

The Structure of Cosymmetries and a Simple Proof of Locality for Hierarchies of Symmetries of Odd Order Evolution Systems

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We derive a formula for the leading terms of formal conservation laws (and hence for the leading terms of higher order cosymmetries as well) for a large class of odd order evolution systems in (1+1)-dimensions. This result yields a simple proof of locality for hierarchies of symmetries generated using master symmetries and recursion operators of such systems.

Introduction

It is well known that members of integrable hierarchies of systems of partial differential equations (PDEs) usually have an infinite number of common conservation laws, see e.g. [1–4]. The latter are typically extracted from the Lax pair or zero curvature representation for the system in question, cf. e.g. [4].

On the other hand, one often has to deal with the inverse problem, that is, to check whether a given conserved density or, more broadly, a cosymmetry [1] (in other terminology, a conserved covariant [5]) is indeed shared by the evolution systems associated with the symmetries of the original system. For instance, we may need to verify whether a given physical quantity (charge, momentum, etc.) is invariant under the higher flows of the hierarchy. This is also required when we wish to verify locality of the hierarchy of symmetries produced by the repeated application of a recursion operator or by the repeated commutation with a master symmetry, cf. [2, 3, 6–11].

Checks of this kind can be quite a difficult task, especially when no Lax pair or zero curvature representation for the system in question is yet found. A possible workaround is to use scaling-based arguments, see e.g. [12], but then one has to know the weights of all (homogeneous) cosymmetries for the system in question. Finding these weights is a fairly nontrivial problem, and in the present paper we solve it for the odd order evolution systems in (1+1) dimensions.

First of all, we derive, under some minor technical assumptions, an explicit formula (8) for the leading terms of formal conservation laws for such systems, see Lemma 1 for details. As the directional derivative of a cosymmetry is a formal conservation law, this yields the formula (10) for the weight of any homogeneous cosymmetry of sufficiently high order.

With this in mind, we can find when a homogeneous cosymmetry is shared by the flow associated with a given (homogeneous) symmetry, i.e., when the Lie derivative of this cosymmetry along the symmetry in question vanishes, see Lemma 2 below for details. Using this result, we obtain new easily verifiable sufficient conditions ensuring the locality of symmetries generated with usage of recursion operator or master symmetry, see Propositions 1 and 3 and Corollaries 1 and 2.

For the hierarchies generated using hereditary recursion operators one can [7, 8, 10] prove locality without finding weights of all cosymmetries. Unfortunately, the check of hereditariness can be quite difficult by itself, especially for the multicomponent systems. Proposition 1 and

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Corollary 1 of the present work enable one to avoid this check altogether and prove the locality of corresponding hierarchy in a simple and straightforward manner.

The absence of immediate analog of hereditariness for master symmetries makes the technique of [7,8] useless for proving locality of the associated hierarchies. If the only nonlocalities in master symmetries are linear combinations of potentials of the so-called canonical conservation laws, we can prove locality of generated symmetries using the results from [11]. However, if master symmetries involve other nonlocalities as well, one needs other methods for proving locality of symmetries, and the one presented below in Proposition 3 and Corollary 2 provides a convenient and easy-to-use alternative.

1 Basic definitions and structures

Consider an evolution system with constraints (cf. [13–15])

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}(x, t, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_{n'}, \vec{\omega}), \quad (1)$$

$$\partial \omega_\kappa / \partial x = X_\kappa(x, t, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_h, \vec{\omega}), \quad (2)$$

$$\partial \omega_\kappa / \partial t = T_\kappa(x, t, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_h, \vec{\omega}). \quad (3)$$

Here $\mathbf{u} = (u^1, \dots, u^s)^T$, $\mathbf{u}_0 \equiv \mathbf{u}$, $\mathbf{u}_l = \partial^l \mathbf{u} / \partial x^l$, $l = 1, 2, \dots$, $\vec{\omega} = (\omega_1, \dots, \omega_c)^T$, $\kappa = 1, \dots, c$. The superscript T here and below stands for the matrix transposition.

Let \mathcal{A}_j be an algebra (with respect to the standard multiplication) of locally analytic functions of $x, t, \mathbf{u}, \dots, \mathbf{u}_j, \omega_1, \dots, \omega_c$, cf. [13]. Set $\mathcal{A} = \bigcup_{j=1}^{\infty} \mathcal{A}_j$. In what follows we assume that $X_\kappa, T_\kappa \in \mathcal{A}$ for all $\kappa = 1, \dots, c$.

Let $D_x \equiv D = \partial / \partial x + \sum_{i=0}^{\infty} \mathbf{u}_{i+1} \partial / \partial \mathbf{u}_i + \sum_{\kappa=1}^c X_\kappa \partial / \partial \omega_\kappa$ and $D_t = \partial / \partial t + \sum_{i=0}^{\infty} D^i(\mathbf{F}) \partial / \partial \mathbf{u}_i + \sum_{\kappa=1}^c T_\kappa \partial / \partial \omega_\kappa$ be the operators of total derivatives on \mathcal{A} , cf. [13–15].

As in [13–15], we require that $[D, D_t] = 0$ or, equivalently, $D_t(X_\kappa) = D(T_\kappa)$ for all $\kappa = 1, \dots, c$. In what follows we tacitly assume that the kernel of D in \mathcal{A} is exhausted by the functions of t alone. We shall employ the notation $\text{Im } D$ for the image of D in \mathcal{A} .

