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On the Liouville–Arnold Integrable Flows Related with Quantum Algebras and Their Poissonian Representations

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Based on the structure of Casimir elements associated with general Hopf algebras there Liouville–Arnold integrable flows, related with naturally induced Poisson structures on arbitrary co-algebra and their deformations, are constructed. Some interesting special cases including the oscillatory Heisenberg–Weil algebra, related co-algebra structures and adjoint with them integrable Hamiltonian systems are considered.

1 Hopf algebras and co-algebras: main definitions

Consider a Hopf algebra $\mathcal{A}$ over $\mathbb{C}$ endowed with two special homomorphisms called coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and counit $\varepsilon : \mathcal{A} \to \mathbb{C}$, as well an antihomomorphism (antipode) $\nu : \mathcal{A} \to \mathcal{A}$, such that for any $a \in \mathcal{A}$

$$(\mathrm{id} \otimes \Delta)\Delta(a) = (\Delta \otimes \mathrm{id})\Delta(a), \qquad (\mathrm{id} \otimes \varepsilon)\Delta(a) = (\varepsilon \otimes \mathrm{id})\Delta(a) = a,$$
$$m((\mathrm{id} \otimes \nu)\Delta(a)) = m((\nu \otimes \mathrm{id})\Delta(a)) = \varepsilon(a)I, \tag{1}$$

where $m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ is the usual multiplication mapping, that is for any $a, b \in \mathcal{A}$ $m(a\otimes b) = ab$. The conditions (1) were introduced by Hopf [1] in a cohomological context. Since most of the Hopf algebras properties depend on the coproduct operation $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and related with it Casimir elements, below we shall dwell mainly on the objects called co-algebras endowed with this coproduct.

The most interesting examples of co-algebras are provided by the universal enveloping algebras $U(\mathcal{G})$ of Lie algebras $\mathcal{G}$. If, for instance, a Lie algebra $\mathcal{G}$ possesses generators $X_i \in \mathcal{G}$, $i = \overline{1, n}$, $n = \dim \mathcal{G}$, the corresponding enveloping algebra $U(\mathcal{G})$ can be naturally endowed with a Hopf algebra structure by defining

$$\Delta(X_i) = I \otimes X_i + X_i \otimes I, \qquad \Delta(I) = I \otimes I, \qquad \varepsilon(X_i) = -X_i, \qquad \nu(I) = -I. \tag{2}$$

These mappings acting only on the generators of $\mathcal{G}$ are straightforwardly extended to any monomial in $U(\mathcal{G})$ by means of the homomorphism condition $\Delta(XY) = \Delta(X)\Delta(Y)$ for any $X, Y \in \mathcal{G} \subset U(\mathcal{G})$. In general an element $Y \in U(\mathcal{G})$ of a Hopf algebra such that $\Delta(Y) = I \otimes Y + Y \otimes I$ is called primitive, and the known Friedrichs theorem [2] ensures that in $U(\mathcal{G})$ the only primitive elements are exactly generators $X_i \in \mathcal{G}$, $i = \overline{1, n}$.

On the other hand, the homomorphism condition for the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ implies the compatibility of the coproduct with the Lie algebra commutator structure:

$$[\Delta(X_i), \Delta(X_j)]_{\mathcal{A}\otimes\mathcal{A}} = \Delta([X_i, X_j]_{\mathcal{A}}) \tag{3}$$

for any $X_i, X_j \in \mathcal{G}$, $i, j = \overline{1, n}$. Since the Drinfeld report [3] the co-algebras defined above are also often called "quantum" groups due to their importance [4] in studying many two-dimensional quantum models of modern field theory and statistical physics.

It was also observed (see for instance [4]) that the standard co-algebra structure (2) of the universal enveloping algebra $U(\mathcal{G})$ can be nontrivially extended by means of some of its infinitesimal deformations saving the co-associativity (3) of the deformed coproduct $\Delta : U_z(\mathcal{G}) \to U_z(\mathcal{G}) \otimes U_z(\mathcal{G})$ with $U_z(\mathcal{G})$ being the corresponding universal enveloping algebra deformation by means of a parameter $z \in \mathbb{C}$, such that $\lim\limits_{z\to 0} U_z(\mathcal{G}) = U(\mathcal{G})$ is subject to some natural topology on $U_z(\mathcal{G})$.

