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Twisted Product of Fock Spaces

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We discuss the construction of the twisted product of the interacting Fock spaces.

1 Introduction

Recall that the C^*-probability space is the pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a C^*-algebra and ϕ is a state. The self-adjoint element $a \in \mathcal{A}$ is called a self-adjoint quantum random variable. The distribution of a with respect to ϕ is a probability measure μ on $\mathbb{R}$ such that

$$\phi(a^k) = \int_{\mathbb{R}} t^k d\mu(t).$$

Remark 1. If $\mathcal{A}$ is a $*$-algebra or W^*-algebra then the corresponding spaces are called the $*$-algebraic and W^*-algebraic quantum probability spaces respectively.

The notion of monotone independence was introduced by N. Muraki in [10].

Definition 1. The family $\{Y_i,\ i = 1, \ldots, n\}$ of random variables in $(\mathcal{A}, \phi)$ is called anti-monotone independent if the following two conditions hold

1. $Y_i^p Y_j^r Y_k^s = \phi(Y_j^r) Y_i^p Y_k^s$ for any $i > j < k$ and $p, s \in \mathbb{N}$.
2. For any $i_1 < i_2 < \cdots < i_s < j > j_t > \cdots > j_2 > j_1$

$$\phi\big(Y_{i_1}^{k_1} \cdots Y_{i_s}^{k_s} Y_j^l Y_{j_t}^{r_t} \cdots Y_{j_1}^{r_1}\big) = \prod_{\nu=1}^{s} \phi\big(Y_{i_\nu}^{k_\nu}\big) \phi\big(Y_j^l\big) \prod_{\omega=1}^{t} \phi\big(Y_{i_\omega}^{k_\omega}\big).$$

The notation $i < j > k$ means $i < j$, $k < j$ and $i > j < k$ means $i > j$, $k > j$.

The notion of boolean independence was studied by many authors, see for example [11].

Definition 2. Let $(\mathcal{B}, \phi)$ be the $*$-algebraic probability space. The family of elements $\{X_i,\ i \in \mathcal{I}\}$ is called boolean independent if one has

$$\phi\big(X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_n}^{k_n}\big) = \phi\big(X_{i_1}^{k_1}\big) \phi\big(X_{i_2}^{k_2}\big) \cdots \phi\big(X_{i_n}^{k_n}\big)$$

for any $i_1 \neq i_2 \neq \cdots \neq i_n \in \mathcal{I}$.

The notion of the interacting Fock space was introduced by L. Accardi, Y. Lu and I. Volovich, see [2]. Let $\mathcal{H}$ be the Hilbert space. Denote by $\mathcal{T}(\mathcal{H})$ the full tensor space over $\mathcal{H}$ and by ω the vacuum vector. Then construct operators of the creation $a(f)$, $f \in \mathcal{H}$,

$$a(f) f_1 \otimes f_2 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n, \qquad n \in \mathbb{N}, \quad a(f)\omega = f.$$

Finally for any $n \in \mathbb{N}$ we supply $\mathcal{H}^{\otimes n}$ with some scalar product $\langle \cdot \mid \cdot \rangle_n$ and define the scalar product $\langle \cdot \mid \cdot \rangle$ on the $\mathcal{T}(\mathcal{H})$ by the rule

$$\mathcal{H}^{\otimes n} \perp \mathcal{H}^{\otimes m}, \quad m \neq n, \qquad \langle x \mid y \rangle = \langle x \mid y \rangle_n, \quad x, y \in \mathcal{H}^{\otimes n}.$$

The operators $a^*(f)$, $f \in \mathcal{H}$ are called the annihilation operators. The following properties of the annihilation operators are obvious

$$a^*(f)\omega = 0, \qquad a^*(f)\colon \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n-1}, \qquad n \in \mathbb{N}.$$

Definition 3. The system $\mathcal{F} = \Big(\mathcal{T}(H), \langle \cdot \mid \cdot \rangle, a(f), a^*(f),\ f \in \mathcal{H}\Big)$ is called the interacting Fock space.

