

Multi-Soliton Solutions with Discrete Symmetries in the Chiral Quark Soliton Model

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We discuss multi-soliton solutions with discrete symmetries in the chiral quark soliton model using the rational map ansatz. The solutions exhibit degenerate bound spectra of the quark orbits depending on the background pion field configurations. It is shown that resultant baryon densities inherit the same discrete symmetries as the chiral fields. Evaluating the radial component of the baryon density, shell-like structure of the valence quark spectra is also observed.

1 Introduction

The chiral quark soliton model (CQSM) was developed in 1980's as a low-energy effective theory of QCD. Since it includes the Dirac Sea quark contribution and explicit valence quark degrees of freedom, the model interpolates between the constituent quark model and the Skyrme model [1–4]. The CQSM is derived from the instanton liquid model of QCD vacuum and incorporates the non-perturbative feature of the low-energy QCD, spontaneous chiral symmetry breaking. It has been shown that the $B = 1$ solution provides correct observables such as a nucleon including mass, electromagnetic value, spin carried by quarks, parton distributions and octet SU(3) baryon spectra. For $B = 2$, the stable axially symmetric soliton solution was found in [5]. The solution exhibits doubly degenerate bound spectrum of the quark orbits in the background of axially symmetric chiral fields with winding number two. Upon quantization, various dibaryon spectra were obtained, showing that the quantum number of the ground state coincide with that of physical deuteron [6, 7]. For $B > 2$, the Skyrme model predicts that minimum energy solutions have only discrete, crystal-like symmetries [8–10]. According to this prediction, we studied the CQSM with $B = 3$ tetrahedrally symmetric chiral fields and obtained triply degenerate spectrum of the quark orbits [11]. Its large degeneracy indicates that the tetrahedrally symmetric solution may be the lowest-lying configuration. Thus, for $B > 3$, one can also expect that the lowest-lying solutions in the CQSM inherits the discrete symmetries predicted in the Skyrme model.

Following the $B = 3$ case, we shall study soliton solutions with $B \geq 3$ in the CQSM using the rational map ansatz with discrete symmetries obtained in the Skyrme model. We will show obtained classical self-consistent soliton solutions with $B = 1-9, 17$. These solutions exhibit various degenerate spectra of the quark orbits depending on the symmetry of the background chiral fields. Such degeneracy generates large shell gaps, which suggests that the solutions are stable local minima. It is shown that resultant baryon number densities inherit the same symmetries as the chiral fields. Evaluating the radial component of the baryon density, shell-like structure of the valence quarks can be observed.

2 The model

The CQSM is derived from the instanton liquid model of the QCD vacuum and incorporates the nonperturbative feature of the low-energy QCD, spontaneous chiral symmetry breaking. The

vacuum functional is defined by [1]

$$\mathcal{Z} = \int \mathcal{D}\pi \mathcal{D}\psi \mathcal{D}\psi^\dagger \exp \left[i \int d^4x \bar{\psi} (i\rlap{\not{\partial}} - MU^{\gamma_5}) \psi \right], \quad (1)$$

where the SU(2) matrix

$$U^{\gamma_5} = \frac{1 + \gamma_5}{2} U + \frac{1 - \gamma_5}{2} U^\dagger \quad \text{with} \quad U = \exp(i\boldsymbol{\tau} \cdot \boldsymbol{\pi} / f_\pi)$$

describes chiral fields, ψ is quark fields and M is the constituent quark mass. f_π is the pion decay constant and experimentally $f_\pi \sim 93$ MeV.

In the CQSM, the number of valence quark is associated with the baryon number such that a soliton with baryon number B consists of $N_c \times B$ valence quarks. If the correlation between quarks is sufficiently strong, their binding energy becomes large and the valence quarks cannot be observed as positive energy particles [12, 13]. Thus, one gets the picture of the topological soliton model in the sense that the baryon number coincides with the winding number of the background chiral field when the valence quarks occupy all the levels diving into negative energy region.