2 Casimir elements and their special properties

Take any Casimir element $C \in U_z(\mathcal{G})$ that is an element satisfying the condition $[C, U_z(\mathcal{G})] = 0$, and consider the action on it of the coproduct mapping Δ:

$$\Delta(C) = C(\{\Delta(X)\}), \tag{4}$$

where we put, by definition, $C := C(\{X\})$ with a set $\{X\} \subset \mathcal{G}$. It is a trivial consequence that for $\mathcal{A} := U_z(\mathcal{G})$

$$[\Delta(C), \Delta(X_i)]_{\mathcal{A}\otimes\mathcal{A}} = \Delta([C, X_i]_{\mathcal{A}}) = 0 \tag{5}$$

for any $X_i \in \mathcal{G}$, $i = \overline{1, n}$.

Define now recurrently the following N-th coproduct $\Delta^{(N)} : \mathcal{A} \to \overset{(N+1)}{\otimes} \mathcal{A}$ for any $N \in \mathbb{Z}_+$, where $\Delta^{(2)} := \Delta$ and $\Delta^{(1)} := \mathrm{id}$ and

$$\Delta^{(N)} := \big((\mathrm{id}\,\otimes)^{N-2} \otimes \Delta\big) \cdot \Delta^{(N-1)}, \tag{6}$$

or as

$$\Delta^{(N)} := \big(\Delta \otimes (\mathrm{id}\,\otimes)^{N-2} \otimes \mathrm{id} \otimes \mathrm{id}\big) \cdot \Delta^{(N-1)}. \tag{7}$$

One can straightforwardly verify that

$$\Delta^{(N)} := \big(\Delta^{(m)} \otimes \Delta^{(N-m)}\big) \cdot \Delta \tag{8}$$

for any $m = \overline{0, N}$, and the mapping $\Delta^{(N)} : \mathcal{A} \to \overset{(N+1)}{\otimes} \mathcal{A}$ is an algebras homomorphism, that is

$$\big[\Delta^{(N)}(X), \Delta^{(N)}(Y)\big]_{\overset{(N+1)}{\otimes}\mathcal{A}} = \Delta^{(N)}([X, Y]_{\mathcal{A}}) \tag{9}$$

for any $X, Y \in \mathcal{A}$. In a particular case if $\mathcal{A} = U(\mathcal{G})$, the following exact expression

$$\Delta^{(N)}(X) = X(\otimes\,\mathrm{id})^{N-1} \otimes \mathrm{id} + \mathrm{id} \otimes X(\otimes\,\mathrm{id})^{N-1} \otimes \mathrm{id} + \cdots + (\otimes\,\mathrm{id})^{N-1} \otimes \mathrm{id} \otimes X \tag{10}$$

holds for any $X \in \mathcal{G}$.

3 Poisson co-algebras and their realizations

As is well known [5, 6], a Poisson algebra $\mathcal{P}$ is a vector space endowed with a commutative multiplication and a Lie bracket $\{\cdot,\cdot\}$ including a derivation on $\mathcal{P}$ in the form

$$\{a, bc\} = b\{a, c\} + \{a, b\}c \tag{11}$$

for any a, b and $c \in \mathcal{P}$. If $\mathcal{P}$ and $\mathcal{Q}$ are Poisson algebras one can naturally define the following Poisson structure on $\mathcal{P} \otimes \mathcal{Q}$:

$$\{a \otimes b, c \otimes d\}_{\mathcal{P}\otimes\mathcal{Q}} = \{a, c\}_{\mathcal{P}} \otimes (bd) + (ac) \otimes \{b, d\}_{\mathcal{Q}} \tag{12}$$

for any $a, c \in \mathcal{P}$ and $b, d \in \mathcal{Q}$. We shall also say that $(\mathcal{P};\Delta)$ is a Poisson co-algebra if $\mathcal{P}$ is a Poisson algebra and $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ is a Poisson algebras homomorphism, that is

$$\{\Delta(a), \Delta(b)\}_{\mathcal{P}\otimes\mathcal{P}} = \Delta(\{a, b\}_{\mathcal{P}}) \tag{13}$$

for any $a, b \in \mathcal{P}$.