Consider the set $\text{Mat}_p(\mathcal{A})[[D^{-1}]]$ of *formal series* in powers of D of the form $\mathfrak{H} = \sum_{j=-\infty}^q h_j D^j$, where h_j are $p \times p$ matrices with entries from \mathcal{A} . The greatest $m \in \mathbb{Z}$ such that $h_m \neq 0$ is called the *degree* of \mathfrak{H} ($\deg \mathfrak{H} \stackrel{\text{def}}{=} m$). If $\det h_m \neq 0$, then \mathfrak{H} is called *nondegenerate*. We assume that $\deg 0 = -\infty$, cf. e.g. [2].

For $\mathfrak{A} = \sum_{i=-\infty}^k a_i D^i$ define [16, 2] its formal adjoint $\mathfrak{A}^\dagger \stackrel{\text{def}}{=} \sum_{i=-\infty}^k (-D)^i \circ a_i^T$. Here \circ stands for the multiplication generalizing the well-known Leibniz rule of calculus. For monomials it reads

$$aD^i \circ bD^j = a \sum_{q=0}^{\infty} \frac{i(i-1) \cdots (i-q+1)}{q!} D^q(b) D^{i+j-q}.$$

It is extended by linearity to the whole $\text{Mat}_p(\mathcal{A})[[D^{-1}]]$ and makes it into an algebra [2, 16]. The commutator $[\mathfrak{A}, \mathfrak{B}] = \mathfrak{A} \circ \mathfrak{B} - \mathfrak{B} \circ \mathfrak{A}$ further makes $\text{Mat}_p(\mathcal{A})[[D^{-1}]]$ into a Lie algebra. In the sequel we shall omit \circ unless this leads to confusion.

The directional derivative \vec{f}^\dagger of $\vec{f} \in \mathcal{A}^q$ is defined as (cf. [13])

$$\vec{f}^\dagger = \sum_{\beta, \kappa=1}^c \frac{\partial \vec{f}}{\partial \omega_\beta} ((D - W)^{-1})_{\beta\kappa} \circ \sum_{j=0}^h \partial X_\kappa / \partial \mathbf{u}_j \circ D^j + \sum_{i=0}^{\infty} \frac{\partial \vec{f}}{\partial \mathbf{u}_i} D^i.$$

Here W is a $c \times c$ matrix with the entries $\partial X_\alpha / \partial \omega_\beta$, and $(D - W)^{-1} = D^{-1} \circ (\mathbb{I} - W \circ D^{-1})^{-1} = \mathbb{I}D^{-1} + D^{-1} \circ W \circ D^{-1} + \dots \equiv \mathbb{I}D^{-1} + \sum_{j=-\infty}^{-2} W_j D^j$, where W_j are $c \times c$ matrices with entries from \mathcal{A} and \mathbb{I} denotes a $c \times c$ unit matrix. In other words, for the entries of $(D - W)^{-1}$ we have

$$((D - W)^{-1})_{\alpha\beta} \equiv \delta_{\alpha\beta} D^{-1} + \sum_{j=-\infty}^{-2} (W_j)_{\alpha\beta} D^j,$$

where $(W_j)_{\alpha\beta}$ is the (α, β) -th entry of the matrix W_j , and $\delta_{\alpha\beta}$ is Kronecker delta.

Clearly, for $\mathbf{K} \in \mathcal{A}^s$ we have $\mathbf{K}' \in \text{Mat}_s(\mathcal{A})[[D^{-1}]]$, so we can define the *formal order* of $\mathbf{G} \in \mathcal{A}^s$ as $\text{ford } \mathbf{G} \stackrel{\text{def}}{=} \text{deg } \mathbf{G}'$.

Next, define [1, 2] the *Lie bracket* for $\mathbf{K}, \mathbf{H} \in \mathcal{A}^s$ as $[\mathbf{K}, \mathbf{H}] = \mathbf{H}'[\mathbf{K}] - \mathbf{K}'[\mathbf{H}]$. It is skew-symmetric, but in general for $\mathbf{K}, \mathbf{H} \in \mathcal{A}^s$ the Lie bracket $[\mathbf{K}, \mathbf{H}]$ is not obliged to belong to \mathcal{A}^s unless $\mathcal{A} = \mathcal{A}_{\text{loc}}$, where \mathcal{A}_{loc} is a subalgebra of *local* (that is, independent of $\vec{\omega}$) functions in \mathcal{A} . Note that the restriction of the Lie bracket $[\cdot, \cdot]$ to $\mathcal{A}_{\text{loc}}^s$ satisfies the Jacobi identity, see e.g. [2].