The vacuum functional in (1) can be integrated over the quark fields to obtain the effective action

$$S_{\text{eff}}[U] = -iN_c \text{Indet} (i\rlap{\not{\partial}} - MU^{\gamma_5}) \quad (2)$$

$$= -\frac{i}{2} N_c \text{Spln} D^\dagger D, \quad (3)$$

where $D = i\rlap{\not{\partial}} - MU^{\gamma_5}$. We introduce the eigenequation

$$H(U^{\gamma_5})\phi_\mu(\mathbf{x}) = E_\mu\phi_\mu(\mathbf{x}), \quad H(U^{\gamma_5}) = -i\boldsymbol{\alpha} \cdot \nabla + \beta MU^{\gamma_5}. \quad (4)$$

The effective action $S_{\text{eff}}(U)$ is ultraviolet divergent and hence must be regularized. Using the proper-time regularization scheme [14], we can write

$$\begin{aligned} S_{\text{eff}}^{\text{reg}}[U] &= \frac{i}{2} N_c \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{Sp} \left(e^{-D^\dagger D \tau} - e^{-D_0^\dagger D_0 \tau} \right) \\ &= \frac{i}{2} N_c T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{Sp} \left[e^{-\tau(H^2 + \omega^2)} - e^{-\tau(H_0^2 + \omega^2)} \right], \end{aligned} \quad (5)$$

where T is the Euclidean time separation, D_0 and H_0 are operators with $U = 1$.

At $T \rightarrow \infty$, we have $e^{iS_{\text{eff}}} \sim e^{-iE_{\text{field}}T}$. The total energy then is given by

$$E_{\text{static}}[U] = E_{\text{val}}[U] + E_{\text{field}}[U] - E_{\text{field}}[U = 1], \quad (6)$$

where

$$E_{\text{val}} = N_c \sum_i E_{\text{val}}^{(i)} \quad (7)$$

is the valence quark contribution with the valence energy of the i -th valence quark $E_{\text{val}}^{(i)}$, and the vacuum part is

$$E_{\text{field}} = N_c \sum_\mu \left\{ \mathcal{N}(E_\mu) |E_\mu| + \frac{\Lambda}{\sqrt{4\pi}} \exp \left[- \left(\frac{E_\mu}{\Lambda} \right)^2 \right] \right\}$$

with

$$\mathcal{N}(E_\mu) = -\frac{1}{\sqrt{4\pi}} \Gamma\left(\frac{1}{2}, \left(\frac{E_\mu}{\Lambda}\right)^2\right).$$

Λ is a cutoff parameter evaluated by the condition that the derivative expansion of (5) reproduces the pion kinetic term with the correct coefficient, *i.e.*,

$$f_\pi^2 = \frac{N_c M^2}{4\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} e^{-\tau M^2}. \quad (8)$$

The constituent quark mass M is only one free parameter and we adopt the value $M = 400$ MeV, which reproduces the observables of the nucleon and the delta. From (8), one obtains $\Lambda \sim 637$ MeV. The contribution for the total energy in the absence of the chiral fields ($U = 1$) can be estimated using the eigenstates of eigenequation,

$$H_0 \phi_\mu^{(0)}(\mathbf{x}) = E_\mu^{(0)} \phi_\mu^{(0)}(\mathbf{x}), \quad H_0 = -i\alpha \cdot \nabla + \beta M. \quad (9)$$

In the Skyrme model, it is known that solitons with $B \geq 3$ have particular discrete symmetries [8]. Therefore, we expect that soliton solutions of the CQSM inherit the same discrete symmetry as skyrmions. Houghton, Manton and Sutcliffe proposed remarkable ansatz for the chiral fields, the rational map ansatz [10]. According to this ansatz, the chiral fields are expressed in a rational map as

$$U(r, z) = \exp(iF(r)\hat{\mathbf{n}}_R \cdot \boldsymbol{\tau}), \quad \hat{\mathbf{n}}_R = \frac{1}{1 + |R(z)|^2} (2\text{Re}[R(z)], 2\text{Im}[R(z)], 1 - |R(z)|^2) \quad (10)$$

