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Universal Structure of Jet Space

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Operators of total differentiation D, Cartan forms ω and infinitesimal symmetries P constitute the structure of infinite jet space $J_{n,m}$. We describe these notions compactly for the space $J_{1,1}$ though reserve the possibility to pass with the help of multi-indices to general case $J_{n,m}$. Our aim is to show the universality of this structure. Every time when we differentiate a function f with respect to the vector field X on a manifold M we can determine a map $\varphi : M \to J_{1,1}$ and connect the triple (X, s, F) with the triple (D, t, U) in $J_{1,1}$, where F is the set of derivatives $f^{(k)} = X^k f$, $k = 0, 1, 2, \ldots$; s is canonical parameter of X, U is the set of fiber coordinates $u^{(k)} = D^k u$, $k = 0, 1, 2, \ldots$, and t is canonical parameter of D. Then all the invariants and symmetries of D as well as all the covariant tensors including Cartan forms can be transformed from $J_{1,1}$ onto the manifold M. The structure is universal as final object in the category of triples (X, s, F).

Let $f : V_n \to V_m$ be a smooth mapping. The infinite jet of the map f is determined by the coordinates t^i, u^α of the points $t \in V_n$ and $u = f(t) \in V_m$, and by the values of partial derivatives at t:

$$u_i^\alpha = \frac{\partial f^\alpha}{\partial u^i}(t), \quad u_{ij} = \frac{\partial^2 f^\alpha}{\partial u^i \partial u^j}(t), \quad \ldots,$$
$$i, j = 1, 2, \ldots, \quad n = \dim V_n, \quad \alpha = 1, 2, \ldots, \quad m = \dim V_m.$$

The set of the jets of f is called *jet space* $J_{m,n}$ where the quantities

$$t^i, \quad u^\alpha, \quad u_i^\alpha, \quad u_{ij}^\alpha, \quad \ldots \tag{1}$$

are *jet coordinates.*

In the space $J_{1,1}$ we have the coordinates

$$t, \quad u, \quad u', \quad u'', \quad \ldots \tag{2}$$

or briefly (t, U) where U is the column of elements $u, u', u'', \ldots$.

In $J_{1,1}$ one has the *natural basis* $\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial U}; dt, dU\right)$ associated with the coordinates (2). Here $\frac{\partial}{\partial U}$ is the row of elements $\frac{\partial}{\partial u}, \frac{\partial}{\partial u'}, \ldots$ and dU is the column of elements $du, du', du'', \ldots$.

Let us introduce the infinite-dimensional unit matrix E and the shift matrix C as follows:

$$E = \begin{pmatrix} 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & \cdot \\ 0 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 & \cdot \\ 0 & 0 & 1 & \cdot \\ 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and define in $J_{1,1}$ the *total differentiation operator* by formula:

$$D = \frac{\partial}{\partial t} + \frac{\partial}{\partial U} U', \qquad \text{where} \qquad U' = CU. \tag{3}$$

Proposition 1. *The operator D is a linear vector field in the jet space $J_{1,1}$ and its flow is determined by exponential law (see [5]),*

$$U' = CU \quad \Rightarrow \quad U_t = e^{Ct} U. \tag{4}$$

The curves (t, U_t) are the trajectories of D.

Proposition 2. *If the operator* $\frac{\partial}{\partial t}$ *in the frame* $\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial U}\right)$ *is replaced by* D *then the differentials* dU *in the coframe* (dt, dU) *have to be replaced by Cartan forms*

$$\omega = dU - U' dt. \tag{5}$$

The new basis in the matrix form

$$\begin{pmatrix} D & \frac{\partial}{\partial U} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial t} & \frac{\partial}{\partial U} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ U' & E \end{pmatrix}, \qquad \begin{pmatrix} dt \\ \omega \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -U' & E \end{pmatrix} \cdot \begin{pmatrix} dt \\ dU \end{pmatrix}.$$

is called *adapted basis* in $J_{1,1}$. The term "adapted basis" proceeds from the theory of connections (see [4, p. 23]).