It is useful to note here that any Lie algebra $\mathcal{G}$ generates naturally a Poisson co-algebra $(\mathcal{P};\Delta)$ by defining a Poisson bracket on $\mathcal{P}$ by means of the following expression: for any $a, b \in \mathcal{P}$

$$\{a, b\}_{\mathcal{P}} := \langle \mathrm{grad}, \vartheta\, \mathrm{grad}\, b\rangle. \tag{14}$$

Here $\mathcal{P} \simeq C^\infty(\mathbb{R}^n;\mathbb{R})$ is a space of smooth mappings linked with base variables of the Lie algebra $\mathcal{G}$, $n = \dim \mathcal{G}$, and the implectic [6] matrix $\vartheta : T^*(\mathcal{P}) \to T(\mathcal{P})$ is given as

$$\vartheta(x) = \left\{ \sum_{k=1}^{n} c_{ij}^k x_k : \ i, j = \overline{1, n} \right\}, \tag{15}$$

where c_{ij}^k, $i, j, k = \overline{1, n}$, are the corresponding structure constants of the Lie algebra $\mathcal{G}$ and $x \in \mathbb{R}^n$ are the corresponding linked coordinates. It is easy to check that the coproduct (1.2) is a Poisson algebras homomorphism between $\mathcal{P}$ and $\mathcal{P}\otimes\mathcal{P}$. If one can find a "quantum" deformation $U_z(\mathcal{G})$, then the corresponding Poisson co-algebra $\mathcal{P}_z$ can be constructed with the naturally deformed implectic matrix $\vartheta_z : T^*(\mathcal{P}_z) \to T(\mathcal{P}_z)$. For instance, if $\mathcal{G} = so(2, 1)$, there exists a deformation $U_z(so(2, 1))$ defined by the following deformed commutator relations with a parameter $z \in \mathbb{C}$:

$$[\tilde{X}_2, \tilde{X}_1] = \tilde{X}_3, [\tilde{X}_2, \tilde{X}_3] = -\tilde{X}_1, \qquad [\tilde{X}_3, \tilde{X}_1] = \frac{1}{z}\sinh(z\tilde{X}_2), \tag{16}$$

where at $z = 0$ elements $\tilde{X}_i\big|_{z=0} = X_i \in so(2, 1)$, $i = \overline{1, 3}$, compile a base of generators of the Lie algebra $so(2, 1)$. Then, based on expressions (16) one can easily construct the corresponding Poisson co-algebra $\mathcal{P}_z$, endowed with the implectic matrix

$$\vartheta_z(\tilde{x}) = \begin{pmatrix} 0 & -\tilde{x}_3 & -\frac{1}{z}\sinh(z\tilde{x}_2) \\ \tilde{x}_3 & 0 & -\tilde{x}_1 \\ \frac{1}{z}\sinh(z\tilde{x}_2) & \tilde{x}_1 & 0 \end{pmatrix} \tag{17}$$

for any point $\tilde{x} \in \mathbb{R}^3$, linked naturally with the deformed generators $\tilde{X}_i$, $i = \overline{1, 3}$, taken above. Since the corresponding coproduct on $U_z(so(2, 1))$ acts on this deformed base of generators as

$$\begin{gathered} \Delta(\tilde{X}_2) = I \otimes \tilde{X}_2 + \tilde{X}_2 \otimes I, \qquad \Delta(\tilde{X}_1) = \exp\left(-\frac{z}{2}\tilde{X}_2\right) \otimes \tilde{X}_1 + \tilde{X}_1 \otimes \exp\left(\frac{z}{2}\tilde{X}_2\right), \\ \Delta(\tilde{X}_2) = \exp\left(-\frac{z}{2}\tilde{X}_2\right) \otimes \tilde{X}_3 + \tilde{X}_3 \otimes \exp\left(\frac{z}{2}\tilde{X}_2\right), \end{gathered} \tag{18}$$

satisfying the main homomorphism property for the whole deformed universal enveloping algebra $U_z(so(2,1))$.

Consider now some realization of the deformed generators $\tilde{X}_i \in U_z(\mathcal{G})$, $i=\overline{1,n}$, that is a homomorphism mapping $D_z : U_z(\mathcal{G}) \to \mathcal{P}(M)$, such that

$$D_z(\tilde{X}_i) = \tilde{e}_i, \qquad i=\overline{1,n}, \tag{19}$$

are some elements of a Poisson manifold $\mathcal{P}(M)$ realized as a space of functions on a finite-dimensional manifold M, satisfying the deformed commutator relationships $\{\tilde{e}_i,\tilde{e}_j\}_{\mathcal{P}(M)} = \vartheta_{z,ij}(\tilde{e})$, where, by definition, expressions $[\tilde{X}_i,\tilde{X}_j] = \vartheta_{z,ij}(\tilde{X})$, $i,j=\overline{1,n}$, generate a Poisson co-algebra structure on the function space $\mathcal{P}_z := \mathcal{P}_z(\mathcal{G})$ linked with a given Lie algebra $\mathcal{G}$. Making use of the homomorphism property (13) for the coproduct mapping $\Delta : \mathcal{P}_z(\mathcal{G}) \to \mathcal{P}_z(\mathcal{G}) \otimes \mathcal{P}_z(\mathcal{G})$, one finds that for all $i,j=\overline{1,n}$

