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On Spectrum of Matrix-Valued Continuous Functions of a Family of Commuting Operators

Yevgenii Ye. SAMOILENKO

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *sam7@imath.kiev.ua*

A spectrum of matrix-valued continuous functions of a family of self-adjoint commuting bounded operators on a Hilbert space is studied.

1 Introduction

Let $\widetilde{A} = \{A_i = A_i^*\}_{i=\overline{1,m}} \subset L(H)$ be a family of self-adjoint commuting operators and $\{E_i\}_{i=\overline{1,m}}$ be a family of their spectral measures. A direct product of spectral measures is $\widetilde{E}(\alpha_1 \times \alpha_2 \times \ldots \times \alpha_m) = \mathop{\times}\limits_{i=1}^{m} E_i(\alpha_i) = E_1(\alpha_1)E_2(\alpha_2)\cdots E_m(\alpha_m)$, a measure on a measurable space. A support of decomposition of unit E is called $\operatorname{Supp} E = \{\cap\varphi | \varphi = \overline{\varphi} : E(\varphi) = \mathbb{I}\}$, i.e. intersection of all closed sets of full measure. A common spectrum of a family of self-adjoint commuting bounded operators is called $S(\widetilde{A}) = S(\{A_i \,|\, i = \overline{1,m}\}) := \operatorname{Supp} \widetilde{E}$, i.e. the support of product of spectral measures. By definition, the following conclusion may be drawn

$$S(\widetilde{A}) = \operatorname{Supp} \widetilde{E} \subseteq \mathop{\times}\limits_{i=1}^{m} \operatorname{Supp} E_i = \mathop{\times}\limits_{i=1}^{m} \sigma(A_i),$$

where $\sigma(A_i)$ is a spectrum of operator A_i, $i = \overline{1,m}$. A continuous function of a family of self-adjoint operators $\widetilde{A}$ is

$$f(\widetilde{A}) = \int_{S(\widetilde{A})} f(\lambda_1, \ldots, \lambda_m) d\widetilde{E}(\Lambda),$$

where $\Lambda = (\lambda_1, \ldots, \lambda_m) \in S(\widetilde{A}) \subset \mathbb{R}^m$ (see e.g. [1–4]).

Let us consider a matrix-valued continuous function $\{f_{ij}(t_1, t_2, \ldots, t_m)\}_{i,j=\overline{1,n}}$, where $f_{ij}(t_1, t_2, \ldots, t_m) \in C(\mathbb{R}^m, \mathbb{C})$. The result of this paper is the formula for spectrum of matrix-valued continuous functions of a family of self-adjoint commuting bounded operators.

2 On spectrum of matrix operators

Let $G = \widetilde{F}(\widetilde{A}) = \{F_{ij}(\widetilde{A}) = F_{ij}(A_1, \ldots, A_m) \,|\, i, j = \overline{1,n}\}$, $G : H^n \to H^n$. Let $\widetilde{F}(\Lambda) = \{F_{ij}(\Lambda) = F_{ij}(\lambda_1, \ldots, \lambda_m) \,|\, i, j = \overline{1,n}\}$ be a continuous matrix function $\widetilde{F}(\Lambda) : U(S(\widetilde{A})) \to \mathbb{C}^{n\times n}$, where U is a neighborhood of S and $\widetilde{F}(\widetilde{A})$ is a function on family of self-adjoint commuting bounded operators. Denote $\Delta(G, \lambda) = \Delta(\widetilde{F}(\widetilde{A}), \lambda) := \det(\{F_{ij}(\widetilde{A}) - \delta_i^j \lambda\}_{i,j=1}^n)$, where δ_i^j is a Kroneker symbol and $\Delta(\Lambda, \lambda) := (F_{ij}(\Lambda) - \delta_i^j \lambda)$, where $\Lambda = (\lambda_1, \ldots, \lambda_m) \in S(\widetilde{A})$.

Theorem 1. *A spectrum of the operator G is equal to $\sigma(G) = \{\lambda \in \mathbb{C} \,|\, \exists\, \Lambda \in S : \Delta(\Lambda, \lambda) = 0\}$.*

Proof. ($\Rightarrow$) Let $\forall\, \Lambda \in S(\widetilde{A}) : \Delta(\Lambda, \lambda) \neq 0$, then existence of operator $(G - \lambda\mathbb{I})^{-1}$ means existence of a solution of the system of operator equations $\sum\limits_{j=1}^{n} F_{ij}(\widetilde{A}) x_j = y_i$, $i = \overline{1,n}$, where $x_j, y_i \in H$. The sufficient condition for it is existence of the operator $(\Delta(\widetilde{F}(\widetilde{A}), \lambda))^{-1}$.