Following [1, 2], we say that $\mathbf{G} \in \mathcal{A}^s$ is a *symmetry* for (1)–(3), if

$$\partial \mathbf{G} / \partial t + [\mathbf{F}, \mathbf{G}] = 0, \quad (4)$$

or equivalently [14]

$$D_t(\mathbf{G}) - \mathbf{F}'[\mathbf{G}] = 0.$$

Let $S_F^{(k)}(\mathcal{A}) = \{\mathbf{G} \in \mathcal{A}^s \mid \frac{\partial \mathbf{G}}{\partial t} + [\mathbf{F}, \mathbf{G}] = 0 \text{ and } \text{ford } \mathbf{G} \leq k\}$, $S_F(\mathcal{A}) = \bigcup_{j=0}^{\infty} S_F^{(j)}(\mathcal{A})$, $\text{Ann}_F(\mathcal{A}) = \{\mathbf{G} \in S_F(\mathcal{A}) \mid \partial \mathbf{G} / \partial t = 0\}$, and $\mathbf{F}' \equiv \sum_{i=-\infty}^n \phi_i D^i$, where $n = \text{ford } \mathbf{F}$. Set

$$n_0 = \begin{cases} 1 - j, & \text{if } \phi'_i = 0 \text{ for } i = n - j, \dots, n, \\ 2 & \text{otherwise.} \end{cases}$$

A *formal conservation law* of rank m for (1)–(3) is [16, 2] a formal series $\mathcal{L} \in \text{Mat}_s(\mathcal{A})[[D^{-1}]]$ such that

$$\text{deg}(D_t(\mathcal{L}) + \mathcal{L} \circ \mathbf{F}' + (\mathbf{F}')^\dagger \circ \mathcal{L}) \leq \text{deg } \mathbf{F}' + \text{deg } \mathcal{L} - m. \quad (5)$$

Likewise, a *formal symmetry* of rank m for (1)–(3) is [16, 2] a formal series $\mathcal{L} \in \text{Mat}_s(\mathcal{A})[[D^{-1}]]$ such that

$$\text{deg}(D_t(\mathcal{L}) - [\mathbf{F}', \mathcal{L}]) \leq \text{deg } \mathbf{F}' + \text{deg } \mathcal{L} - m. \quad (6)$$

Recall [5, 1] that $\gamma \in \mathcal{A}^s$ is called a *cosymmetry* (or a *conserved covariant*) for (1)–(3) if

$$D_t(\gamma) + (\mathbf{F}')^\dagger(\gamma) = 0. \quad (7)$$

Set $CS_F(\mathcal{A}) = \{\gamma \in \mathcal{A}^s \mid D_t(\gamma) + (\mathbf{F}')^\dagger(\gamma) = 0\}$ and $SCS_F(\mathcal{A}) = \{\gamma \in CS_F(\mathcal{A}) \mid \partial \gamma / \partial t = 0\}$.

If $\text{ford } \gamma \equiv \text{deg } \gamma' = r \geq n_0$, then γ' is easily seen to be a formal conservation law of rank $r - n_0 + 2$. Likewise, if \mathbf{G} is a symmetry of (1)–(3) of formal order $k \geq n_0$, then \mathbf{G}' is a formal symmetry of rank $k - n_0 + 2$, cf. [16, 17].

Define (see e.g. [2, 3]) the operator of variational derivative

$$\delta / \delta \mathbf{u} = \sum_{j=0}^{\infty} (-D)^j \frac{\partial}{\partial \mathbf{u}_j}.$$

If $\rho \in \mathcal{A}_{\text{loc}}$ is a conserved density for (1)–(3), that is, $D_t(\rho) = D(\sigma)$ for some $\sigma \in \mathcal{A}_{\text{loc}}$, then $\gamma = \delta \rho / \delta \mathbf{u}$ is a cosymmetry for (1)–(3), cf. e.g. [5, 16].

2 NWD systems

Let Φ denote the leading coefficient of \mathbf{F}' (so $\mathbf{F}' \equiv \Phi D^n + \dots$). Suppose that $n = \deg \mathbf{F}' \geq 2$ and Φ is diagonalizable, i.e., there exists a matrix Γ such that $\Lambda = \Gamma\Phi\Gamma^{-1}$ is diagonal, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_s)$. Further assume that $\det \Phi \neq 0$ and $\lambda_i \neq \lambda_j$ if $i \neq j$. Following [17], we shall call the systems (1)–(3) that enjoy these properties *nondegenerate weakly diagonalizable (NWD)*, because for such systems there exists [16, 17] a unique formal series

$$\mathfrak{T} = \Gamma + \sum_{j=-1}^{-\infty} \Gamma_j \Gamma D^j \in \text{Mat}_s(\mathcal{A})[[D^{-1}]]$$

with the property $\text{diag } \Gamma_j = 0$, $j = -1, -2, \dots$, such that all coefficients of the formal series $\mathfrak{W} = \mathfrak{T}\mathbf{F}'\mathfrak{T}^{-1} + D_t(\mathfrak{T})\mathfrak{T}^{-1}$ are diagonal matrices. Notice that any system (1)–(3) with $s = 1$ is (trivially) NWD with $\mathfrak{T} = 1$.

We have $\mathfrak{W} \equiv \Lambda D^n + \Psi D^{n-1} + \dots$, where Ψ is a diagonal matrix by construction, so $\Psi \equiv \text{diag}(\psi_1, \dots, \psi_s)$, $\psi_i \in \mathcal{A}$. For i such that $\psi_i/\lambda_i \in \text{Im } D$ let $z_i \in \mathcal{A}$ be solutions of the equations $D(z_i) = \psi_i/\lambda_i$.