and $R(z)$ is the rational map. The complex coordinate z is given by $z = \tan(\theta/2)e^{i\varphi}$ via stereographic projection. Rational maps are maps from $CP(1)$ to $CP(1)$ (equivalently, from S^2 to S^2) classified by winding number. In [10], Manton *et al.* showed that $B = N$ skyrmions can be well-approximated by rational maps with winding number N . The rational map with winding number N possesses $(2N + 1)$ complex parameters whose values can be determined by imposing the symmetry of the skyrmion. We shall use this ansatz for the background chiral fields in the CQSM. We employ the explicit forms of $R(z)$ for various B proposed in [10] to our analysis. Since the chiral fields in (10) is parameterized by polar coordinates, one can apply the numerical technique developed for $B = 1$ to find solutions with higher B .

Field equations for the chiral fields can be obtained by demanding that the total energy in equation (6) be stationary with respect to variation of the profile function $F(r)$,

$$\frac{\delta}{\delta F(r)} E_{\text{static}} = 0,$$

which gives

$$S(r) \sin F(r) = P(r) \cos F(r), \quad (11)$$

where

$$S(r) = N_c \sum_{\mu} (n_{\mu} \theta(E_{\mu}) + \text{sign}(E_{\mu}) \mathcal{N}(E_{\mu})) \langle \mu | \gamma^0 \delta(|x| - r) | \mu \rangle, \quad (12)$$

$$P(r) = N_c \sum_{\mu} (n_{\mu} \theta(E_{\mu}) + \text{sign}(E_{\mu}) \mathcal{N}(E_{\mu})) \langle \mu | i\gamma^0 \gamma^5 \hat{\mathbf{n}}_R \cdot \boldsymbol{\tau} \delta(|x| - r) | \mu \rangle. \quad (13)$$

The procedure to obtain self-consistent solutions of equation (11) is that 1) solve the eigenequation in (4) under an assumed initial profile function $F_0(r)$, 2) use the resultant eigenfunctions and

eigenvalues to calculate $S(r)$ and $P(r)$, 3) solve equation (11) to obtain a new profile function, 4) repeat 1)–3) until the self-consistency is attained.

The baryon density $b(\mathbf{x})$ is defined by the zeroth component of the baryon current [2];

$$b(\mathbf{x}) = \frac{1}{N_c} \langle \bar{\psi} \gamma_0 \psi \rangle = b_{\text{val}}(\mathbf{x}) + b_{\text{field}}(\mathbf{x}), \quad (14)$$

where

$$b_{\text{val}}(\mathbf{x}) = \sum_i b_{\text{val}}^{(i)}(\mathbf{x}) = \frac{1}{N_c} \sum_i \phi_i(\mathbf{x})^\dagger \phi_i(\mathbf{x}),$$

$$b_{\text{field}}(\mathbf{x}) = \frac{1}{N_c} \sum_\mu \left[\text{sign}(E_\mu) \mathcal{N}(E_\mu) \phi_\mu(\mathbf{x})^\dagger \phi_\mu(\mathbf{x}) - \text{sign}(E_\mu^{(0)}) \mathcal{N}(E_\mu^{(0)}) \phi_\mu^{(0)}(\mathbf{x})^\dagger \phi_\mu^{(0)}(\mathbf{x}) \right]. \quad (15)$$

To examine the shell structure of the quarks, we evaluated the radial density for the i th valence quark $\rho^{(i)}(r)$ in which the angular degrees of freedom are integrated via,

$$\rho^{(i)}(r) = \int d\varphi \int \sin\theta d\theta b_{\text{val}}^{(i)}(r, \theta, \varphi) \quad (16)$$

with the baryon number

$$B = \sum_i \int dr r^2 \rho^{(i)}(r). \quad (17)$$

3 Results and discussions

Let us first show some results of the spectral flow analysis. For convenience, we shall take

$$F(r) = \begin{cases} -\pi + \pi r/X & \text{for } r < X, \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

as a trial function for the profile function. In Figs. 1, 2, we show the spectral flow with $B = 3, 7$. As can be seen, the number of B positive energy levels are diving into negative energy region and thus we obtain the baryon number B soliton solutions. In Table 1 there are the results for the valence quark levels as well as the vacuum sea contributions. The valence quark spectra show various degenerate patterns depending on the background configuration. The total energy indicates that all the solitons are deep bound states. From (15), we estimated the baryon number density (see Fig. 3). The density inherits the same symmetry as the corresponding skyrmion.