From $\Delta(\Lambda,\lambda)\neq 0$ the following conclusion ($\widetilde{E}=\underset{i,j=1}{\overset{m}{\times}}E_{ij}$) may be drawn

$$\mathbb{I}=\int_{\mathbb{R}^m}d\widetilde{E}=\int_{\mathbb{R}^m}\Delta(\Lambda,\lambda)(\Delta(\Lambda,\lambda))^{-1}d\widetilde{E}(\Lambda)=(G-\lambda\mathbb{I})B.$$

It follows that $\exists(G-\lambda\mathbb{I})^{-1}=B\Rightarrow\lambda\in\rho(G)$, where ρ is a resolvent set.

($\Leftarrow$) Let $\exists\Lambda\in S(\widetilde{A}):\Delta(\widetilde{F}(\Lambda,\lambda))=0$, then $\exists\, f$, where f is the eigenvector of operator $\widetilde{F}(\Lambda)$. Denote $\forall\, z\geq 1$:

$$0\neq f^{(z)}:=\widetilde{E}\left(\left(\lambda_1-\frac{1}{z},\lambda_1+\frac{1}{z}\right)\times\cdots\times\left(\lambda_m-\frac{1}{z},\lambda_m+\frac{1}{z}\right)\right)f$$

(notice that E is a projector) and $y^{(z)}:=\frac{f^{(z)}}{\|f^{(z)}\|}$. It follows that

$$(G-\lambda\mathbb{I})y^{(z)}=(\{F_{ij}(\Lambda)-\lambda_{ij}\}_{i,j=\overline{1,n}})y^{(z)}-\Delta(\Lambda,\lambda)y^{(z)}.$$

The condition $\Delta(\Lambda,\lambda)y^{(z)}=0$ is clearly fulfilled, and

$$\|(F_{ij}(\widetilde{A})-\lambda_{ij})y_j^{(z)}\|^2=\int\limits_{\mathbb{R}}|\alpha_{ij}-\lambda_{ij}|^2d(E(\alpha_{ij})y^{(z)},y^{(z)})\to 0,\qquad z\to 0.$$

And finally $\|(G-\lambda\mathbb{I})y^{(z)}\|\to 0$, $z\to 0$, where $\|y^{(z)}\|=1\Rightarrow\lambda\in\sigma(G)$. ■

The following proposition will be useful in examples.

Proposition 1. *Let $A^*=A\subset L(H)$ be a self-adjoint bounded operator. Let $\widetilde{A}=\{f_1(A),\ldots,f_n(A)\}$, where $\{f_i|i=\overline{1,m}\}\subset C(U(\sigma(A)),\mathbb{R})$, U is a neighborhood of $\sigma(A)$, then*

$$S(\{f_1(A),f_2(A),\ldots,f_m(A)\})=\{(f_1(\lambda),f_2(\lambda),\ldots,f_m(\lambda))|\lambda\in\sigma(A)\}.$$

Proof. There is $E_i(\alpha)=\int_{\sigma(A)}\chi(f_i(\lambda))dE$, where $\chi(A)$ is the characteristic function, $i=\overline{1,m}$. Then denote a decomposition $\underset{i=1}{\overset{m}{\times}}\sigma_i=\operatorname{Supp}\widetilde{E}\cup\Theta$, where the common measure of any opening set with Θ is equal to zero, and the common measure of any opening set is not equal to zero if this set contains a point from $\operatorname{Supp}\widetilde{E}$. Since $\widetilde{E}(\alpha_1\alpha_2\cdots\alpha_m)=\int_{\mathbb{R}}\chi_{\alpha_1}(\lambda)\cdots\chi_{\alpha_m}(\lambda)dE(\lambda)=0$ and $f(\sigma(A))=\sigma(f(A))$, then proposition is proved. ■

3 Examples

Example 1. Let $H=L_2([-\pi,\pi],dt)$ and $G=\begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix}$, and $K_j(t):=je^{it}$, $j=\overline{1,4}$, $i=\sqrt{-1}$, and $\widetilde{A}:=\{A_j=\int_{-\pi}^{\pi}K_j(t-\tau)x(\tau)d\tau\,|\,j=\overline{1,4}\}$ family of commuting self-adjoint bounded operators. The common spectrum of $\widetilde{A}$ is equal to

$$S(\widetilde{A})=\{(0,0,0,0),(2\pi,4\pi,6\pi,8\pi)\}=:\{\Lambda_1,\Lambda_2\}.$$

Let us solve the following equations $\Delta(\Lambda_1,\lambda)=0,\Delta(\Lambda_2,\lambda)=0$:

$$\det\begin{pmatrix}-\lambda & 0\\ 0 & -\lambda\end{pmatrix}=0\qquad\text{and}\qquad\det\begin{pmatrix}2\pi-\lambda & 4\pi\\ 6\pi & 8\pi-\lambda\end{pmatrix}=0\quad\Rightarrow$$

$\Rightarrow$ The spectrum of matrix operator is equal to $\sigma(G)=\{0,\pi(5\pm\sqrt{33})\}$.