Lemma 1. *Let (1)–(3) be an NWD system with odd $n \equiv \deg \mathbf{F}' \geq 3$. Then any its formal conservation law \mathfrak{L} of degree k and of rank $m \geq 2$ can be written in the form*

$$\mathfrak{L} = \mathfrak{T}^\dagger \circ (b_k \Lambda^{k/n} D^k + d_k D^{k-1}) \circ \mathfrak{T} + \mathfrak{N}, \quad \deg \mathfrak{N} < k - 1. \quad (8)$$

Here $\Lambda^{k/n} = \text{diag}(\lambda_1^{k/n}, \dots, \lambda_s^{k/n})$, $b_k = \text{diag}(c_1(t) \exp(2z_1/n), \dots, c_s(t) \exp(2z_s/n))$, d_k is a diagonal $s \times s$ matrix; $c_i(t) = 0$ if $\psi_i/\lambda_i \notin \text{Im } D$.

Proof. Set $\tilde{\mathfrak{L}} = (\mathfrak{T}^\dagger)^{-1} \circ \mathfrak{L} \circ \mathfrak{T}^{-1}$. Then equation (5) is [16] equivalent to the following:

$$\deg(D_t(\tilde{\mathfrak{L}}) + \tilde{\mathfrak{L}} \circ \mathfrak{W} + \mathfrak{W}^\dagger \circ \tilde{\mathfrak{L}}) \leq \deg \mathbf{F}' + \deg \tilde{\mathfrak{L}} - m. \quad (9)$$

Let $\tilde{\mathfrak{L}} \equiv L_k D^k + L_{k-1} D^{k-1} + \dots$. Then equating to zero the coefficient at D^{n+k} in (9) yields $[L_k, \Lambda] = 0$, hence L_k is diagonal: $L_k = \text{diag}(h_1, \dots, h_s)$.

Diagonal entries of the matrix $[L_{k-1}, \Lambda]$ are zeros, so equating to zero the diagonal part of the coefficient at D^{m+k-1} in (9) yields

$$kh_i D(\lambda_i) - n\lambda_i D(h_i) + 2h_i \psi_i = 0, \quad i = 1, \dots, s.$$

Solving these equations yields $h_i = c_i(t) \lambda_i^{k/n} \exp(2z_i/n)$, where c_i are arbitrary functions of t ; if $\psi_i/\lambda_i \notin \text{Im } D$ for some i , then we have $c_i = 0$ for these values of i .

Finally, equating to zero the antidiagonal part of the coefficient at D^{m+k-1} in (9) yields $[L_{k-1}, \Lambda] = 0$, so L_{k-1} is a diagonal matrix. ■

Notice that using (8) for $\mathfrak{L} = \gamma'$ one can readily find the form of leading terms for cosymmetries γ of formal order $q \geq n_0$. This complements known results (cf. e.g. [16, 18, 19]) and can be very useful e.g. while searching for cosymmetries and conserved densities for (1)–(3).

Remark 1. If $\psi_i/\lambda_i \notin \text{Im } D$ for all $i = 1, \dots, s$, then the system (1)–(3) has no formal conservation laws from $\text{Mat}_s(\mathcal{A})[[D^{-1}]]$ of rank $m \geq 2$ [16]. However, if (1)–(3) has a nondegenerate formal symmetry of rank not lower than $n + 2$, then [16] $\chi_i \equiv \psi_i/\lambda_i$ satisfy $D_t(\chi_i) = D(\xi_i)$, where $\xi_i \in \mathcal{A}$, for all $i = 1, \dots, c$, and we can proceed as follows.

Let I_0 be the set of all $i \in \{1, \dots, c\}$ such that $\chi_i \notin \text{Im } D$. Next, let I be a maximal subset of I_0 such that χ_i for $i \in I$ are linearly independent. Introduce additional nonlocal variables $\tilde{\omega}_i$, $i \in I$, such that $\partial \tilde{\omega}_i / \partial t = \xi_i$ and $\partial \tilde{\omega}_i / \partial x = \chi_i$. Then $\chi_i \in \text{Im } D$, as $\chi_i = D(\tilde{\omega}_i)$, so we can set $z_i = \tilde{\omega}_i$, and moreover [20], $\ker D$ is still exhausted by functions of t , so Lemma 1 remains valid in this extended setting.

3 Homogeneous cosymmetries of NWD systems

Following the literature (cf. e.g. [5,1,7]), define the Lie derivatives along $\mathbf{G} \in \mathcal{A}^s$ by the following formulae: a) $L_{\mathbf{G}}(f) = f'[\mathbf{G}]$ for $f \in \mathcal{A}$, b) $L_{\mathbf{G}}(\mathbf{H}) = [\mathbf{G}, \mathbf{H}]$ for a symmetry \mathbf{H} , c) $L_{\mathbf{G}}(\gamma) = \gamma'[\mathbf{G}] + (\mathbf{G}')^\dagger(\gamma)$ for a cosymmetry γ .

Let (1)–(3) be an NWD system such that $\partial X_\kappa/\partial t = \partial T_\kappa/\partial t = 0$ for all $\kappa = 1, \dots, c$. Consider a scaling vector field $\mathbf{S} = x\mathbf{u}_1 + \beta\mathbf{u}$, where β is a constant. A function $f \in \mathcal{A}$ such that $\partial f/\partial t = 0$ (resp. some other quantity \mathcal{X} (a symmetry, a cosymmetry etc.) such that $\partial \mathcal{X}/\partial t = 0$) is called *\mathbf{S} -homogeneous of weight μ* if $L_{\mathbf{S}}(f) - xD(f) = \mu f$ (resp. $L_{\mathbf{S}}(\mathcal{X}) = \mu \mathcal{X}$) for some constant μ . We write this as $\text{wt } f = \mu$ (resp. $\text{wt } \mathcal{X} = \mu$), cf. e.g. [7,8].

