar}{4}(B(t)D_\nu B^+(t) + B^*(t)D_\nu B^T(t)),$$

$$\sigma_{px}(t) = \frac{\hbar}{4}(B(t)D_\nu C^+(t) + B^*(t)D_\nu C^T(t)),$$

where the diagonal matrix $D_\nu = \mathrm{diag}\,(2\nu_1 + 1, 2\nu_2 + 1, 2\nu_3 + 1)$, $\nu_1, \nu_2, \nu_3 = 0, 1, 2, \ldots$ and the symbol $(^+)$ implies Hermitian conjugation. The matrices $\sigma_{xx}$ and $\sigma_{pp}$ are diagonal as well and are equal to

$$\sigma_{xx}(t) = \frac{\hbar}{m}\,\mathrm{diag}\left(\frac{\nu_1 + \nu_2 + 1}{\omega_+ + \omega_-}, \frac{\nu_1 + \nu_2 + 1}{\omega_+ + \omega_-}, \frac{2\nu_3 + 1}{2\omega_s}\right),$$

$$\sigma_{pp}(t) = \frac{\hbar m}{4}\,\mathrm{diag}\,((\omega_+ + \omega_-)(\nu_1 + \nu_2 + 1), (\omega_+ + \omega_-)(\nu_1 + \nu_2 + 1), 2\omega_s(2\nu_3 + 1)), \tag{46}$$

The nonzero elements of the matrix $\sigma_{xp}(t)$ are $\sigma_{p_1 x_2}(t) = -\sigma_{p_2 x_1} = \frac{\hbar}{2}(\nu_1 - \nu_2)$. Thus, the solution (37) and the relevant matrix of the second moments $\Delta_2$ constitute a periodic solution of the HEE's (35), (36).

## 4 The quasi-energies and geometric phases

Let us construct the quasi-energy TCS's of the associated linear Schrödinger equation (25) for the periodic solutions of the HEE's (35), (36).

Based on these solutions, we can find, in a semiclassical approximation, the corresponding solutions of the Hartree type equation (1), which coincide with the quasi-energy TCS's of equation (25) for the time zero. We shall refer to these solutions of equation (1) as semiclassical quasi-energy trajectory coherent states (SQETCS's).

Let us construct the SQETCS's following [11, 12]. The linear Schrödinger equation (25) with the squared Hamiltonian (34) admits a solution in the form of a Gaussian wave packet:

$$\Phi(\vec{x}, t) = N_\hbar \exp\left\{\frac{i}{\hbar}\left[S(t, \hbar) + \langle \vec{P}_0(t, \hbar), \Delta\vec{x}\rangle + \frac{1}{2}\langle \Delta\vec{x}, Q(t, \hbar)\Delta\vec{x}\rangle\right]\right\}, \tag{47}$$

where the normalization factor $N_\hbar$, the real function $S(t, \hbar)$, and the complex matrix $Q(t, \hbar)$ are to be determined. The Hamiltonian (34) of the linear Schrödinger equation (25), whose coefficients (26)–(33) are specified by solving the HEE's (37), (46), is periodic in time with period $T = 2\pi/\omega$. In this case, there exists a solution of the form (47) satisfying the quasi-energy condition (17). Following quantum mechanical terms, we treat this solution as a vacuum state and denote it as $|0, t\rangle$. One can directly verify that

$$
\begin{aligned}
|0, t\rangle = N_\hbar \exp \Bigg\{ & \frac{i}{\hbar} \left[ \frac{1}{2} eE\xi - \tilde{\varkappa} V_0 - \frac{\hbar}{2}(\omega_+ + \omega_- + \omega_s) + \eta \hbar \omega_{\mathrm{nl}}^2 \left( \frac{\nu_1 + \nu_2 + 1}{\omega_+ + \omega_-} + \frac{2\nu_3 + 1}{4\omega_s} \right) \right] t \\
& - \frac{i}{\hbar} \left( \frac{\omega_H}{2} + \omega \right) m\xi(\Delta x_1 \sin \omega t - \Delta x_2 \cos \omega t) \\
& - \frac{m}{4\hbar} [(\omega_+ + \omega_-)(\Delta x_1^2 + \Delta x_2^2) + 2\omega_s \Delta x_3^2] \Bigg\},
\end{aligned}
\tag{48}
$$

where $\xi$ is of the form (38).