The operators $X(f) = a(f) + a^*(f)$, $f \in \mathcal{H}$ are called the field operators and the operator N, defined by the rule

$$N\omega = 0, \qquad N(f_1 \otimes \cdots \otimes f_k) = k f_1 \otimes \cdots \otimes f_k, \qquad k \in \mathbb{N}$$

is called the number operator. The vector state associated with vacuum vector ω is called Fock state. If $\dim \mathcal{H} = 1$ then the corresponding interacting Fock space is called a one-mode interacting Fock space.

The role of the one-mode interacting Fock spaces in quantum probability was clarified by L. Accardi and M. Bożejko in [1]. Namely, it was shown that any self-adjoint quantum variable can be realized in the form $a + a^* + f(N)$, where a, a^*, N are the creation, annihilation and the number operators acting on some one-mode interacting Fock space.

Let us discuss some type of general central limit theorem of the type considered by R. Speicher and W. von Waldenfels, see [15].

Theorem 1. *Consider the $*$-algebra $\mathcal{A}$ and state ϕ and the sequence of elements $a_i, a_i^* \in \mathcal{A}$, $i \in \mathbb{N}$. Denote by $b_{2i-1} := a_i$, $b_{2i} := a_i^*$ and $X_i := a_i + a_i^*$. Put S_N to be*

$$S_N = \frac{X_1 + \cdots + X_N}{\sqrt{N}}.$$

Suppose that the following assumptions are satisfied.

(i) For any odd $n \in \mathbb{N}$ one has $\phi(b_{\sigma(1)} \cdots b_{\sigma(n)}) = 0$.

(ii) For even n the mixed moment $\phi(b_{\sigma(1)} \cdots b_{\sigma(n)}) \neq 0$ only if $(\sigma(1), \ldots, \sigma(n))$ is the permutation (with replications) of the collection

$$\{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \ldots, 2i_k - 1, 2i_k,\ k \le n/2\}.$$

(iii) Let $(\sigma(1), \ldots, \sigma(n))$ be the permutation of the collection

$$\{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \ldots, 2i_k - 1, 2i_k,\ k \le n/2\},$$

where $i_1 < i_2 < \cdots < i_k$. Then for any $\{j_1 < j_2 < \cdots < j_k\}$ construct $(\widetilde{\sigma}(1), \cdots, \widetilde{\sigma}(n))$ by the rule

$$\widetilde{\sigma}(i) = 2j_s, \quad if \quad \sigma(i) = 2i_s, \qquad \widetilde{\sigma}(i) = 2j_s - 1, \quad if \quad \sigma(i) = 2i_s - 1.$$

Obviously the relation $\sigma \sim \widetilde{\sigma}$ is equivalence. Then if σ and $\widetilde{\sigma}$ are equivalent permutations with property $\sharp\sigma^{-1}(i) = 1$, we have the equality

$$\phi(b_{\sigma(1)} \cdots b_{\sigma(n)}) = \phi(b_{\widetilde{\sigma}(1)} \cdots b_{\widetilde{\sigma}(n)}).$$

(iv) For any even $n \in \mathbb{N}$ there exists the constant $C_n > 0$ such that

$$\forall\, \sigma \quad |\phi(b_{\sigma(1)} \cdots b_{\sigma(n)}| < C_n.$$

Then for even n

$$\lim_{N\to\infty} \phi(S_N^n) = \frac{1}{(n/2)!} \sum_{\sigma\in S_n} \phi(b_{\sigma(1)} \cdots b_{\sigma(n)}).$$

For odd n

$$\phi(S_N^n) = 0 \quad \forall\, N.$$

Proof. We use the standard arguments, see, for example [15]. For any even n on has

$$\phi(S_N^n) = N^{-\frac{n}{2}} \sum_{\sigma\colon \{1,\dots,n\}\to\{1,\dots 2N\}} \phi(b_{\sigma(1)} \cdots b_{\sigma(n)}).$$

By the condition (ii) we have consider only σ which are the permutations of the collections

$$\{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \dots, 2i_k - 1, 2i_k,\ k \le n/2\}.$$