Motivated by the prototypic example of the Korteweg–de Vries equation $u_t = u_3 + uu_1$, let us introduce the following

Definition 1. Let (1)–(3) be an NWD system such that a) $\partial X_\kappa/\partial t = \partial T_\kappa/\partial t = 0$ for all $\kappa = 1, \dots, c$; b) $n \equiv \deg \mathbf{F}'$ is odd and $n \geq 3$; c) $\partial \Phi/\partial t = 0$ (recall that Φ is the leading coefficient of \mathbf{F}'). Further assume that there exists a scaling $\mathbf{S} = x\mathbf{u}_1 + \beta\mathbf{u}$, where β is a constant, such that a) all entries of Φ are \mathbf{S} -homogeneous, all of them have the same weight $\tilde{\alpha}$, and $\tilde{\alpha} \neq -n$; b) it is possible to choose z_i , introduced in the previous section, in such a way that $\partial z_i/\partial t = 0$, $\exp(2z_i/n)$ are \mathbf{S} -homogeneous, and $\text{wt}(\exp(2z_i/n)) = \text{wt}(\exp(2z_j/n))$ for all $i, j = 1, \dots, s$. Then we shall call the system (1)–(3) *weakly KdV-like* with the scaling \mathbf{S} .

Remark 2. Let $\zeta \in SCS_F(\mathcal{A})$ be an \mathbf{S} -homogeneous time-independent cosymmetry of a weakly KdV-like system (1)–(3), and $k \equiv \deg \zeta' \geq n_0$. Then it is immediate from (8) that

$$\text{wt}(\zeta) = \text{wt}(\exp(2z_1/n)) + ((\tilde{\alpha} + n)/n)k + 2\beta - 1. \quad (10)$$

Note that this formula is very useful on its own, for instance, for finding nonlocal parts of recursion operators and master symmetries for weakly KdV-like systems, cf. [7,8].

Lemma 2. Assume that (1)–(3) is weakly KdV-like with a scaling \mathbf{S} . Let \mathcal{L} be a subspace of $SCS_F(\mathcal{A})$, and $\mathcal{M}_{\mathcal{L}} = \{\zeta \in \mathcal{L} \mid \deg \zeta' < n_0 \text{ and } \zeta \text{ is } \mathbf{S}\text{-homogeneous}\}$.

Then $L_{\mathbf{P}}(\gamma) = 0$ for any \mathbf{S} -homogeneous $\mathbf{P} \in \text{Ann}_F(\mathcal{A})$ and any \mathbf{S} -homogeneous $\gamma \in SCS_F(\mathcal{A})$ such that $L_{\mathbf{P}}(\gamma) \in \mathcal{L}$, $p \equiv \text{ford } \mathbf{P} \geq n_0$, $\text{wt } \zeta \neq p(\tilde{\alpha} + n)/n + \text{wt } \gamma$ for all $\zeta \in \mathcal{M}_{\mathcal{L}}$, $\zeta \neq 0$, and either p is odd and $\deg \gamma' \geq n_0$ or there exists no integer j such that $j = p + n(\text{wt } \gamma - 2\beta + 1 - \text{wt}(\exp(2z_1/n)))/(\tilde{\alpha} + n)$ and $n_0 \leq j \leq \max(p + n_0 - 2, p + \deg \gamma')$.

Proof. Consider first the case when p is odd and $\deg \gamma' \geq n_0$. Using (10) and the formula $\text{wt}(\mathbf{P}) = p(\tilde{\alpha} + n)/n$ from [17], we obtain $\text{wt } L_{\mathbf{P}}(\gamma) = \text{wt}(\mathbf{P}) + \text{wt}(\gamma) = \text{wt}(\exp(2z_1/n)) + 2\beta - 1 + (p + \deg \gamma')(\tilde{\alpha} + n)/n$. On the other hand, as p is odd and $\deg \gamma' \geq n_0$, a straightforward computation with usage of (8) shows that $\deg(L_{\mathbf{P}}(\gamma))' \leq p + \deg \gamma' - 1$. Along with our assumption that $L_{\mathbf{P}}(\gamma) \in \mathcal{L}$, this implies that $\deg(L_{\mathbf{P}}(\gamma))' < n_0$, and thus $L_{\mathbf{P}}(\gamma) \in \mathcal{M}_{\mathcal{L}}$. Indeed, otherwise (10) yields $\text{wt } L_{\mathbf{P}}(\gamma) = \text{wt}(\exp(2z_1/n)) + 2\beta - 1 + \deg(L_{\mathbf{P}}(\gamma))'(\tilde{\alpha} + n)/n \neq \text{wt}(\exp(2z_1/n)) + 2\beta - 1 + (p + \deg \gamma')(\tilde{\alpha} + n)/n$. Thus, $L_{\mathbf{P}}(\gamma) \in \mathcal{M}_{\mathcal{L}}$, but $\text{wt } \zeta \neq \text{wt } L_{\mathbf{P}}(\gamma) = \text{wt}(\exp(2z_1/n)) + 2\beta - 1 + (p + \deg \gamma')(\tilde{\alpha} + n)/n$ for all $\zeta \in \mathcal{M}_{\mathcal{L}}$, $\zeta \neq 0$, and hence $L_{\mathbf{P}}(\gamma) = 0$.