In Fig. 4, we display the valence quark spectra together with the results of $B = 1, 2$ [6]. It is interesting that the results strongly suggest the existence of shell structure for the valence quarks. The spectra show (i) four-fold degeneracy of the ground state labeled by \mathcal{G} and various degenerate pattern for excited levels labeled by $\mathcal{A}_1, \mathcal{A}_2, \dots$, (ii) a large energy gap between the ground state \mathcal{G} and the first excited level \mathcal{A}_1 . We suspect that these large degeneracy should contribute to the minimization of the total energy. Note that the small dispersions of the spectra observed here are caused by the finite size effect of the basis. Increasing the size r_{max} together with increase of the number of the basis, more accurate degeneracy will be attained.

The bunch of the valence spectra caused by symmetry have also been observed within the study of heavier nuclear system. As discussed in Ref. [10], the group theory should predict the level structure for pion fluctuation. However, our problem is more complicated due to the presence of quarks.

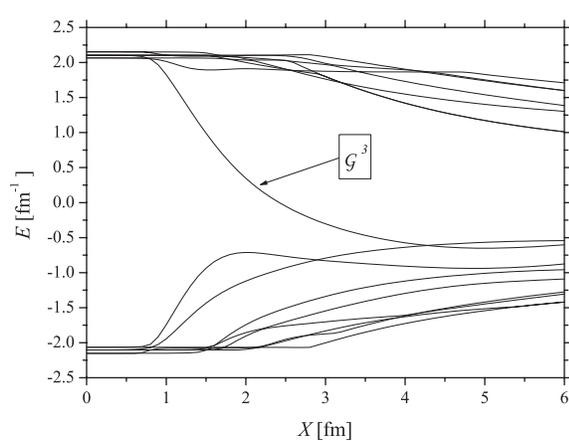


Figure 1. Spectral flow of $B = 3$ with the occupation number.

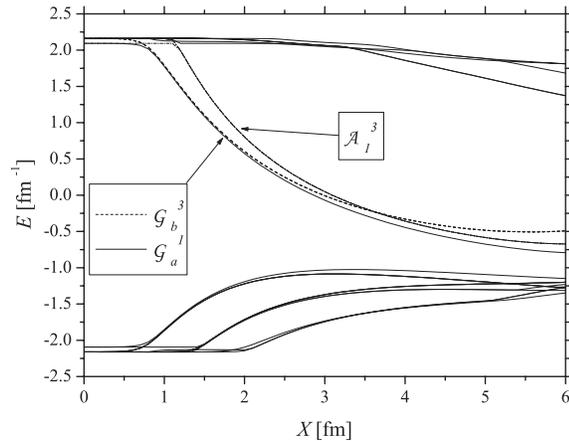


Figure 2. Spectral flow of $B = 7$ with the occupation number.

Table 1. Mass spectra for $B = 1-9, 17$ (in MeV). The data for $B = 1, 2$ are taken from Ref. [6]. The ratio of the mass E_{static} to $B \times E_{\text{static}}^{(B=1)}$ are compared to that of the Skyrme model [10].