$$\{\Delta(\tilde{x}_i),\Delta(\tilde{x}_j)\}_{\mathcal{P}_z(\mathcal{G})\otimes\mathcal{P}_z(\mathcal{G})} = \Delta(\{\tilde{x}_i,\tilde{x}_j\}_{\mathcal{P}_z(\mathcal{G})} = \vartheta_{z,ij}(\Delta(\tilde{x})) \tag{20}$$

and for the corresponding coproduct $\Delta : \mathcal{P}(M) \to \mathcal{P}(M) \otimes \mathcal{P}(M)$ one gets similarly

$$\{\Delta(\tilde{e}_i),\Delta(\tilde{e}_j)\}_{\mathcal{P}(M)\otimes\mathcal{P}(M)} = \Delta(\{\tilde{e}_i,\tilde{e}_j\}_{\mathcal{P}(M)} = \vartheta_{z,ij}(\Delta(\tilde{e})), \tag{21}$$

where $\{\cdot,\cdot\}_{\mathcal{P}(M)}$ is some, eventually, canonical Poisson structure on a finite-dimensional manifold M.

Let $q \in M$ be a point of M and consider its coordinates as elements of $\mathcal{P}(M)$. Then one can define the following elements

$$q_j := (I\otimes)^{j-1} q (\otimes I)^{N-j} \in \overset{(N)}{\otimes} \mathcal{P}(M), \tag{22}$$

where $j=\overline{1,N}$ by means of which one can construct the corresponding N-tuple realization of the Poisson co-algebra structure (21) as follows:

$$\{\tilde{e}_i^{(N)},\tilde{e}_j^{(N)}\}_{\overset{(N)}{\otimes}\mathcal{P}(M)} = \vartheta_{z,ij}\big(\tilde{e}^{(N)}\big), \tag{23}$$

with $i,j=\overline{1,n}$ and

$$\overset{(N)}{\otimes} D_z\big(\Delta^{(N-1)}(\tilde{e}_i)\big) := \tilde{e}_i^{(N)}(q_1,q_2,\ldots,q_N). \tag{24}$$

For instance, for the $U_z(so(2,1))$ case (16), one can take [7] the realization Poisson manifold $\mathcal{P}(M)=\mathcal{P}(\mathbb{R}^2)$ with the standard canonical Heisenberg–Weil Poissonian structure on it:

$$\{q,q\}_{\mathcal{P}(\mathbb{R}^2)} = 0 = \{p,p\}_{\mathcal{P}(\mathbb{R}^2)}, \qquad \{p,q\}_{\mathcal{P}(\mathbb{R}^2)} = 1, \tag{25}$$

where $(q,p)\in\mathbb{R}^2$. Then expressions (24) for $N=2$ give rise to the following relationships

$$\begin{aligned}
\tilde{e}_1^{(2)}(q_1,q_2,p_1,p_2) &:= (D_z\otimes D_z)\Delta(\tilde{X}_1)\\
&= 2\frac{\sinh\left(\frac{z}{2}p_1\right)}{z}\cos q_1 \exp\left(\frac{z}{2}p_1\right) + 2\exp\left(-\frac{z}{2}p_1\right)\frac{\sinh\left(\frac{z}{2}p_2\right)}{z}\cos q_2,\\
\tilde{e}_2^{(2)}(q_1,q_2,p_1,p_2) &:= (D_z\otimes D_z)\Delta(\tilde{X}_2) = p_1+p_2,\\
\tilde{e}_3^{(2)}(q_1,q_2,p_1,p_2) &:= (D_z\otimes D_z)\Delta(\tilde{X}_3)\\
&= 2\frac{\sinh\left(\frac{z}{2}p_1\right)}{z}\sin q_1 \exp\left(\frac{z}{2}p_2\right) + 2\exp\left(-\frac{z}{2}p_1\right)\frac{\sinh\left(\frac{z}{2}p_2\right)}{z}\sin q_2,
\end{aligned} \tag{26}$$

where elements $(q_1,q_2,p_1,p_2)\in\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)$ satisfy the induced by (25) Heisenberg–Weil commutator relations:

$$\{q_i,q_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)} = 0 = \{p_i,p_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)}, \qquad \{p_i,q_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)} = \delta_{ij} \tag{27}$$

for any $i,j=\overline{1,2}$.