It can easily be seen that the operators

$$
\hat{a}_j(t) = N_j \langle a_j(t), J^T \Delta \hat{z} \rangle, \qquad j = \overline{1, 3},
\tag{49}
$$

where $N_j$ is a normalization factor and the vector function $a_j(t)$ is the $j$th column of (42), are the symmetry operators for the linear Schrödinger equation (25) commuting with the equation operator,

$$
-i\hbar \partial_t \hat{a}_j(t) + [\hat{\tilde{\mathcal{H}}}(t), \hat{a}_j(t)] = 0.
$$

The trajectory coherent states $|\nu, t\rangle$ for equation (25) are determined as a result of the action of the creation operators $\hat{a}_j^+(t)$ on the vacuum state (48)

$$
|\nu, t\rangle = \prod_{j=1}^{3} \frac{1}{\sqrt{\nu_j!}} (\hat{a}_j^+(t))^{\nu_j} |0, t\rangle.
\tag{50}
$$

Here, the operators $\hat{a}_j^+(t)$ are Hermite conjugate to $\hat{a}_j(t)$ with respect to the scalar product (2). Substituting equations (48) and (49) into (50), we obtain explicit expressions of the TCS's for equation (25):

$$
\begin{aligned}
|\nu, t\rangle = \exp \left[ -i(\omega_+ \nu_1 + \omega_- \nu_1 + \omega_s \nu_3)t \right] & \prod_{j=1}^{3} N_{a_j}^{\nu_j} \frac{\hbar^{\frac{1}{2}(\nu_1 + \nu_2 + \nu_3)}}{2^{\frac{\nu_1 + \nu_2}{2}}} \sqrt{\frac{\nu_1! \nu_2!}{\nu_3!}} \\
& \times \sum_{k_1=0}^{\nu_1} \sum_{k_2=0}^{\nu_2} \frac{(-1)^{\nu_2 + \nu_3 - k_1} \, i^{\nu_1 + \nu_2 - k_1 - k_2}}{k_1! k_2! (\nu_1 - k_1)! (\nu_2 - k_2)!} H_{k_1 + k_2} \left( \frac{\Delta x_1 g_0}{\sqrt{\hbar}} \right) \\
& \times H_{\nu_1 + \nu_2 - k_1 - k_2} \left( \frac{\Delta x_2 g_0}{\sqrt{\hbar}} \right) H_{\nu_3} \left( \frac{\Delta x_3 g_s}{\sqrt{\hbar}} \right) |0, t\rangle.
\end{aligned}
\tag{51}
$$

The states (51) satisfy the condition (17) and they are the quasi-energy TCS's for equation (25). When $|\nu, 0\rangle$ coincides with the initial conditions for equation (1), expression (51) is a semiclassical approximation for the quasi-energy TCS of the Hartree type equation (1) with an accuracy of $O(\hbar^{3/2})$, $\hbar \to 0$. For example, expression (48) is a solution of the linear Schrödinger equation (25) for any $\nu = (\nu_1, \nu_2, \nu_3)$. However, for $t = 0$ the matrix $\Delta_2$ of the form (26), (46) with $\nu = (\nu_1, \nu_2, \nu_3) = (0, 0, 0)$ corresponds to the state $|0, 0\rangle$. Thus, the function (48) will be an asymptotic solution, up to $O(\hbar^{3/2})$, of the Hartree type equation (1) only if $\nu = (\nu_1, \nu_2, \nu_3) = (0, 0, 0)$.

Substituting the function (51) into condition (17)

$$|\nu, t + T\rangle = \exp\left[-\frac{i}{\hbar}\mathcal{E}_\nu T\right]|\nu, t\rangle,$$

we obtain the spectrum of quasi-energies $\mathcal{E}_\nu$ for equation (1)

$$\mathcal{E}_\nu = -\frac{eE}{2}\xi + \tilde{\varkappa}V_0 + \hbar\left[\left(\omega_+ - \frac{\eta\omega_{\mathrm{nl}}^2}{\omega_+ + \omega_-}\right)\left(\nu_1 + \frac{1}{2}\right)\right.$$
$$\left. + \left(\omega_- - \frac{\eta\omega_{\mathrm{nl}}^2}{\omega_+ + \omega_-}\right)\left(\nu_2 + \frac{1}{2}\right) + \left(\omega_s - \frac{\eta\omega_{\mathrm{nl}}^2}{2\omega_s}\right)\left(\nu_3 + \frac{1}{2}\right)\right] + O(\hbar^{3/2}). \tag{52}$$