Likewise, if there is no integer j such that $j = p + n(\text{wt } \gamma - 2\beta + 1 - \text{wt}(\exp(2z_1/n)))/(\tilde{\alpha} + n)$ and $n_0 \leq j \leq \max(p + n_0 - 2, p + \deg \gamma')$, then there exists no \mathbf{S} -homogeneous cosymmetry $\zeta \in \mathcal{L}$ with $\deg \zeta' \geq n_0$ such that $\text{wt } L_{\mathbf{P}}(\gamma) = \text{wt } \zeta$, so $L_{\mathbf{P}}(\gamma) \in \mathcal{M}_{\mathcal{L}}$, and proceeding as above we conclude that $L_{\mathbf{P}}(\gamma) = 0$. \blacksquare

4 Applications to recursion operators and master symmetries

Recall [5, 2, 1] that $\mathfrak{R} \in \text{Mat}_s(\mathcal{A})[[D^{-1}]]$ is called a (*adjoint*) *recursion operator* for (1)–(3), if $D_t(\mathfrak{R}) - [\mathbf{F}', \mathfrak{R}] = 0$ (resp. $D_t(\mathfrak{R}) + [(\mathbf{F}')^\dagger, \mathfrak{R}] = 0$), and a (*inverse*) *Noether operator*, if $D_t(\mathfrak{R}) - \mathfrak{R} \circ (\mathbf{F}')^\dagger - \mathbf{F}' \circ \mathfrak{R} = 0$ (resp. $D_t(\mathfrak{R}) + \mathfrak{R} \circ \mathbf{F}' + (\mathbf{F}')^\dagger \circ \mathfrak{R} = 0$).

Consider an operator of the form

$$\mathfrak{R} = \sum_{i=0}^r a_i D^i + \sum_{\kappa=1}^k \mathbf{G}_\kappa \otimes D^{-1} \circ \gamma_\kappa, \tag{11}$$

where $r \geq 0$, a_i are $s \times s$ matrices with local entries, $\mathbf{G}_\kappa \in \mathcal{A}_{\text{loc}}^s$, $\gamma_\kappa = \delta\rho_\kappa/\delta\mathbf{u}$, and $\rho_\kappa \in \mathcal{A}_{\text{loc}}$ are local conserved densities for (1)–(3), i.e., $D_t(\rho_\kappa) = D(\sigma_\kappa)$ for some $\sigma_\kappa \in \mathcal{A}_{\text{loc}}$.

Proposition 1. *Suppose that a weakly KdV-like system (1)–(3) with a scaling \mathbf{S} has a recursion (resp. inverse Noether) operator of the form (11) with $\gamma_\kappa \in \text{SCSF}(\mathcal{A}_{\text{loc}})$, and the requirements of Lemma 2 are met for a symmetry $\mathbf{P} \in \text{Ann}_F(\mathcal{A}_{\text{loc}})$ and for all cosymmetries γ_κ , $\kappa = 1, \dots, k$.*

Then $\mathfrak{R}(\mathbf{P}) \in S_F(\mathcal{A}_{\text{loc}})$ (resp. $\mathfrak{R}(\mathbf{P}) \in \text{CS}_F(\mathcal{A}_{\text{loc}})$), i.e., $\mathfrak{R}(\mathbf{P})$ is again a local symmetry (resp. cosymmetry) for (1)–(3).

Proof. We have to prove that $\gamma_\kappa \cdot \mathbf{P} = D(\eta_\kappa)$ for some local η_κ , or equivalently [2] that $\delta(\gamma_\kappa \cdot \mathbf{P})/\delta\mathbf{u} = (\gamma'_\kappa)^\dagger(\mathbf{P}) + (\mathbf{P}')^\dagger(\gamma_\kappa) = 0$. But by Lemma 2 $L_{\mathbf{P}}(\gamma_\kappa) = 0$, so $(\mathbf{P}')^\dagger(\gamma_\kappa) = -\gamma'_\kappa[\mathbf{P}]$, whence $\delta(\gamma_\kappa \cdot \mathbf{P})/\delta\mathbf{u} = ((\gamma'_\kappa)^\dagger - \gamma'_\kappa)(\mathbf{P})$. As $\gamma_\kappa = \delta\rho_\kappa/\delta\mathbf{u}$ implies (see e.g. [5]) $(\gamma'_\kappa)^\dagger = \gamma'_\kappa$, we have $\delta(\gamma_\kappa \cdot \mathbf{P})/\delta\mathbf{u} = 0$, as desired. ■

Consider now an operator of the form

$$\mathfrak{N} = \sum_{i=0}^r b_i D^i + \sum_{\kappa=1}^k \gamma_\kappa \otimes D^{-1} \circ \mathbf{G}_\kappa, \tag{12}$$

where $r \geq 0$, b_i are $s \times s$ matrices with local entries, $\gamma_\kappa \in \mathcal{A}_{\text{loc}}^s$, $\mathbf{G}_\kappa \in \text{Ann}_F(\mathcal{A}_{\text{loc}})$ are time-independent local symmetries for (1)–(3). In complete analogy with Proposition 1 we obtain the following result.