B	$E_{\text{val}}^{(i)}$					E_{field}	E_{static}	$E_{\text{static}}/BE_{\text{static}}^{(B=1)}$					
								Ours	Skyrme				
1	173					674	1192	1.00	1.00				
2	173	173				1166	2204	0.92	0.95				
3	210	210	210			1633	3522	0.98	0.96				
4	144	146	146	146		2628	4378	0.92	0.92				
5	123	131	131	139	210	3265	5467	0.92	0.93				
6	120	124	150	150	206	3740	6603	0.92	0.92				
7	115	120	120	120	166	166	166	4554	7478	0.90	0.90		
8	97	97	115	120	139	139	203	203	5229	8565	0.90	0.91	
9	99	105	105	121	142	142	210	210	210	5700	9742	0.91	0.89
17	83	95	95	95	153	156	157	173	175	10586	18650	0.93	0.88
5*	177	178	179	192	194	194	196	196		2874	5680	0.95	1.00

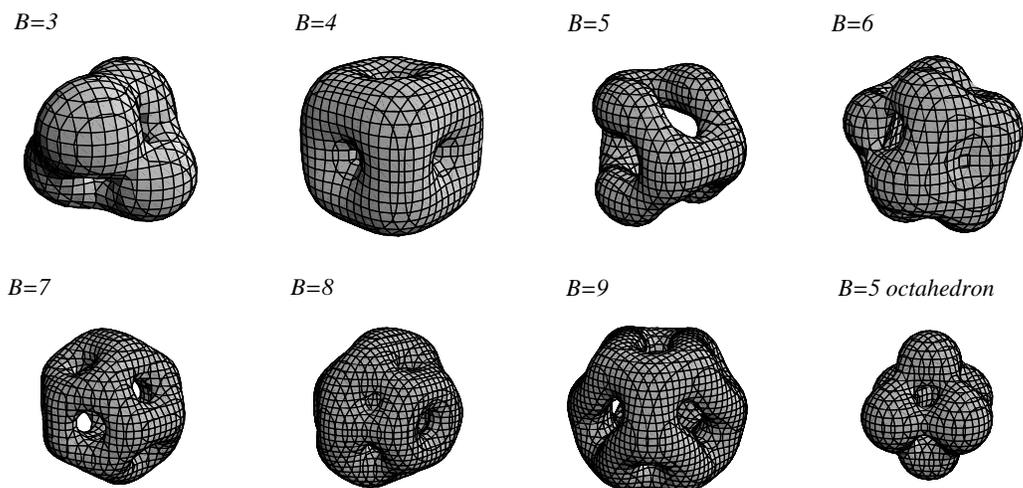


Figure 3. Baryon number densities.