4 Casimir elements and the Heisenberg–Weil algebra related algebraic structures

Consider any Casimir element $\tilde{C} \in U_z(\mathcal{G})$ related with an $\mathbb{R} \ni z$-deformed Lie algebra $\mathcal{G}$ structure in the form

$$[\tilde{X}_i, \tilde{X}_j] = \vartheta_{z,ij}(\tilde{X}), \tag{28}$$

where $i, j = \overline{1,n}$, $n = \dim \mathcal{G}$, and, by definition, $[\tilde{C}, \tilde{X}_i] = 0$. The following general lemma holds.

Lemma 1. *Let* $(U_z(\mathcal{G}); \Delta)$ *be a co-algebra with generators satisfying* (28) *and* $\tilde{C} \in U_z(\mathcal{G})$ *be its Casimir element; then*

$$[\Delta^{(m)}(\tilde{C}), \Delta^{(N)}(\tilde{X}_i)]_{\overset{(N+1)}{\otimes} U_z(\mathcal{G})} = 0 \tag{29}$$

for any $i = \overline{1,n}$ *and* $m = \overline{1,N}$.

As a simple corollary of this Lemma one finds from (29) that

$$[\Delta^{(m)}(\tilde{C}), \Delta^{(N)}(\tilde{C})]_{\overset{(N+1)}{\otimes} U_z(\mathcal{G})} = 0$$

for any $k, m \in \mathbb{Z}_+$.

Consider now some realization (19) of our deformed Poisson co-algebra structure (28) and check that the expression

$$\left[\Delta^{(m)}(C(\tilde{e})), \Delta^{(N)}(\mathcal{H}(\tilde{e}))\right]_{\overset{(N+1)}{\otimes} \mathcal{P}(M)} = 0 \tag{30}$$

too for any $m = \overline{1,N}$, $N \in \mathbb{Z}_+$, if $C(\tilde{e}) \in I(\mathcal{P}(M))$, that is $\{C(\tilde{e}), q\}_{\mathcal{P}(M)} = 0$ for any $q \in M$. Since

$$\mathcal{H}^{(N)}(q) := \Delta^{(N-1)}(\mathcal{H}(\tilde{e})) \tag{31}$$

are in general, smooth functions on $\overset{(N+1)}{\otimes} M$, which can be used as Hamilton ones subject to the Poisson structure on $\overset{(N+1)}{\otimes} \mathcal{P}(M)$, the expressions (31) mean nothing else that functions

$$\gamma^{(m)}(q) := \Delta^{(N)}(C(\tilde{e})) \tag{32}$$

are their invariants, that is

$$\left\{\gamma^{(m)}(q), \mathcal{H}^{(N)}(q)\right\}_{\overset{(N+1)}{\otimes} \mathcal{P}(M)} = 0 \tag{33}$$

for any $m = \overline{1,N}$. Thereby, the functions (31) and (32) generate under some additional but natural conditions a hierarchy of a priori Liouville–Arnold integrable Hamiltonian flows on the Poisson manifold $\overset{(N+1)}{\otimes} \mathcal{P}(M)$.

Consider now a case of a Poisson manifold $\mathcal{P}(M)$ and its co-algebra deformation $\mathcal{P}_z(\mathcal{G})$. Thus for any coordinate points $x_i \in \mathcal{P}(\mathcal{G})$, $i = \overline{1,n}$, the following relationships

$$\{x_i, x_j\} = \sum_{k=1}^{n} c_{ij}^k x_k := \vartheta_{ij}(x) \tag{34}$$

define a Poisson structure on $\mathcal{P}(\mathcal{G})$, related with the corresponding Lie algebra structure of $\mathcal{G}$, and there exists a representation (19), such that elements $\tilde{e}_i := D_z(\tilde{X}_i) = \tilde{e}_i(x)$ satisfy the relationships $\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}_z(\mathcal{G})} = \vartheta_{z,ij}(\tilde{e})$ for any $i = \overline{1,n}$, with the limiting conditions

$$\lim_{z\to 0} \vartheta_{z,ij}(\tilde{e}) = \sum_{k=1}^{n} c_{ij}^k x_k, \qquad \lim_{z\to 0} \tilde{e}_i(x) = x_i \tag{35}$$