Proposition 2. *Let a weakly KdV-like system (1)–(3) with a scaling \mathbf{S} have an adjoint recursion (resp. Noether) operator \mathfrak{N} of the form (12). Assume that the requirements of Lemma 2 are met for a local time-independent cosymmetry $\zeta \in \text{SCSF}(\mathcal{A}_{\text{loc}})$ and for all symmetries \mathbf{G}_κ , $\kappa = 1, \dots, k$, and let $\zeta = \delta\chi/\delta\mathbf{u}$ for some local function χ .*

Then $\mathfrak{N}(\zeta)$ is again a local cosymmetry (resp. symmetry) for (1)–(3).

Now let us turn to the case when (1)–(3) has a master symmetry of degree one or equivalently [6] a symmetry linear in time t .

Proposition 3. *Let a weakly KdV-like system (1)–(3) with a scaling \mathbf{S} be such that $\partial\mathbf{F}/\partial t = 0$, $\partial\mathbf{F}/\partial\omega_\kappa = 0$ and $X_\kappa, T_\kappa \in \mathcal{A}_{\text{loc}}$ for all $\kappa = 1, \dots, c$. Further suppose that (1)–(3) has a symmetry $\mathbf{K} = \boldsymbol{\tau} + t[\boldsymbol{\tau}, \mathbf{F}] \in S_F(\mathcal{A})$, where $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \sum_{\kappa=1}^c \mathbf{W}_\kappa \omega_\kappa$, $\boldsymbol{\tau}_0 \in \mathcal{A}_{\text{loc}}^s$, $\partial\boldsymbol{\tau}_0/\partial t = 0$, and $\mathbf{W}_\kappa \in \text{Ann}_F(\mathcal{A}_{\text{loc}})$ for all $\kappa = 1, \dots, c$. Let the requirements of Lemma 2 be met for a local time-independent symmetry $\mathbf{P} \in \text{Ann}_F(\mathcal{A}_{\text{loc}})$ and for all cosymmetries $\gamma_\kappa = \delta X_\kappa/\delta\mathbf{u} \in \text{SCSF}(\mathcal{A}_{\text{loc}})$ with $\kappa \in \{1, \dots, c\}$ such that $\mathbf{W}_\kappa \neq 0$. Finally, suppose that $[\mathbf{P}, \mathbf{W}_\kappa] = 0$ for all $\kappa = 1, \dots, c$.*

Then $[\mathbf{K}, \mathbf{P}]$ is again a local symmetry for (1)–(3).

Proof. By virtue of the assumptions made $[\boldsymbol{\tau}, \mathbf{F}]$ is local [11], and hence so is $[[\boldsymbol{\tau}, \mathbf{F}], \mathbf{P}]$. Since $[\mathbf{P}, \mathbf{W}_\kappa] = 0$, it remains to prove that $\omega'_\kappa[\mathbf{P}]$ are local or, equivalently, that $\delta(\gamma_\kappa \cdot \mathbf{P})/\delta\mathbf{u} = 0$ for all $\kappa = 1, \dots, c$ such that $\mathbf{W}_\kappa \neq 0$. This is done as in the proof of Proposition 1 above. ■

Let $\text{ad}_{\mathbf{H}}(\mathbf{G}) \equiv [\mathbf{H}, \mathbf{G}]$. Set $\mathbf{Q}_j = \text{ad}_{\mathbf{K}}^j(\mathbf{P})$. Clearly, under minor technical assumptions the repeated application of Proposition 1 (resp. Proposition 3) enables one to prove locality for all symmetries $\mathfrak{R}^j(\mathbf{P})$ (resp. \mathbf{Q}_j), $j = 1, 2, \dots$, where \mathfrak{R} is a recursion operator for (1)–(3). In particular, the following assertions hold.

Corollary 1. *Under the assumptions of Proposition 1 suppose that \mathfrak{R} is a recursion operator for (1)–(3), r is even, the coefficients of \mathfrak{R} are time-independent, and \mathfrak{R} is \mathbf{S} -homogeneous of some weight μ , i.e., $L_{\mathbf{S}}(\mathfrak{R}) \equiv \mathfrak{R}'[\mathbf{S}] - [\mathbf{S}', \mathfrak{R}] = \mu\mathfrak{R}$. Further assume that $p \equiv \text{ford } \mathbf{P}$ is odd, and for any $j \in \mathbb{N}$ and all $\kappa = 1, \dots, k$ $L_{\mathfrak{R}^j(\mathbf{P})}(\gamma_\kappa) \in \mathcal{L}$, $\text{deg } \gamma_\kappa \geq n_0$, and $\text{wt } \zeta \neq (p + rj + \text{deg } \gamma'_\kappa)(\tilde{\alpha} + n)/n + \text{wt}(\exp(2z_1/n)) + 2\beta - 1$ for all $\zeta \in \mathcal{M}_{\mathcal{L}}$, $\zeta \neq 0$.*

Then the symmetries $\mathfrak{R}^j(\mathbf{P})$ are local for all $j = 1, 2, 3, \dots$.

