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# Asymptotical Expansions for One-Phase Soliton-Type Solution to Perturbed Korteweg–de Vries Equation

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Korteweg–de Vries equation with coefficients depending on a small parameter is studied. The asymptotical expansions for one-phase soliton-type solutions are constructed.

## 1 Introduction

Korteweg–de Vries equation is one of the most known nonlinear differential equation and is a fundamental one of modern physics. It was deduced by D.J. Korteweg and G. de Vries [1] in order to describe so-called solitary waves firstly observed by J. Scott-Russel [2] in 1834.

The secondary discovery of this outstanding equation is connected with attempt to solve famous Fermi–Pasta–Ulam problem [3] in 1965, when M.D. Kruskal and N.J. Zabusky [4] found relation of the system of nonlinear oscillators of Toda type to the Korteweg–de Vries equation. Later it appeared as object for studying in different fields of physics, hydrodynamics, solid body theory, plasma theory, quantum mechanics and others [5, 6].

During the last 35 years a lot of papers were devoted to consideration of different properties of solutions to the Korteweg–de Vries equation, in particular, to finding its solutions in exact form via the inverse scattering transform initiated by C.S. Gardner, J.M. Green, M.D. Kruskal, R.M. Miura and P.D. Lax [7, 8] and developed in [9–12].

On the other hand, while considerating processes with small dispersion a problem of studying Korteweg–de Vries equation with varying coefficients and a small parameter arises. Similar problems can be researched through one of approaches developed on the base of a small parameter technique being well-known to be effective in studying different nonlinear problems [13, 14].

## 2 Statement of a problem and preliminary notes

We consider Korteweg–de Vries equation with varying coefficients and a small parameter of the following form

$$u_{xxx} = a(x,\varepsilon)u_t + b(x,\varepsilon)uu_x. \tag{1}$$

Functions $a(x,\varepsilon)$, $b(x,\varepsilon)$ are assumed to be represented as

$$a(x,\varepsilon) = \frac{1}{\varepsilon}\sum_{k=0}^{\infty} a_k(x)\varepsilon^k, \qquad b(x,\varepsilon) = \frac{1}{\varepsilon}\sum_{k=0}^{\infty} b_k(x)\varepsilon^k,$$

where $x \in \mathbb{R}^1$, $t \in (0;T)$, $a_k(x), b_k(x) \in C^{(\infty)}(\mathbb{R}^1)$, $k \geq 0$.

By using the approach proposed in [15–18] we develop an algorithm for constructing asymptotical solutions to problem (1).

Let us consider infinite differentiable functions $f = f(x,t,\tau)$, $(x,t,\tau) \in (\mathbb{R}^1_x \times (0;T) \times \mathbb{R}^1_t)$, such that for any compact $K \subset \mathbb{R}^1_x \times (0;T)$ and any non-negative integer numbers $n$, $m$, $q$, $\alpha$ uniformly with respect to variables $(x,t)$ the following conditions are fulfilled:

1) the relation

$$\lim_{\tau\to\infty}\tau^n\frac{\partial^{m+q+\alpha}}{\partial x^m\partial t^q\partial\tau^\alpha}f(x,t,\tau)=0 \tag{2}$$

takes place;

2) there exists an infinite differentiable function $f^-(x,t)$ such that

$$\lim_{\tau\to-\infty}\tau^n\frac{\partial^{m+q+\alpha}}{\partial x^m\partial t^q\partial\tau^\alpha}\left[f(x,t,\tau)-f^-(x,t)\right]=0,\qquad (x,t)\in K. \tag{3}$$

Let the linear space of such functions $f=f(x,t,\tau)$, $(x,t,\tau)\in(\mathbb{R}^1_x\times(0;T)\times\mathbb{R}^1_t)$ be denoted $G=G(\mathbb{R}^1_x\times(0;T)\times\mathbb{R}^1_\tau)$.

Let $G_0=G_0(\mathbb{R}^1_x\times(0;T)\times\mathbb{R}^1_\tau)$ be a linear subspace of the space $G=G(\mathbb{R}^1_x\times(0;T)\times\mathbb{R}^1_\tau)$ consisting of functions $f(x,t,\tau)$ satisfying on every compact $K\subset\mathbb{R}^1_x\times(0;T)$ additionally to the conditions (2), (3) the equality

$$\lim_{\tau\to-\infty}f(x,t,\tau)=0,\qquad (x,t)\in K,$$

is satisfied uniformly with respect to $(x,t)$.