Proposition 3. *The derivation formulae valid for the adapted basis (vertical part):*

$$\left(\frac{\partial}{\partial U}\right)' = -\frac{\partial}{\partial U} C, \qquad \omega' = C\omega. \tag{6}$$

The stroke means Lie derivative with respect to D. The frame $\frac{\partial}{\partial U}$ and the coframe ω are transported by the flow of D according to the law (4):

$$\left(\frac{\partial}{\partial U}\right)' = -\frac{\partial}{\partial U} C \qquad \Rightarrow \qquad \left(\frac{\partial}{\partial U}\right)_t = \frac{\partial}{\partial U} e^{-Ct},$$
$$\omega' = C\omega \qquad \Rightarrow \qquad \omega_t = e^{Ct}\omega.$$

Proposition 4. *The quantities*

$$I = e^{-Ct} U \tag{7}$$

are the invariants of D *because* $I' = e^{-Ct}(U' - CU) = 0$*. Replacing* U *by* I *in the fibers of* $J_{1,1}$ *we have the invariant basis:*

$$\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial I}; dt, dI\right).$$

The exponential e^{-Ct} is *integrating matrix* for Cartan forms ω and the operators $\frac{\partial}{\partial I}$ are *infinitesimal symmetries* of D (*infinitesimals* after [1]) in the following sense:

$$dI = e^{-Ct}\omega, \qquad \frac{\partial}{\partial I} = \frac{\partial}{\partial U} e^{Ct}. \tag{8}$$

Infinitesimal symmetries of D are called *Lie vector fields* in $J_{1,1}$.

Proposition 5. *A vector field* P *written in three frames of* $J_{1,1}$*, natural, adapted and invariant as follows*

$$P = \frac{\partial}{\partial t}\xi + \frac{\partial}{\partial U}\lambda = D\xi + \frac{\partial}{\partial U}\mu = \frac{\partial}{\partial t}\xi + \frac{\partial}{\partial I}\nu \tag{9}$$

has for components the entities

$$\xi = Pt, \qquad \lambda = PU, \qquad \mu = \omega(P), \qquad \nu = PI, \tag{10}$$

with relations

$$\nu = e^{-Ct}\mu, \qquad \mu = \lambda - U'\xi. \tag{11}$$

The field P *is a Lie vector field if and only if one of equivalent conditions is satisfied:*

$$\nu' = 0, \qquad \mu' = C\mu, \qquad \lambda' = C\lambda + U'\xi'. \tag{12}$$

It is obvious from the Lie derivatives:

$$P' = D\xi' + \frac{\partial}{\partial U}(\lambda' - C\lambda - U'\xi') = D\xi' + \frac{\partial}{\partial U}(\mu' - C\mu) = D\xi' + \frac{\partial}{\partial I}\nu',$$
$$L_p\omega = (\lambda' - C\lambda - U'\xi')dt + \left(\frac{\partial\lambda}{\partial U} - U'\frac{\partial\xi}{\partial U}\right)\omega = (\mu' - C\mu)dt + \frac{\partial\mu}{\partial U}\omega = e^{Ct}\nu' dt + \frac{\partial\nu}{\partial I}\omega.$$

The most simple condition $\nu' = 0$ says that the components ν in invariant frame are invariants of D.

The condition $\mu' = C\mu$ means that each entry of column μ is the derivative of preceding entry. Thus all entries of column μ in adapted frame are generated by the first entry $\mu_0 = f$ (*generating function*, see [1, p. 454]) by means of differentiation:

$$\mu_k = f^{(k)} = D^k f, \qquad k = 0, 1, 2, \ldots.$$

There is an obvious analogy between two equations $I = e^{-Ct}U$ and $v = e^{-Ct}\mu$.

The most complicated condition $\lambda' = C\lambda + U'\xi'$ is principal for the calculation of symmetries in natural basis (see [1, p. 244], [2, p. 110], [3, p. 55]).