Corollary 2. *Under the assumptions of Proposition 3 suppose that $q - n$ is even, τ is \mathbf{S} -homogeneous, i.e., $[\mathbf{S}, \tau] = \mu\tau$ for some constant μ , $q = \text{ford}[\tau, \mathbf{F}] > \max(\text{ford } \tau, n)$, $p \equiv \text{ford } \mathbf{P} > n$, p is odd, $[\mathbf{Q}_j, \mathbf{W}_\kappa] = 0$ and $[\mathbf{Q}_j, [\tau, \mathbf{F}]] = 0$ for all $\kappa = 1, \dots, c$ and all $j = 0, 1, \dots, b - 1$. Further assume that for all $j = 0, \dots, b - 1$ and all $\kappa = 1, \dots, c$ such that $\mathbf{W}_\kappa \neq 0$ we have $L_{\mathbf{Q}_j}(\gamma_\kappa) \in \mathcal{L}$, $\text{deg } \gamma_\kappa \geq n_0$, and $\text{wt } \zeta \neq (p + (q - n)j + \text{deg } \gamma'_\kappa)(\tilde{\alpha} + n)/n + \text{wt}(\exp(2z_1/n)) + 2\beta - 1$ for all $\zeta \in \mathcal{M}_{\mathcal{L}}$, $\zeta \neq 0$.*

Then $\mathbf{Q}_j = \text{ad}_{\mathbf{K}}^j(\mathbf{P})$ are local for all $j = 1, 2, \dots, b$.

Thus, proving locality for the hierarchies generated using master symmetries is slightly more complicated, as we must proceed inductively and make sure that the conditions of Corollary 2 are satisfied for all $b = 1, 2, 3, \dots$. However, this considerably simplifies verification of the relations $[\mathbf{Q}_j, \mathbf{W}_\kappa] = 0$ and $[\mathbf{Q}_j, [\tau, \mathbf{F}]] = 0$, because we can use the information on locality obtained at the previous step. There are various ways of proving these relations, see e.g. Proposition 5 of [17] for a scaling-based approach. Note that the conditions $q = \text{ford}[\tau, \mathbf{F}] > \max(\text{ford } \tau, n)$ and $p > n$ in Corollary 2 ensure [17] that $\text{ford } \mathbf{Q}_{j+1} > \text{ford } \mathbf{Q}_j$.

5 Examples

Example 1. Consider the Hirota–Satsuma system (see [21, 7] and references therein)

$$u_t = (1/2)u_3 + 3uu_1 - 6vv_1, \quad v_t = -v_3 - 3uv_1.$$

and its recursion operator

$$\mathfrak{R} = \begin{pmatrix} D^3/2 + D \circ u + uD & D \circ v + vD \\ D \circ v + vD & D^3/2 + D \circ u + uD \end{pmatrix} \circ \begin{pmatrix} D/2 + D^{-1} \circ u + uD^{-1} & -2D^{-1} \circ v \\ -2vD^{-1} & -2D \end{pmatrix}.$$

The nonlocal part of this operator can be written [7] as $\mathfrak{R}_- = \mathbf{G}_1 \otimes D^{-1} \circ \gamma_1 + \mathbf{G}_2 \otimes D^{-1} \circ \gamma_2$, where $\mathbf{G}_1 = ((1/2)u_3 + 3uu_1 - 6vv_1, -v_3 - 3uv_1)^T$, $\mathbf{G}_2 = (u_1, v_1)^T$, $\gamma_1 = (1, 0) = \delta\rho_1/\delta\mathbf{u}$ for $\rho_1 = u$, $\gamma_2 = (u, -2v) = \delta\rho_2/\delta\mathbf{u}$ for $\rho_2 = u^2/2 - v^2$. It is immediate that the system in question is weakly KdV-like with the scaling $\mathbf{S} = x\mathbf{u}_1 + 2\mathbf{u}$, and the requirements of Proposition 1 are met for any x, t -independent local \mathbf{S} -homogeneous generalized symmetry \mathbf{P} of odd formal order not lower than one, if we set $\mathcal{L} = \{\zeta \in SCS_F(\mathcal{A}_{\text{loc}}) | \partial\zeta/\partial x = 0\}$, and thus the symmetries $\mathfrak{R}^j(\mathbf{P})$ are local for all $j = 1, 2, \dots$. In particular, $\mathfrak{R}^j(\mathbf{G}_\kappa)$ are local for all $j \in \mathbb{N}$ and $\kappa = 1, 2$.

Example 2. Consider an NWD system with constraints made up from the Schwarzian Korteweg–de Vries equation $u_t = u_3 - (3/2)u_2^2/u_1$, see e.g. [3, 7] and references therein, and the equations defining the potential ω of the conserved density $\rho = (1/2)u_2^2/u_1^2$:

$$u_t = u_3 - (3/2)u_2^2/u_1, \\ \partial\omega/\partial t = u_2u_4/u_1^2 - (1/2)u_3^2/u_1^2 - 2u_2^2u_3/u_1^3 + (9/8)u_2^4/u_1^4, \quad \partial\omega/\partial x = (1/2)u_2^2/u_1^2.$$

It is weakly KdV-like with the scaling $\mathbf{S} = x\mathbf{u}_1$ and satisfies the requirements of Proposition 3 with \mathbf{P} being any x, t, u -independent local \mathbf{S} -homogeneous symmetry of odd formal order not lower than three, $\mathcal{L} = \{\zeta \in SCS_F(\mathcal{A}_{\text{loc}}) | \partial\zeta/\partial x = 0 \text{ and } \partial\zeta/\partial u = 0\}$, and master symmetry $\tau = u_2 + x(u_3 - (3/2)u_2^2/u_1) - u_1\omega$. Hence by Proposition 3 and Corollary 2 the symmetries $\text{ad}_{\mathbf{K}}^j(u_3 - (3/2)u_2^2/u_1) = \text{ad}_{\tau}^j(u_3 - (3/2)u_2^2/u_1)$ are local for all $j = 1, 2, \dots$

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