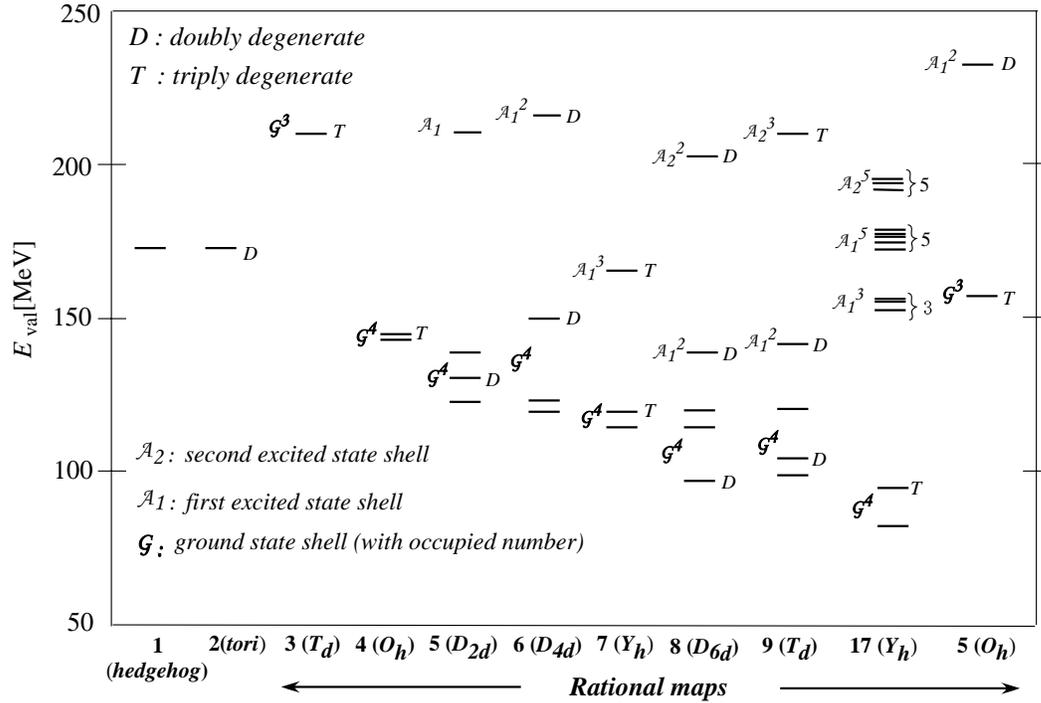


Figure 4. Valence quark spectra for $B = 1-9, 17$.

Let us show how the shell deformation is related to the degeneracy of the spectrum. In general, if eigenequation

$$H\psi_\mu = E_\mu\psi_\mu \quad (19)$$

is invariant under a symmetric operation $R \in G$, the equation transforms as

$$RH\psi_\mu = H(R\psi_\mu) = E_\mu\psi_\mu. \quad (20)$$

Thus the states $\{\psi_\mu, R\psi_\mu\}$ have the same energy E_μ . The set of d_μ eigenfunctions $\{\psi_i^{(\mu)}\}$ ($i = 1, \dots, d_\mu$) belonging to a given eigenvalue E_μ will provide the basis for an irreducible representation of the group G of the Hamiltonian [15]:

$$R\psi_j^{(\mu)} = \sum_i \psi_i^{(\mu)} D_{ij}^{(\mu)}(R). \quad (21)$$

The operator R are constructed as follows. If chiral fields have some particular point group symmetry i.e., $U(\mathbf{x}') = G(\mathbf{a})U(\mathbf{x})G(\mathbf{a})^\dagger$ ($G(\mathbf{a}) \in SU(2)_I$), the Dirac equation is invariant under the rotation

$$x' = ax \quad (22)$$

with

$$x' = (t, \mathbf{x}'), \quad a = (1, \mathbf{a}), \quad x = (t, \mathbf{x}), \quad (23)$$

accompanying the iso-rotation

$$(i\gamma^\mu\partial_\mu - MU(\mathbf{x}))\psi(x) = 0 \Rightarrow (i\gamma^\nu\partial'_\nu - MU(\mathbf{x}'))\psi'(x') = 0 \quad (24)$$

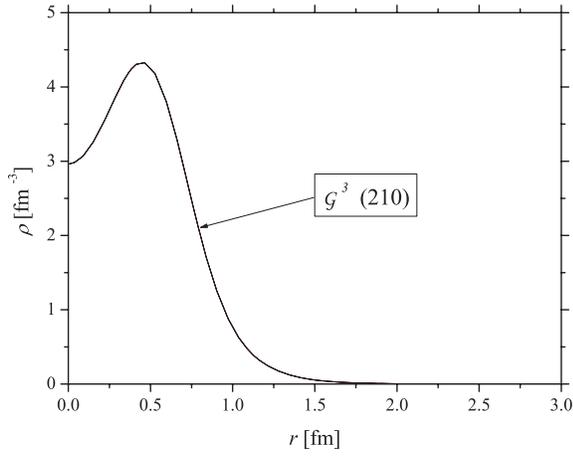


Figure 5. Baryon number density of i th valence quarks $\rho^{(i)}(r)$ of $B = 3$, with the occupation numbers and their eigenvalues (in MeV).

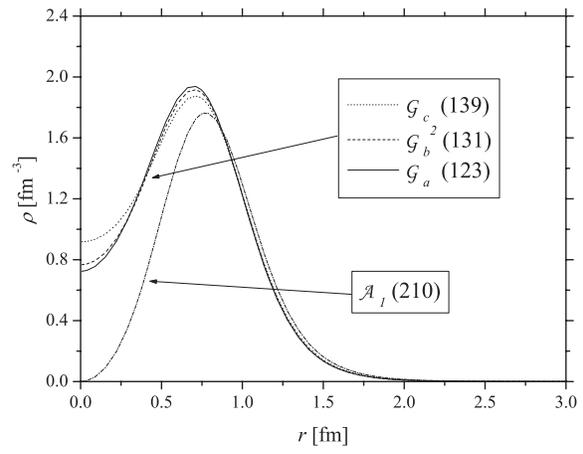


Figure 6. Baryon number density of i th valence quarks $\rho^{(i)}(r)$ of $B = 5$, with the occupation numbers and their eigenvalues (in MeV).