Remark 1. In $J_{1,1}$ the invariants $I = e^{-Ct}U$ are described as follows:

$$I_k = \sum_{\ell=0}^{\infty} u^{(k+\ell)} \frac{(-t)^\ell}{\ell!}, \qquad k = 0, 1, 2, \ldots.$$

The operators $\frac{\partial}{\partial I}$ are basic Lie vector fields with generating functions $1, t, \frac{t^2}{2}, \ldots$ respectively, that is

$$\frac{\partial}{\partial I_0} = \frac{\partial}{\partial u},$$
$$\frac{\partial}{\partial I_1} = t\frac{\partial}{\partial u} + \frac{\partial}{\partial u'},$$
$$\frac{\partial}{\partial I_2} = \frac{t^2}{2}\frac{\partial}{\partial u} + t\frac{\partial}{\partial u'} + \frac{\partial}{\partial u''}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Remark 2. In $J_{n,m}$ instead of D we have a system of n operators D_i, $i = 1, 2, \ldots, n$, and instead of 1-dimensional trajectories we have n-dimensional orbits of the additive group $\mathbb{R}^n$. For example, in the space $J_{2,1}$ there are the 2-dimensional time $t = (t_1, t_2)$ and two operators

$$D_1 = \frac{\partial}{\partial t_1} + u_1\frac{\partial}{\partial u} + u_{11}\frac{\partial}{\partial u_1} + u_{12}\frac{\partial}{\partial u_2} + \cdots,$$
$$D_2 = \frac{\partial}{\partial t_2} + u_2\frac{\partial}{\partial u} + u_{12}\frac{\partial}{\partial u_1} + u_{22}\frac{\partial}{\partial u_2} + \cdots.$$

Herewith 2-dimensional orbits of $\mathbb{R}^2$ are determined by the series

$$u_t = u + u_1 t_1 + u_2 t_2 + \frac{1}{2}\left[u_{11}(t_1)^2 + 2u_{12}t_1 t_2 + u_{22}(t_2)^2\right] + \cdots$$

and its partial derivatives of all orders with respect to t_1 and t_2.

Remark 3. In $J_{2,1}$ the Lie field P with the generating function f can be represented in adapted and natural basis as follows:

$$P = \xi^1 D_1 + \xi^2 D_2 + f\frac{\partial}{\partial u} + f_1\frac{\partial}{\partial u_1} + f_2\frac{\partial}{\partial u_2} + \cdots$$

$$= \xi^1 \frac{\partial}{\partial t_1} + \xi^2 \frac{\partial}{\partial t_2} + \left(f + u_1\xi^1 + u_2\xi^2\right) \frac{\partial}{\partial u} + \left(f_1 + u_{11}\xi^1 + u_{12}\xi^2\right) \frac{\partial}{\partial u_1} + \left(f_2 + u_{12}\xi^1 + u_{22}\xi^2\right) \frac{\partial}{\partial u_2} + \cdots ,$$

where $f_i = D_i f$, $i = 1, 2$. The components $\lambda_k = f_k + u_{ki}\xi^i$, $k = 0, 1, 2, \ldots$ are consistent with the relation $\lambda = \mu + U'\xi$.

Theorem 1. *Any smooth vector field X without singularities on a manifold M can be connected with the total differentiation operator D in the jet space $J_{1,1}$, i.e. there exists a smooth map $\varphi : M \longrightarrow J_{1,1}$ such that the vector field X is φ-connected with the operator D.*

Proof. Let s be the canonical parameter of the vector field X, herewith $Xs = 1$. Take a smooth function f and calculate its derivatives with respect to X, $f^{(k)} = X^k f$, $k = 1, 2, \ldots$. Let F be the infinite column of elements $f, f', f'', \ldots$ and let us define the mapping φ by the relations

$$t \circ \varphi = s, \qquad U \circ \varphi = F. \tag{13}$$

At some step n we get the conditions

$$\Theta = df \wedge df' \wedge df'' \wedge \cdots \wedge df^{(n-1)} \neq 0 \qquad \text{and} \qquad \Theta \wedge df^{(n)} = 0. \tag{14}$$

There are two possible cases: a) $n = \dim M$, or b) $n < \dim M$.