for any $i, j = \overline{1,n}$ being held. For instance, take the Poisson co-algebra $\mathcal{P}_z(so(2,1))$ for which there exists a realization (19) in the following form:

$$\tilde{e}_1 := D_z(\tilde{X}_1) = \frac{\sinh\left(\frac{z}{2}x_2\right)}{zx_2} x_1, \qquad \tilde{e}_2 := D_z(\tilde{X}_2) = x_2, \qquad \tilde{e}_3 := D_z(\tilde{X}_3) = \frac{\sinh\left(\frac{z}{2}x_2\right)}{zx_2} x_3,$$

where $x_i \in \mathcal{P}(so(2,1))$, $i = \overline{1,3}$, satisfy the $so(2,1)$-commutator relationships

$$\{x_2, x_1\}_{\mathcal{P}(so(2,1))} = x_3, \qquad \{x_2, x_3\}_{\mathcal{P}(so(2,1))} = -x_1, \qquad \{x_3, x_1\}_{\mathcal{P}(so(2,1))} = x_2, \tag{36}$$

with the coproduct operator $\Delta : \mathcal{U}_z(so(2,1)) \to \mathcal{U}_z(so(2,1)) \otimes \mathcal{U}_z(so(2,1))$ being given by (18). It is easy to check that conditions (34) and (35) hold.

The next example is related with the co-algebra $\mathcal{U}_z(\pi(1,1))$ of the Poincaré algebra $\pi(1,1)$ for which the following non-deformed relationships

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_2, \qquad [X_3, X_2] = 0 \tag{37}$$

hold. The corresponding coproduct $\Delta : \mathcal{U}_z(\pi(1,1)) \to \mathcal{U}_z(\pi(1,1)) \otimes \mathcal{U}_z(\pi(1,1))$ is given by the Woronowicz [8] expressions

$$\begin{aligned} &\Delta(\tilde{X}_1) = I \otimes \tilde{X}_1 + \tilde{X}_1 \otimes I, \qquad \Delta(\tilde{X}_2) = \exp\left(-\frac{z}{2}\tilde{X}_1\right) \otimes \tilde{X}_1 + \tilde{X}_1 \otimes \exp\left(\frac{z}{2}\tilde{X}_1\right), \\ &\Delta(\tilde{X}_3) = \exp\left(-\frac{z}{2}\tilde{X}_1\right) \otimes \tilde{X}_3 + \tilde{X}_3 \otimes \exp\left(\frac{z}{2}\tilde{X}_1\right), \end{aligned} \tag{38}$$

where $z \in \mathbb{R}$ is a parameter. Under the deformed expressions (38) the elements $\tilde{X}_j \in \mathcal{U}_z(\pi(1,1))$, $j = \overline{1,3}$, still satisfy undeformed commutator relationships, that is $\vartheta_{z,ij}(\tilde{X}) = \vartheta_{ij}(X)\big|_{X \Rightarrow \tilde{X}}$ for any $z \in \mathbb{R}$, $i, j = \overline{1,3}$, being given by (37). As a result, we can state that $\tilde{e}_i := D_z(\tilde{X}_i) = \tilde{e}_i(x) = x_i$, where for $x_i \in \mathcal{P}(\pi(1,1))$, $i = \overline{1,3}$, the following Poisson structure

$$\{x_1, x_2\}_{\mathcal{P}(\pi(1,1))} = x_3, \qquad \{x_1, x_3\}_{\mathcal{P}(\pi(1,1))} = x_2, \qquad \{x_3, x_2\}_{\mathcal{P}(\pi(1,1))} = 0 \tag{39}$$

holds. Moreover, since $C = x_2^2 - x_3^2 \in I(\mathcal{P}(\pi(1,1)))$, that is $\{C, x_i\}_{\mathcal{P}(\pi(1,1))} = 0$ for any $i = \overline{1,3}$, one can construct, making use of (31) and (32), integrable Hamiltonian systems on $\overset{(N)}{\otimes} \mathcal{P}(\pi(1,1))$. The same one can do subject to the discussed above Poisson co-algebra $\mathcal{P}_z(so(2,1))$ realized by means of the Poisson manifold $\mathcal{P}(so(2,1))$, taking into account that the following element $C = x_2^2 - x_1^2 - x_3^2 \in I(\mathcal{P}(so(2,1)))$ is a Casimir one.