**Definition 1.** A function $u=u(x,t,\varepsilon)$ is called one-phase soliton-type one if for any integer $N\geq 0$ function $u(x,t,\varepsilon)$ can be represented as

$$u(x,t,\varepsilon)=\sum_{k=0}^{2N}\varepsilon^{\frac{k}{2}}\left[u_k(x,t)+v_k(x,t,\tau)\right]+O\big(\varepsilon^{N+1}\big), \tag{4}$$

where $\tau=\frac{x-\varphi(t)}{\sqrt{\varepsilon}}$, $\varphi(t)\in C^{(\infty)}(0;T)$ is a scalar real function; functions $u_k(x,t)$ are infinite differentiable; $v_0(x,t,\tau)\in G_0$, $v_k(x,t,\tau)\in G$, $k=1,2,\ldots,2N$.

Function $S=x-\varphi(t)$ is called a phase of one-phase soliton-type solution (4).

Below we propose algorithm of constructing asymptotical solution to the equation (1). Firstly we establish a form of asymptotical solution to problem (1). In particular, it is not complicated to prove that asymptotical solution to the problem (1) can be represented as

$$u(x,t,\varepsilon)=\bar{U}(x,t,\varepsilon)+\sqrt{\varepsilon}G_\varepsilon(S,x,t)+\sqrt{\varepsilon}F_\varepsilon(S,x,t),$$

where $\bar{U}(x,t,\varepsilon)$ is infinite differentiable function; $G_\varepsilon(S,x,t)$, $F_\varepsilon(S,x,t)$ are infinite differentiable functions sufficiently quickly tending to zero as $|x|\to\infty$ and satisfying conditions

$$\|\sqrt{\varepsilon}G_\varepsilon(S,x,t)\|_C\leq C_1,\qquad \|F_\varepsilon(S,x,t)\|_C\leq C_2$$

with constants $C_1$, $C_2$ not depending on $\varepsilon$ and norm $\|\cdot\|_C$ of continuous in $(x,t)\in\mathbb{R}^1_x\times[0;T]$ function space.

## 3 Scheme of constructing asymptotical solution

Solution to the equation (1) is searched in the form of following asymptotical series:

$$u(x,t,\varepsilon)=Y_N(x,t,\tau,\varepsilon)+O\big(\varepsilon^{N+1}\big), \tag{5}$$

where

$$Y_N(x,t,\tau,\varepsilon)=\sum_{k=0}^{2N}\varepsilon^{\frac{k}{2}}\left[u_k(x,t)+v_k(x,t,\tau)\right],\qquad \tau=\frac{x-\varphi(t)}{\sqrt{\varepsilon}}.$$

Function $U_N(x,t,\varepsilon)=\sum\limits_{k=0}^{2N}\varepsilon^{\frac{k}{2}}u_k(x,t)$ is a regular part of asymptotic (5), and function $V_N(x,t,\varepsilon)$ $=\sum\limits_{k=0}^{2N}\varepsilon^{\frac{k}{2}}v_k(x,t,\tau)$ is a singular part of asymptotic (5), i.e. $Y_N=U_N+V_N$.

At the first step the asymptotical solution to the problem (1) is constructed in some $\mu$-neighborhood of curve $\Gamma=\{(x,t)\in\mathbb{R}^1\times(0;T):x=\varphi(t)\}$, i.e. in the domain:

$$\Omega_\mu(\Gamma)=\{(x,t)\in\mathbb{R}^1\times(0,T):|x-\varphi(t)|<2\mu\},$$

where $\mu$ is a small positive parameter, as well as a function $\varphi(t)$ is defined later.

By direct calculations we find derivatives $u_t(x,t,\varepsilon)$, $u_x(x,t,\varepsilon)$, $u_{xxx}(x,t,\varepsilon)$, substitute them into equation (1) and obtain the relation:

$$\frac{\partial^3 Y_N}{\partial x^3}+\frac{3}{\sqrt{\varepsilon}}\frac{\partial^3 Y_N}{\partial x^2\partial\tau}+\frac{3}{\varepsilon}\frac{\partial^3 Y_N}{\partial x\partial\tau^2}+\frac{1}{\varepsilon^{\frac{3}{2}}}\frac{\partial^3 Y_N}{\partial\tau^3}$$
$$=a(x,\varepsilon)\left(\frac{\partial Y_N}{\partial t}-\frac{1}{\sqrt{\varepsilon}}\frac{\partial Y_N}{\partial\tau}\varphi'(t)\right)+b(x,\varepsilon)\left(\frac{\partial Y_N}{\partial x}+\frac{1}{\sqrt{\varepsilon}}\frac{\partial Y_N}{\partial\tau}\right)Y_N+g_N(x,t,\tau,\varepsilon). \quad (6)$$

Here $g_N(x,t,\tau,\varepsilon)=O\big(\varepsilon^{N+1}\big)$ is an infinite differentiable function of its arguments that are defined recursively in $N$.