Case a) $n = \dim M$. Let the functions $f, f', f'', \ldots, f^{(n-1)}$ be the coordinates on M and let us represent the field X as follows:

$$X = f' \frac{\partial}{\partial f} + f'' \frac{\partial}{\partial f'} + \cdots + f^{(n)} \frac{\partial}{\partial f^{(n-1)}}.$$

The Jacobian matrix of φ relate the components of X to the components of D (the subscripts mean the partial derivatives):

$$\begin{pmatrix} s_1 & \cdots & s_n \\ 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 1 \\ f_1^{(n)} & \cdots & f_n^{(n)} \\ \cdots & \cdots & \cdots \end{pmatrix} \cdot \begin{pmatrix} f' \\ \cdots \\ f^{(n)} \end{pmatrix} = \begin{pmatrix} s' \\ f' \\ \cdots \\ f^{(n)} \\ f^{(n+1)} \\ \cdots \end{pmatrix} = \begin{pmatrix} 1 \\ u' \\ \cdots \\ u^{(n)} \\ u^{(n+1)} \\ \cdots \end{pmatrix} \circ \varphi.$$

The rank of the Jacobian matrix is equal to n and φ is an immersion of M into $J_{1,1}$. The triple (X, s, F) on the manifold M is φ-connected with the triple (D, t, U) in the jet space $J_{1,1}$.

Case b) $n < N = \dim M$. It follows from $\Theta \wedge df^{(n)} = 0$ that $df^{(n)}$ is a linear combination of $df, df', df'', \ldots, df^{(n-1)}$,

$$df^{(n)} = \sum_{i=1}^{n} \alpha_i df^{(n-i)} \qquad \text{and} \qquad L_X\Theta = \alpha_1\Theta.$$

The functions $f, f', f'', \ldots, f^{(n-1)}$ determine a submersion $\pi : M \longrightarrow W$. The vector field X transports the fibers of π into the fibers of the same bundle and because of this the field X can be projected on the n-dimensional manifold W. In the coordinates $v^{(i)}$,

$$v^{(i)} \circ \pi = f^{(i)}, \qquad i = 0, 1, 2, \ldots, n-1,$$

the projection of X is a vector field

$$T\pi X = v'\frac{\partial}{\partial v} + v''\frac{\partial}{\partial v'} + \cdots + v^{(n-1)}\frac{\partial}{\partial v^{(n-2)}} + f^{(n)}\frac{\partial}{\partial v^{(n-1)}}$$

which can be connected by a map $\tilde{\varphi} : W \longrightarrow J_{1,1}$ with the operator D. Then the vector field X is φ-connected with D, where $\varphi = \tilde{\varphi} \circ \pi$.

General case. How to make the correspondence between a system of n vector fields Y_i on a manifold M with the operators of total differentiation D_i in the jet space $J_{n,m}$? Let u^α be the coordinates on M, u^i the canonical parameters of Y_i, $Y_i u^j = \delta_i^j$, and y_i^α the natural components of the fields Y_i. The operators

$$X_i = \frac{\partial}{\partial u^i} + Y_i = \frac{\partial}{\partial u^i} + y_i^\alpha \frac{\partial}{\partial u^\alpha}$$

determine a n-dimensional distribution in the "space-time" $R^n \times M$ with the coordinates (u^i, u^α), $i = 1, 2, \ldots, n$; $\alpha = n+1, \ldots, n+m$; $m = \dim M$. This is a particular case of connection in the fiber space, see [4], where the operators

$$X_i = \frac{\partial}{\partial u^i} + \Gamma_i^\alpha \frac{\partial}{\partial u^\alpha} \tag{15}$$

form in the coordinates (u^i, u^α) an adapted frame of the horizontal distribution Δ_h, with the components

$$\Gamma_i^\alpha = \Gamma_i^\alpha(u^j, u^\beta).$$

In our case we have $\Gamma_i^\alpha = y_i^\alpha(u^j)$. Let us immerse the operators X_i in the space $J_{n,m}$ with the help of the map $\varphi : M \longrightarrow J_{n,m}$ supposing

$$\begin{aligned} &t^i \circ \varphi = u^i, \quad u^\alpha \circ \varphi = u^\alpha, \quad u_i^\alpha \circ \varphi = \Gamma_i^\alpha, \\ &u_{ij}^\alpha \circ \varphi = X_{(i}\Gamma_{j)}^\alpha, \quad u_{ijk}^\alpha \circ \varphi = X_{(i}X_j\Gamma_{k)}^\alpha, \quad \ldots . \end{aligned} \tag{16}$$