Now we will consider a special extended Heisenberg–Weil co-algebra $\mathcal{U}_z(h_4)$, called still the oscillator co-algebra. The undeformed Lie algebra h_4 commutator relationships take the form:

$$[n, a_+] = a_+, \qquad [n, a_-] = -a_-, \qquad [a_-, a_+] = m, \qquad [m, \cdot] = 0, \tag{40}$$

where $\{n, a_\pm, m\} \subset h_4$ compile a basis of h_4, $\dim h_4 = 4$. The Poisson co-algebra $\mathcal{P}(h_4)$ is naturally endowed with the Poisson structure like (40) and admits its realization (19) on the Poisson manifold $\mathcal{P}(\mathbb{R}^2)$. Namely, on $\mathcal{P}(\mathbb{R}^2)$ one has

$$e_\pm = D(a_\pm) = \sqrt{p}\exp(\mp q), \qquad e_1 = D(m) = 1, \qquad e_0 = D(n) = p, \tag{41}$$

where $(q, p) \in \mathbb{R}^2$ and the Poisson structure on $\mathcal{P}(\mathbb{R}^2)$ is canonical, that is the same as (25).

Closely related with the relationships (40) there is a generalized $\mathcal{U}_z(su(2))$ co-algebra, for which

$$[x_3, x_\pm] = \pm x_\pm, \qquad [y_\pm, \cdot] = 0, \qquad [x_+, x_-] = y_+ \sin(2zx_3) + y_- \cos(2zx_3))\frac{1}{\sin z}, \tag{42}$$

where $z \in \mathbb{C}$ is an arbitrary parameter. The co-algebra structure is given now as follows:

$$\begin{aligned}
&\Delta(x_\pm) = c^\pm_{1(2)} e^{izx_3} \otimes x_\pm + x_\pm \otimes c^\pm_{2(1)} e^{-izx_3}, \qquad \Delta(x_3) = I \otimes x_3 + x_3 \otimes I,\\
&\Delta(c^\pm_i) = c^\pm_i \otimes c^\pm_i, \qquad \nu(x_\mp) = -(c^\pm_{1(2)})^{-1} e^{-izx_3} x_\mp e^{izx_3} (c^\pm_{2(1)})^{-1},\\
&\nu(c^\pm_i) = (c^\pm_i)^{-1}, \qquad \nu(e^{\pm izx_3}) = e^{\mp izx_3}
\end{aligned} \tag{43}$$

with $c^\pm_i \in \mathcal{U}_z(su(2))$, $i = \overline{1,2}$, being fixed elements. One can check that the Poisson structure on $\mathcal{P}_z(su(2))$ corresponding to (42) can be realized by means of the canonical Poisson structure on the phase space $\mathcal{P}(\mathbb{R}^2)$ as follows:

$$\begin{aligned}
&[q, p] = i, \qquad D_z(x_3) = q, \qquad D_z(x_\mp) = e^{\pm ip} g_z(q),\\
&g_z(q) = (k + \sin[z(s-q)])(y_+ \sin[(q+s+1)] + y_- \cos[z(q+s+1)])^{1/2}\frac{1}{\sin z},
\end{aligned} \tag{44}$$

where $k, s \in \mathbb{C}$ are constant parameters. Thereby making use of (32) and (33), one can construct a new class of Liouville integrable Hamiltonian flows.

5 The Heisenberg–Weil co-algebra structure and related integrable flows

Consider the Heisenberg–Weil algebra commutator relationships (40) and the following homogenous quadratic forms related with them

$$\left.\begin{aligned}
&x_1x_2 - x_2x_1 - \alpha x_3^2 = 0,\\
&x_1x_3 - x_3x_1 = 0, \qquad x_2x_3 - x_3x_2 = 0
\end{aligned}\right\} R(x), \tag{45}$$

where $\alpha \in \mathbb{C}$, $x_i \in A$, $i = \overline{1,3}$, are some elements of a free associative algebra A. The quadratic algebra $A/R(x)$ can be deformed via

$$\left.\begin{aligned}
&x_1x_2 - z_1x_2x_1 - \alpha x_3^2 = 0,\\
&x_1x_3 - z_2x_3x_1 = 0, \qquad x_2x_3 - z_2^{-1}x_3x_2 = 0,
\end{aligned}\right\} R_z(x), \tag{46}$$

where $z_1, z_2 \in \mathbb{C}\backslash\{0\}$ are some parameters.