To find equations for regular part of asymptotics from relation (6) we calculate limit as $\tau\to+\infty$ and equate coefficients at the same powers $\varepsilon$. As result, we have system of partial differential equations for functions $u_k$, $k=0,\ldots,2N$:

$$a_0(x)\frac{\partial u_0}{\partial t}+b_0(x)\frac{\partial u_0}{\partial x}u_0=0,$$
$$a_0(x)\frac{\partial u_k}{\partial t}+b_0(x)u_0(x,t)\frac{\partial u_k}{\partial x}+b_0(x)u_k(x,t)\frac{\partial u_0}{\partial x}=f_k(x,t,u_0,u_1,\ldots,u_{k-1}), \quad (7)$$

where functions $f_k(t,x,u_0,u_1,\ldots,u_{k-1})$, $k=1,\ldots,2N$, are defined recurrently. Additionally the equations (7) for any $k=0,\ldots,2N$ are assumed to have infinite differentiable solutions.

### 3.1 Defining singular part of asymptotics

Differential equations for singular part of asymptotics (5) are deduced from relation (6) after reducing (7). Later functions $v_k(x,t,\tau)$, $k=0,1,\ldots,2N$, are defined on the curve $x=\varphi(t)$ and continued into domain $\Omega_\mu(\Gamma)$. Let values of functions $v_k(x,t,\tau)$, $k=0,1,\ldots,2N$, on the curve $x=\varphi(t)$ be denoted $v_k(t,\tau)$. As result, we have the following system of partial differential equations:

$$\frac{\partial^3 v_0}{\partial\tau^3}+a_0(\varphi)\frac{\partial v_0}{\partial\tau}\varphi'(t)-b_0(\varphi)\left[u_0(\varphi,t)\frac{\partial v_0}{\partial\tau}+v_0\frac{\partial v_0}{\partial\tau}\right]=0,$$
$$\frac{\partial^3 v_k}{\partial\tau^3}+a_0(\varphi)\frac{\partial v_k}{\partial\tau}\varphi'(t)-b_0(\varphi)\left[u_0(\varphi,t)\frac{\partial v_k}{\partial\tau}+v_k\frac{\partial v_0}{\partial\tau}+v_0\frac{\partial v_k}{\partial\tau}\right]=\mathcal{F}_k(t,\tau), \quad (8)$$

where $k=1,\ldots,2N$, functions

$$\mathcal{F}_k(t,\tau)=F_k(t,v_0(x,t,\tau),\ldots,v_{k-1}(x,t,\tau),u_0(x,t),\ldots,u_k(x,t))\Big|_{x=\varphi(t)}$$

are defined recurrently. Namely, first we find function $v_0(t,\tau)$ from (8), later extend it into domain $\Omega_\mu(\Gamma)$. At the next step we calculate the function

$$\mathcal{F}_1(t,\tau)=F_1(t,v_0(x,t,\tau),u_0(x,t),u_1(x,t))\Big|_{x=\varphi(t)},$$

define function $v_1(t,\tau)$, extend it into the domain $\Omega_\mu(\Gamma)$, and so on.

### 3.2 Solvability of equations for singular part of asymptotics

Consider the first equation in (8). It can be written as follows (after integration in $\tau$):

$$\frac{d^2 v_0}{d\tau^2} = -a_0(\varphi)v_0(\varphi,\tau)\varphi'(t) + b_0(\varphi)u_0(\varphi,t)v_0(t,\tau) + \frac{1}{2}b_0(\varphi)v_0^2(\varphi) + c_1(t). \tag{9}$$

Due to condition $v_0(t,\tau) \in G_0$, we can take $c_1(t) \equiv 0$.

Multiplying equation (9) by $dv_0/d\tau$ and integrating it in $\tau$ one receives:

$$\left(\frac{dv_0}{d\tau}\right)^2 = -a_0(\varphi)v_0^2(t,\tau)\varphi'(t) + \frac{1}{2}b_0(\varphi)v_0^3(t,\tau) + c_2(t).$$

Due to condition $v_0(t,\tau) \in G_0$ we can take $c_2(t) \equiv 0$.

Thus, a solution to the equation (9) in the space $G_0$ is represented as

$$v_0(t,\tau) = A[\varphi]\,\mathrm{ch}^{-2}((\tau + C_0)H[\varphi]), \tag{10}$$

where

$$A[\varphi] = -2\frac{a_0(\varphi(t))\varphi'(t) - b_0(\varphi(t))u_0(\varphi(t),t)}{b_0(\varphi(t))}, \qquad H[\varphi] = \frac{2\sqrt{A}}{b_0(\varphi(t))}$$

provided that $A[\varphi] > 0$.