The operators X_i and the vector fields Y_i are φ-connected with the operators D_i. ■

As corollaries we have the next Propositions.

Proposition 6. *If the vector field X is φ-connected with the operator D then for any function I in $J_{1,1}$ the derivatives $X(I \circ \varphi)$ and DI are φ-connected, i.e. $X(I \circ \varphi) = (DI) \circ \varphi$. From this it follows that $DI = 0 \Longrightarrow X(I \circ \varphi) = 0$ and all the invariants of D can be transported on the manifold M in the invariants of the vector field X. In particular the invariants $I = e^{-Ct}U$ are transported from $J_{1,1}$ on M in the invariants $I \circ \varphi = e^{-Cs}F$.*

Proposition 7. *If the vector field X is φ-connected with the operator D then all the covariant tensors can be transported from $J_{1,1}$ on the manifold M. For example, the Cartan forms $\omega = dU - U'dt$ can be transported in the forms $\omega \circ T\varphi = dF - F'dt$, where $F' = XF$. The sequence of Lie derivatives with respect to D (Cartan forms)*

$$\omega_0 = du - u'dt, \quad \omega_0' = du' - u''dt, \quad \omega_0'' = du'' - u''dt, \quad \ldots$$

induces the sequence of Lie derivatives with respect to X:

$$\omega_0 \circ T\varphi = df - f'ds, \quad \omega_0' \circ T\varphi = df' - f''ds, \quad \omega_0'' \circ T\varphi = df'' - f'''ds, \quad \ldots .$$

Proposition 8. *In the general case (16) the Cartan forms in $J_{n,m}$*

$$\omega^\alpha = du^\alpha - u^\alpha dt^i, \quad \omega_i^\alpha = du_i^\alpha - u_{ij}^\alpha dt^j, \quad \dots$$

induce on the manifold $\mathbb{R}^n \times M$ the sequence of 1-forms

$$\theta^\alpha = \omega^\alpha \circ T\varphi = du^\alpha - \Gamma_i^\alpha du^i, \quad \theta_i^\alpha = \omega_i^\alpha \circ T\varphi = d\Gamma_i^\alpha - X_{(i}\Gamma_{j)}^\alpha du^j, \quad \dots .$$

The horizontal distribution Δ_h is the annulator of the forms θ^α, i.e. $\theta^\alpha(X_i) = 0$. The forms θ_i^α imply the appearance of two important objects:

$$K_{ij}^\alpha = X_{[i}\Gamma_{j]}^\alpha, \quad \text{object of curvature,}$$

$$\Gamma_{i\beta}^\alpha = -\partial_\beta \Gamma_i^\alpha, \quad \text{object of connection.}$$

Namely, because $d\Gamma_i^\alpha = X_j\Gamma_i^\alpha du^j + \partial_\beta\Gamma_i^\alpha\theta^\beta$ and $X_j\Gamma_i^\alpha = X_{(i}\Gamma_{j)}^\alpha - X_{[i}\Gamma_{j]}^\alpha$ we have

$$\theta_i^\alpha = -K_{ij}^\alpha du^j - \Gamma_{i\beta}^\alpha\theta^\beta.$$

For the linear connection the quantities Γ_i^α are linear functions on the fibers: $\Gamma_i^\alpha = -\Gamma_{i\beta}^\alpha u^\beta$, and we have $K_{ij}^\alpha = -K_{ij\beta}^\alpha u^\beta$, where $K_{ij\beta}^\alpha = \partial_{[i}\Gamma_{j]\beta}^\alpha + \Gamma_{[i|\gamma|}^\alpha\Gamma_{j]\beta}^\gamma$, (see [4, p. 26]).