The Aharonov–Anandan geometric phases (20) related to the SQETCS's (51) are written as

$$\gamma_{\mathcal{E}_\nu} = -\frac{T}{\hbar}\mathcal{E}_\nu + \frac{1}{\hbar}\int_0^T \langle\hat{\tilde{\mathcal{H}}}(t)\rangle dt, \tag{53}$$

where $\mathcal{E}_\nu$ and the operator $\hat{\tilde{\mathcal{H}}}(t)$ are of the form (52) and (34), respectively.

Calculation of the mean value for the operator $\hat{\tilde{\mathcal{H}}}(t)$ according to the formula (22) for the functions of the form (51) yields

$$\langle\hat{\tilde{\mathcal{H}}}(t)\rangle = \tilde{\mathcal{H}}(t) + \frac{\hbar}{2}(\omega_+ + \omega_-)(\nu_1 + \nu_2 + 1) + \hbar\omega_s\left(\nu_3 + \frac{1}{2}\right) + \frac{\hbar\omega_H}{4}(\nu_1 - \nu_2),$$
$$\tilde{\mathcal{H}}(t) = \frac{m\xi^2}{2}(3\omega^2 + 2\omega\omega_H - \omega_0^2) + \tilde{\varkappa}V_0 - \frac{\eta\hbar\omega_{\mathrm{nl}}^2}{\omega_+ + \omega_-}(\nu_1 + \nu_2 + 1) - \frac{\eta\hbar\omega_{\mathrm{nl}}^2}{2\omega_s}\left(\nu_3 + \frac{1}{2}\right). \tag{54}$$

Let us now substitute the quasi-energies (52) and (54) into the formula (53). Then we obtain the following expression for the Aharonov–Anandan geometric phase corresponding to the quasi-energy TCS's of equation (1) in the semiclassical approximation:

$$\gamma_{\mathcal{E}_\nu} = \frac{T}{2\hbar}m(2\omega^2 + \omega\omega_H)\xi^2 + O(\hbar^{3/2}), \tag{55}$$

where $\xi$ has the form (39).

For more detailed analysis of quasi-energies and geometric phases it is useful to compare the expressions (52) and (55) with similar ones for the one-dimensional case.

## 5   One-dimensional case

Consider equation (1) where $\Psi$ belongs to the space $L_2(\mathbb{R}_x^1)$, the linear operator $\hat{\mathcal{H}}(t)$ is taken as

$$\hat{\mathcal{H}}(t) = \frac{\hat{p}^2}{2m} - eEx\cos\omega t + \frac{k}{2}x^2, \tag{56}$$

and the nonlocal operator $\hat{V}(t, \Psi(t))$ is

$$\hat{V}(t, \Psi(t))\Psi(x, t) = \int_{-\infty}^{+\infty} V(x, y)|\Psi(y, t)|^2 dy\,\Psi(x, t),$$
$$V(x, y) = V_0\exp\left[-\frac{(x - y)^2}{2\gamma^2}\right]. \tag{57}$$

As the magnetic field is missing ($H = 0$) in the case under consideration, the cyclotron frequency (9) is zero ($\omega_H = 0$) and from (11) we have $\omega_a = \omega_0$. Putting $\omega_H = 0$ into (43), (44) one has

$$\omega_+ = \omega_- = \omega_s = \sqrt{\omega_0^2 - \eta\omega_{nl}^2}.$$

The Hamilton–Ehrenfest system (35), (36) corresponding to (56), (57) accurate to $O(\hbar^{3/2})$ is

$$\dot{p} = -kx + eE\cos\omega t, \qquad \dot{x} = \frac{p}{m}, \tag{58}$$

$$\dot{\sigma}_{xx} = \frac{2}{m}\sigma_{xp}, \qquad \dot{\sigma}_{xp} = \frac{1}{m}\sigma_{pp} - m\omega_s^2\sigma_{xx}, \qquad \dot{\sigma}_{pp} = -2m\omega_s^2\sigma_{xp}. \tag{59}$$