So, the following lemma is proved.

**Lemma 1.** *Let $A[\varphi] > 0$. Then a solution of the first equation of system (8) in the space $G_0$ exists and is given by formula (10).*

Differential equations of system (8) for $k = 1, 2, \ldots, 2N$ can be written in the following operator form

$$\hat{L}v_k = F_k, \tag{11}$$

where operator $\hat{L}$ is

$$\hat{L} = \frac{\partial^3}{\partial\tau^3} + \left[a_0(\varphi(t))\varphi'(t) - b_0(\varphi(t)) - b_0(\varphi(t))u_0(\varphi(t),t)\right]\frac{\partial}{\partial\tau} - \frac{\partial v_0}{\partial\tau}b_0(\varphi(t)).$$

The following lemma is true.

**Lemma 2.** *Suppose functions $F_k(t,\tau) \in G_0$, $k \geq 1$. Then operator equations (11) are solvable in the space $G$ if and only if the conditions*

$$\int_{-\infty}^{+\infty} F_k(t,\tau)v_0(t,\tau)d\tau = 0, \qquad k \geq 1, \tag{12}$$

*are satisfied.*

Relation (12) is called an orthogonality condition. In the case, when the orthogonality condition (12) is fulfilled, the general solution to the equation (8) in the space $G$ can be written as

$$\nu_k(t,\tau) = z_k(t,\tau) + c_k v_{0\tau}, \qquad k \geq 1,$$

where $c_k$ is constant of integrability, $z_k(t,\tau)$ is a particular solution to the non-homogeneous equation (8):

$$z_k(t,\tau) = v_{0\tau}\int_{-\infty}^{\tau} v_{0\tau}^{-2}(t,\tau_1)\int_{-\infty}^{\tau_1}\Phi_k(t,\tau_2)v_{0\tau}(t,\tau_2)d\tau_2 d\tau_1, \qquad k = 1, 2, \ldots, 2N.$$

### 3.3 Definition of the phase function $\varphi(t)$

The function of phase $\varphi(t)$ is defined from ordinary differential equation deduced from orthogonality condition (12). In particular, if one takes the function

$$\mathcal{F}_1(t,\tau) = a_0(\varphi)\frac{\partial v_0}{\partial t} + b_0(\varphi)\frac{\partial u_0(x,t)}{\partial x}\bigg|_{x=\varphi(t)} \cdot v_0 + b_0(\varphi)u_1(\varphi)\frac{\partial v_0}{\partial \tau}, \tag{13}$$

then after substituting function $v_0(t,\tau)$ into (13) and using (12), finds

$$\frac{d\varphi}{dt} = -\frac{\frac{\partial u_0(x,t)}{\partial x}\Big|_{x=\varphi} b_0(\varphi)A[\varphi]H^2[\varphi]}{2a_0(\varphi)A'[\varphi]H[\varphi] - A[\varphi]H'[\varphi]a_0(\varphi)}. \tag{14}$$

Since under rather general conditions on $a_0(x)$, $b_0(x)$ functions $A[\varphi]$, $H[\varphi]$ may be considered to be infinite differentiable ones and consequently, the differential equation (14) may be assumed to have unique infinite differentiable solution.

### 3.4 Continuation of singular part of asymptotics into domain $\Omega_\mu(\Gamma)$

We define functions $v_k(x,t,\tau)$, $k \geq 0$, in the closure of domain $\Omega_\mu(\Gamma)$. For $k = 0$ solution of equation (8) is deduced above (see formula (10)). For $k = 1, 2, \ldots, 2N$ a solution to the equation (8) has a form

$$v_k(t,\tau) = \nu_k(t)\eta_k(t,\tau) + \psi_k(t,\tau), \qquad k = 1, \ldots, 2N,$$

where

$$\lim_{\tau\to-\infty} \eta = 1, \quad \eta \in G,$$
$$\nu_k(t,\tau) = -\left[a_0(\varphi(t))\varphi'(t) - b_0(\varphi(t))\right]^{-1} \lim_{\tau\to-\infty} \Phi_k(t,\tau),$$
$$\Phi_k(t,\tau) = -\int_{-\infty}^{\tau} \mathcal{F}_k(t,\tau)d\tau + E_k(t), \qquad \lim_{\tau\to+\infty} \Phi_k(t,\tau) = 0,$$
$$\psi_k(t,\tau) = \psi_{k,1}(t,\tau) + c_k(t)v_{0\tau}(t,\tau),$$

a function $\psi_{k,1}$ belongs to the space $G_0$, $c_k(t)$ is a constant of integration.