Let V be the vector space of columns $X := (x_1, x_2, x_3)^\intercal$ and define the following action h_T: $V \to (V \otimes V^*) \otimes V$, where, by definition, for any $X \in V$ $h_T(X) = T \otimes X$. It is easy to check that conditions (46) will be satisfied if the following relations [9]

$$\begin{aligned}
&T_{11}T_{33} = T_{33}T_{11}, \qquad T_{12}T_{33} = z_2^{-2}T_{33}T_{12}, \qquad T_{21}T_{33} = z_1^2T_{33}T_{21},\\
&T_{22}T_{33} = T_{33}T_{22}, \qquad T_{31}T_{33} = z_2T_{33}T_{31}, \qquad T_{32}T_{33} = z_1^{-1}T_{33}T_{32},\\
&T_{11}T_{12} = z_1T_{12}T_{11}, \qquad T_{21}T_{22} = z_1T_{22}T_{21},\\
&z_2T_{11}T_{32} - z_2T_{32}T_{11} = z_1z_2T_{12}T_{31} - T_{31}T_{12},\\
&T_{21}T_{32} - z_1z_2T_{32}T_{21} = z_1T_{22}T_{31} - z_2T_{31}T_{22},\\
&T_{11}T_{22} - T_{22}T_{11} = z_1T_{12}T_{21} - z_1^{-1}T_{21}T_{12},\\
&(T_{11}T_{22} - z_1T_{12}T_{21}) = \alpha T_{33}^2 - T_{31}T_{32} + z_1T_{32}T_{31}
\end{aligned} \tag{47}$$

hold. Put now for further convenience $z_1 = z_2^2 := z^2 \in \mathbb{C}$ and compute the "quantum" determinant $D(T)$ of the matrix $T : (A/R_z(x))^3 \to (A/R_z(x))^3$:

$$D(T) = (T_{11}T_{22} - z^{-2}T_{21}T_{12})T_{33}. \tag{48}$$

Remark here that the determinant (48) is not central, that is

$$\begin{aligned} &D^{-1}T_{11} = T_{11}D^{-1}, \qquad D^{-1}T_{12} = z^{-6}T_{12}D^{-1}, \qquad D^{-1}T_{33} = T_{33}D^{-1},\\ &z^{-6}D^{-1}T_{21} = T_{12}D^{-1}, \qquad D^{-1}T_{22} = T_{22}D^{-1}, \qquad z^{-3}D^{-1}T_{31} = T_{31}D^{-1},\\ &D^{-1}T_{32} = z^{-3}T_{32}D^{-1}. \end{aligned} \tag{49}$$

Taking into account properties (47)–(49), one can construct the Heisenberg–Weil related co-algebra $\mathcal{U}_z(h)$ being a Hopf algebra with the following coproduct Δ, counit ε and antipode ν:

$$\begin{aligned} &\Delta(T) := T \otimes T, \qquad \Delta\big(D^{-1}\big) := D^{-1} \otimes D^{-1},\\ &\varepsilon(T) := I, \qquad \varepsilon\big(D^{-1}\big) := I, \qquad \nu(T) := T^{-1}, \qquad \nu(D) := D^{-1}. \end{aligned} \tag{50}$$

Based now on relationships (47), one can easily construct the Poisson tensor

$$\{\Delta(\tilde{T}), \Delta(\tilde{T})\}_{\mathcal{P}_z(h)\otimes\mathcal{P}_z(h)} = \Delta(\{\tilde{T}, \tilde{T}\}_{\mathcal{P}_z(h)}) := \vartheta_z(\Delta(\tilde{T})),$$

subject to which all of functionals (32) will be commuting to each other, and moreover, will be Casimir ones. Choosing some appropriate Hamiltonian functions $\mathcal{H}^{(N)}(\tilde{T}) := \Delta^{(N-1)}(\mathcal{H}(\tilde{T}))$ for $N \in \mathbb{Z}_+$ one makes it possible to present a priori nontrivial integrable Hamiltonian systems. On the other hand, the co-algebra $\mathcal{U}_z(h)$ built by (49) and (50) possesses the following fundamental $\mathcal{R}$-matrix [4] property:

$$\mathcal{R}(z)(T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I)\mathcal{R}(z)$$

for some complex-valued matrix $\mathcal{R}(z) \in \mathrm{Aut}\,(\mathbb{C}^3 \otimes \mathbb{C}^3)$, $z \in \mathbb{C}$. The latter, as is well known [4], gives rise to a regular procedure of constructing of an infinite hierarchy of Liouville-integrable operator (quantum) Hamiltonian systems on related quantum Poissonian phase spaces. On their special cases interesting for applications we plan to go on in another place.

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