Example 2. Let $H = L_2([0,1], dt)$ and $f_1, f_2, f_3, f_4 \in C([0,1])$. Let $\widetilde{A} = \{(A_1x)(t) = f_1(t)x(t), (A_2x)(t) = f_2(t)x(t), (A_3x)(t) = f_3(t)x(t), (A_4x)(t) = f_4(t)x(t)\}$ be a family of self-adjoint commuting operators, then

$$S(A_1, A_2, A_3, A_4) = \{f_1(t), f_2(t), f_3(t), f_4(t) | t \in [0,1]\},$$
$$G = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} F_{11}(\widetilde{A}) & F_{12}(\widetilde{A}) \\ F_{21}(\widetilde{A}) & F_{22}(\widetilde{A}) \end{pmatrix},$$

where

$$F_{11}(t_1, t_2, t_3, t_4) = t_1, F_{12}(t_1, t_2, t_3, t_4) = t_2, F_{21}(t_1, t_2, t_3, t_4) = t_3, F_{22}(t_1, t_2, t_3, t_4) = t_4,$$
$$\sigma(G) = \left\{ \lambda \in \mathbb{C} \,|\, \exists\, t \in [0,1] \,:\, \det \begin{pmatrix} f_1(t) - \lambda & f_2(t) \\ f_3(t) & f_4(t) - \lambda \end{pmatrix} = 0 \right\} \Leftrightarrow$$
$$\Leftrightarrow \sigma(G) = \left\{ \frac{f_1(t) + f_4(t) \pm \sqrt{(f_1(t) + f_4(t))^2 + 4(f_2f_3(t) - f_1(t)f_4(t))}}{2} \,\Big|\, t \in [0,1] \right\}.$$

Example 3. Let $(P_1x)(t) = t_1x(t)$, $(P_2y)(t) = 2\pi t_2 x(t)$ be two commuting self-adjoint bounded operators, where $x(t) \in L_2([0,1]^2, dt)$, $t = (t_1, t_2)$. The operator $N = P_1(\cos P_2 + i \sin P_2)$ is a normal operator ($N^*N = NN^*$). Let us consider the subalgebra B_V of $C(D, \mathbb{C}^{2\times 2})$ of the form $B_V(D) = \{a \in C(D, \mathbb{C}^{2\times 2}) : a(z) = V^{-1}(z)a(1)V(z), if |z| = 1\}$, where $D = \{z \in \mathbb{C} : |z| \le 1\}$ is the unit disk and $V(z) = \operatorname{diag}\{z^{-1}, 1\}$. The algebra $B_V(D)$ is an example of a non-trivial 2-homogeneous C^*-algebra (see [5]).

It follows that $B(N) = \{a(N) \,|\, a \in B_V(D)\}$ is an example of a non-trivial operator C^*-algebra, because the spectrum of operator N is equal to $\sigma(N) = D$. The following conclusion may be drawn

$$\forall\, a = a(N) \in B_V(N) : a(N) = \begin{pmatrix} a_{11}(N) & a_{12}(N) \\ a_{21}(N) & a_{22}(N) \end{pmatrix} = \begin{pmatrix} a_{11}(f(P_1, P_2)) & a_{12}(f(P_1, P_2)) \\ a_{21}(f(P_1, P_2)) & a_{22}(f(P_1, P_2)) \end{pmatrix},$$

where

$$f(x, y) = x(\cos y + i \sin y),$$
$$S(\{a_{ij}(N), i, j = 1, 2\}) = \{\{a_{ij}(t_1(\cos 2\pi t_2 + i \sin 2\pi t_2))\} \,|\, t \in [0,1]^2\},$$
$$\sigma(a(N)) = \left\{ \frac{a_{11}(\tau) + a_{22}(\tau)}{2} \right.$$
$$\left. \pm \frac{\sqrt{(a_{11}(\tau) + a_{22}(\tau))^2 - 4(a_{11}(\tau)a_{22}(\tau) - a_{12}(\tau)a_{21}(\tau))}}{2} \,|\, \tau = f(t_1, 2\pi t_2),\ t \in [0,1]^2 \right\}.$$

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