The periodic solution (37), (38) with the period $T = 2\pi/\omega$ for the Hamilton system (58) takes the form

$$P_0(t,\hbar) = -m\omega\xi\sin\omega t, \qquad X_0(t,\hbar) = \xi\cos\omega t.$$

Here and below

$$\xi = \frac{eE}{m(\omega_0^2 - \omega^2)}.$$

The matrix $A(t)$ in the system of equations in variations (40) has now the size $2 \times 2$ with one-dimensional blocks $B(t)$ and $C(t)$ in (45). Accordingly, the system (40) becomes

$$\dot{B} = -m\omega_s^2 C, \qquad \dot{C} = \frac{B}{m}. \tag{60}$$

The Floquet solution (41)

$$a(t) = \begin{pmatrix} B(t) \\ C(t) \end{pmatrix}$$

of the system (60) normalized by the condition $\langle a, J^T a^* \rangle = 2i$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, is found as

$$a(t) = \frac{\exp(i\omega_s t)}{\sqrt{m\omega_s}} \begin{pmatrix} im\omega_s \\ 1 \end{pmatrix}.$$

Then the quasi-energy TCS's similar to (51) for the one-dimensional HTE (1) with operators (56), (57) have the form

$$|n, t\rangle = \frac{i^n}{\sqrt{2^n n!}} N_\hbar \exp\left\{ \frac{i}{\hbar}\left[ \frac{1}{4}eE\xi t - \frac{3\omega^2 - \omega_0^2}{8\omega}m\xi^2\sin 2\omega t - \tilde{\varkappa}V_0 t \right. \right. \tag{61}$$

$$\left. \left. - \hbar\left(\omega_s - \frac{\eta\omega_{nl}^2}{2\omega_s}\right)\left(n + \frac{1}{2}\right)t \right] - \frac{i}{\hbar}m\omega\xi\sin\omega t\Delta x - \frac{m\omega_s}{2\hbar}\Delta x^2 \right\} H_n\left( \sqrt{\frac{m\omega_s}{\hbar}}\Delta x \right).$$

Here $\Delta x = x - \xi\cos\omega t$, $N_\hbar$ is a normalization factor, $n \in \mathbb{Z}_+$. The condition (17) yields the spectrum of quasi-energies $\mathcal{E}_n$ for the functions (61) as

$$\mathcal{E}_n = -\frac{1}{4}eE\xi + \tilde{\varkappa}V_0 + \hbar\left(\omega_s - \frac{\eta\omega_{nl}^2}{2\omega_s}\right)\left(n + \frac{1}{2}\right) + O(\hbar^{3/2}), \tag{62}$$

and the Aharonov–Anandan geometric phase is

$$\gamma_{\mathcal{E}_n} = \frac{T}{2\hbar}m\omega^2\xi^2 + O(\hbar^{3/2}). \tag{63}$$

# 6   Concluding remarks

The spectrum of quasi-energies (52) includes the sum of energies of three oscillators with shifted combined frequencies $\omega_\pm$ and $\omega_s$ the shift of which depends on nonlinearity. The geometric phase (55) in the approximation $O(\hbar^{3/2})$ under consideration does not include quantum corrections and nonlinearity. Similar is true for the one-dimensional case (62), (63). The following explanation to this can be proposed. The nonlinearity and the quantity $\hbar$ in this approximation do not enter to the solution (37), (38) of the Hamiltonian system (35) that specifies the geometry of the physical system. As a consequence, the geometric phase also does not include these quantities. In addition, the geometric phase for the three-dimensional case (55) (if $\omega_H = 0$) is twice as large as that for the one-dimensional case (63) since the centroid motion of the field $\Psi$ is plane in the three-dimensional case (see (37), (38)).

In the limit $T \to \infty$ ($\omega \to 0$) the operator $\hat{\mathcal{H}}(t)$ of the form (6) (in the three-dimensional case) and (56) (in the one-dimensional case) does not depend on $t$ any more and we denote $\lim_{\omega \to 0} \hat{\mathcal{H}}(t) = \hat{\mathcal{H}}$. Then the expressions (51), (52) and (61), (62) determine the discrete spectral series of the nonlinear spectral problem

$$(\hat{\mathcal{H}} + \varkappa \hat{V}(\Psi)\}\Psi = \mathcal{E}\Psi.$$

This corresponds to a stationary solution of the Hamilton–Ehrenfest system (35), (36) and (58), (59), respectively (see also [10]).

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