Let us consider the Cauchy problem:

$$\begin{aligned} &\Lambda u_k^-(x,t) = f_k(x,t), \\ &u_k^-(x,t)\Big|_\Gamma = \nu_k(t), \qquad k = 1, \ldots, 2N, \end{aligned} \tag{15}$$

where the differential operator $\Lambda$ is

$$\Lambda = a_0(x)\frac{\partial}{\partial t} + b_0(x)u_0(x,t)\frac{\partial}{\partial x} + b_0(x)\frac{\partial u_0(x,t)}{\partial x}.$$

Since the curve $\Gamma$ is transversal to a characteristics of differential operator $\Lambda$ for any $t \in [0;T]$, the problem (15) is correctly posed, and because of it there exists a solution $u_k^-(x,t) \in C^{(\infty)}(\Omega_\mu(\Gamma))$ for small enough $\mu$.

Continuation of functions $v_k(t,\tau)$, $k = 0, 1, \ldots, 2N$ into domain $\Omega_\mu(\Gamma)$ is defined as

$$v_0(x,t,\tau) = v_0(t,\tau), \qquad v_k(x,t,\tau) = u_k^-(x,t)\eta(t,\tau) + \psi_k(t,\tau).$$

# 4 Constructing global asymptotical solution

Let us note domains

$$D^- = \{(x,t) \in \mathbb{R}^1 \times (0;T) : \varphi(t) - x \geq \mu\},$$
$$D^+ = \{(x,t) \in \mathbb{R}^1 \times (0;T) : x - \varphi(t) \geq \mu\}.$$

For all $x < \varphi(t)$, $t \in (0;T)$, functions $u_k^-(x,t)$, $k = 1,\dots,2N$ are defined as infinite differentiable solutions to the problem (15).

Asymptotical solution to the equation (1) is found through gluing functions defined before. Let $\chi(\xi) \in C^\infty(\mathbb{R}^1) : \chi(\xi) = 1$ if $\xi \geq 2$ and $\chi(\xi) = 0$ if $\xi \leq 1$.

Thus, the following result is established.

**Theorem 1.** *Let us suppose:*
1) *functions* $a_k(x), b_k(x) \in C^{(\infty)}(\mathbb{R}^1)$, $k \geq 0$*;*
2) *the inequality* $A[\varphi] > 0$*, where function* $\varphi(t)$ *is a solution to the equation* (14)*, takes place;*
3) *functions* $F_k(t,\tau)$, $k = 1,\dots,N$*, belong to the space* $G_0$*;*
4) *the orthogonality condition* (12) *is fulfilled.*

*Then the function*

$$Y_N(x,t,\varepsilon) = \begin{cases} Y_N^-(x,t,\varepsilon), & (x,t) \in D^- \backslash \Omega_\mu(\Gamma), \\ Y_N^+(x,t,\varepsilon), & (x,t) \in D^+ \backslash \Omega_\mu(\Gamma), \\ Y_N(x,t,\tau,\varepsilon), & (x,t) \in \Omega_\mu(\Gamma), \end{cases}$$

*where*

$$Y_N^-(x,t,\varepsilon) = u_0(x,t) + \sum_{k=1}^{2N} \varepsilon^{\frac{k}{2}} \left[u_k(x,t) + u_k^-(x,t)\right], \qquad (x,t) \in D^-,$$

$$Y_N^+(x,t,\varepsilon) = \sum_{k=1}^{2N} \varepsilon^{\frac{k}{2}} u_k(x,t), \qquad (x,t) \in D^+,$$

$$Y_N(x,t,\tau,\varepsilon) = \sum_{k=0}^{2N} \varepsilon^{\frac{k}{2}} \left[u_k(x,t) + v_k(x,t,\tau)\right], \qquad (x,t) \in \Gamma, \qquad \tau = \frac{x - \varphi(t)}{\sqrt{\varepsilon}},$$

*is an asymptotical expansion for solution to the Korteweg–de Vries equation* (1) *on the interval* $(0;T)$*, i.e. for any compact* $K \subset \mathbb{R}_x^1 \times (0;T)$

$$\max_{(x,t)\in K} |u(x,t,\varepsilon) - Y_N(x,t,\varepsilon)| = O\big(\varepsilon^{N+1}\big), \qquad N \in \mathbb{N}.$$

# 5 Conclusions

In the paper the problem of constructing asymptotical solutions to the Korteweg–de Vries equation with coefficients depending on a small parameter is studied. The algorithm of constructing asymptotical solutions is developed.


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