Extending the linear connection onto the tangent bundle $TM \longrightarrow M$ we get the affine connection on the manifold M in the classical sense.

Proposition 9. *The vertical distribution Δ_v is integrable because $\Delta_v = \operatorname{Ker} T\pi$ and the vector fields* (15) *are infinitesimals of Δ_v. For any coframe θ^i of Δ_v there exists an integrating matrix B_j^i such that $B_j^i\theta^j = du^i$. Then $B_k^i\theta^k(Xj) = \delta_j^i$ is unit matrix and B_j^i is inverse to the matrix $\theta^i(X_j)$.*

Let us mention that from (8) we have the same situation $e^{-Ct}\omega(\frac{\partial}{\partial I}) = E$. This generalizes the known property of integrating factor for $n = 1$ (see [1, p. 60]).

Proposition 10. *The vector field P represented in natural and adapted frames as follows (see [5, p. 286])*

$$P = \xi^i \frac{\partial}{\partial u^i} + \lambda^\alpha \frac{\partial}{\partial u^\alpha} = \xi^i X_i + \mu^\alpha \frac{\partial}{\partial u^\alpha}, \qquad \mu^\alpha = \lambda^\alpha - \Gamma_i^\alpha \xi^i,$$

is an infinitesimal symmetry of horizontal distribution Δ_h if and only if either

$$X_i\lambda^\alpha - P\Gamma_i^\alpha - \Gamma_j^\alpha X_i\xi^j = 0 \tag{17}$$

or

$$X_i\mu^\alpha + \Gamma_{i\beta}^\alpha\mu^\beta + 2K_{ij}^\alpha\xi^j = 0. \tag{18}$$

For the case

$$\Gamma_i^\alpha = -\Gamma_{i\beta}^\alpha u^\beta, \qquad \mu^\alpha = \mu_\beta^\alpha u^\beta, \qquad \lambda^\alpha = \lambda_\beta^\alpha u^\beta, \qquad \mu_\beta^\alpha = \lambda_\beta^\alpha + \Gamma_{i\beta}^\alpha \xi^i$$

the conditions (17) and (18) are equivalent to

$$\partial_i\lambda_\beta^\alpha - \lambda_\gamma^\alpha\Gamma_{i\beta}^\gamma + \Gamma_{i\gamma}^\alpha\lambda_\beta^\gamma + \partial_i\Gamma_{j\beta}^\alpha\xi^j + \Gamma_{j\beta}^\alpha X_i\xi^j = 0, \tag{19}$$

$$\partial_i\mu_\beta^\alpha - \mu_\gamma^\alpha\Gamma_{i\beta}^\gamma + \Gamma_{i\gamma}^\alpha\mu_\beta^\gamma - 2K_{ij\beta}^\alpha\xi^j = 0. \tag{20}$$

On the tangent bundle $TM \longrightarrow M$ we have the correspondence

$$(u^i, u^\alpha) \sim (u^i, du^i), \qquad (\xi^i, \lambda^\alpha) \sim (\xi^i, d\xi^i), \qquad \lambda_\beta^\alpha \sim \frac{\partial\xi^i}{\partial u^j}, \qquad \mu^\alpha \sim \frac{\partial\xi^i}{\partial u^j} + \Gamma_{kj}^i\xi^k$$

and the conditions (19) and (20) define P as an *affine collineation* (infinitesimal movement in the space of affine connection or *Killing's vector field* in Riemannian geometry), see [6, p. 37, formulae (2.30) and (2.31)].

Remark 4. For ODE $y' + p(x)y + q(x) = 0$ we have $\omega = (py + q)dx + dy$ and the condition (18) gives $\mu = e^{-\int p dx}$. The form

$$\frac{\omega}{\mu} = d\left(\frac{y}{\mu}\right) + \frac{q}{\mu}dx$$

is exact and determines the first integral (see [1, p. 160])

$$\frac{y}{\mu} + \int \frac{q}{\mu}dx.$$

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