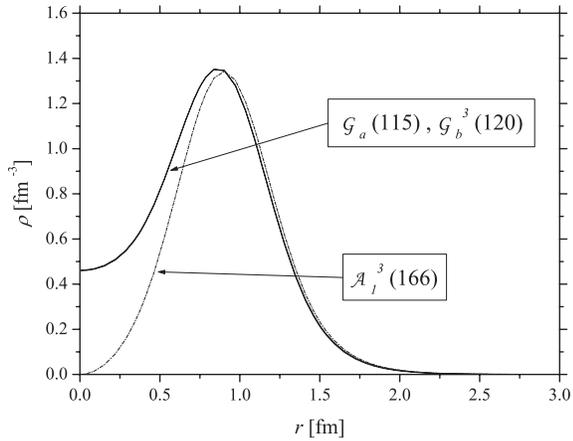


Figure 7. Baryon number density of i th valence quarks $\rho^{(i)}(r)$ of $B = 7$, with the occupation numbers and their eigenvalues (in MeV).

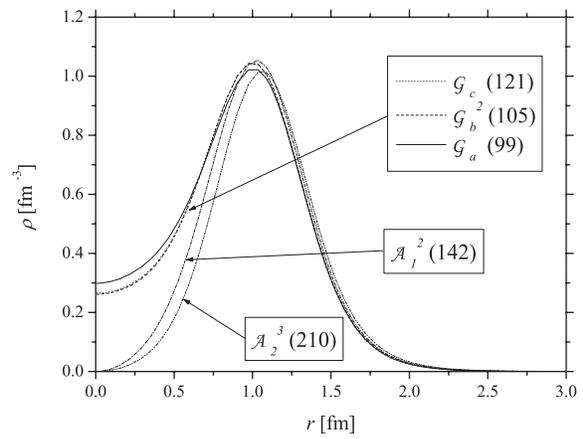


Figure 8. Baryon number density of i th valence quarks $\rho^{(i)}(r)$ of $B = 9$, with the occupation numbers and their eigenvalues (in MeV).

and

$$\psi'(x') = S(\mathbf{a}) \times G(\mathbf{a})\psi(x), \quad (25)$$

where $S(\mathbf{a})$ satisfies $a_\mu^\nu \gamma^\mu = S^{-1} \gamma^\nu S$. The operator R corresponding to this rotation is defined by

$$\psi'(x) \stackrel{\text{def}}{=} R\psi(x) = S(\mathbf{a}) \times G(\mathbf{a})\psi(a^{-1}x). \quad (26)$$

One can easily check that R commutes with the Hamiltonian in (4). Constructing R for each symmetry, one should be able to deduce the degeneracy structure of the spectra occurring in the valence levels. The four-fold degeneracy of the lowest states may be ascribed to the chiral symmetry $SU(2)_L \times SU(2)_R$ of the Hamiltonian. The degenerate structure will be understood if symmetric operators of the Hamiltonian which consist of the angular momentum, spin, isospin and winding number, are explicitly constructed.

In Figs. 5–8, we present the results of $\rho^{(i)}(r)$ for $B = 3, 5, 7$ and 9 . The behavior of the density near the origin confirms at least three shells ($\mathcal{G}, \mathcal{A}_1, \mathcal{A}_2$). \mathcal{G} behaves like “ S -wave”, and others

like “ P -, D -wave” in a hydrogen-like atom. Although the shell structure is emerged, most of the densities are nearly on the same surface and very small (not zero) near the origin. The plateau in the density observed at the center of the nucleus [16] can not be attained in our solutions. Therefore, one may need to employ the multi-shell ansatz [17] even in the case of light nuclei.

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