Denote by $\mathcal{M}$ the union of the classes of equivalence of σ. Let $\mathcal{M}_k$ be the set of classes of equivalence consisting of σ with $\sharp\{\sigma(i)\} = 2k$, $k = 1, \dots, n/2$. For any $\mathfrak{m} \in \mathcal{M}_k$ there exists a unique $\sigma_{\mathfrak{m}} \in \mathfrak{m}$, which is the permutation with replications of the collection $\{1, 2, \dots, 2k\}$. Obviously, for any $\mathfrak{m} \in \mathcal{M}_k$ one has $\sharp\mathfrak{m} = C_N^k$, since any $\sigma \in \mathfrak{m}$ is uniquely determined by the ordered collection $1 \le i_1 < \dots < i_k \le N$. Then

$$\begin{aligned}\phi(S_N^n) &= N^{-\frac{n}{2}} \sum_{k=1}^{\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_k} \sum_{\sigma\in\mathfrak{m}} \phi(b_{\sigma(1)} \cdots b_{\sigma(n)}) \\ &= N^{-\frac{n}{2}} \sum_{k<\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_k} \sum_{\sigma\in\mathfrak{m}} \phi(b_{\sigma_{\mathfrak{m}}(1)} \cdots b_{\sigma_{\mathfrak{m}}(n)}) + N^{-\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_{n/2}} C_N^{n/2} \phi(b_{\sigma_{\mathfrak{m}}(1)} \cdots b_{\sigma_{\mathfrak{m}}(n)}).\end{aligned}$$

Since the number of summands is finite and independent on N and

$$\lim_{N\to\infty} N^{-\frac{n}{2}} C_N^k = 0, \quad k < \frac{n}{2}, \qquad \lim_{N\to\infty} N^{-\frac{n}{2}} C_N^{n/2} = \frac{1}{(n/2)!}$$

and

$$\begin{aligned}&\left| N^{-\frac{n}{2}} \sum_{k<\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_k} \sum_{\sigma\in\mathfrak{m}} \phi(b_{\sigma_{\mathfrak{m}}(1)} \cdots b_{\sigma_{\mathfrak{m}}(n)}) \right| \le N^{-\frac{n}{2}} \sum_{k<\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_k} \sum_{\sigma\in\mathfrak{m}} |\phi(b_{\sigma_{\mathfrak{m}}(1)} \cdots b_{\sigma_{\mathfrak{m}}(n)})| \\ &\quad \le C_n \sum_{k<\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_k} N^{-\frac{n}{2}} C_N^k \to 0, \qquad N \to \infty.\end{aligned}$$

Thus

$$\lim_{N\to\infty} \phi(S_N^n) = \frac{1}{(n/2)!} \sum_{\mathfrak{m}\in\mathcal{M}_{n/2}} \phi(b_{\sigma_{\mathfrak{m}}(1)} \cdots b_{\sigma_{\mathfrak{m}}(n)}).$$

Evidently, the canonical representatives $\sigma_{\mathfrak{m}}$ of classes $\mathfrak{m} \in \mathcal{M}_{n/2}$ are the elements $\sigma_{\mathfrak{m}} \in S_n$, where we denote by S_n the group of permutations on n symbols; please do not confuse with S_N. Finally

$$\lim_{N\to\infty} \phi(S_N^n) = \frac{1}{(n/2)!} \sum_{\sigma\in S_n} \phi(b_{\sigma(1)} \cdots b_{\sigma(n)}).$$

The equality $\phi(S_N^n) = 0$ for odd n is evident. ∎

Example 1. Consider the family of centered identically distributed elements $\{a_i, a_i^*,\ i \in \mathbb{N}\}$ satisfying the following conditions

1. $a_i^* a_j = q a_j a_i^*$, $a_j a_i = q a_i a_j$, $i < j$, $q \neq 0$.
2. If $\sigma(1) < \cdots < \sigma(k)$, one has

$$\phi(y_{\sigma(1)} \cdots y_{\sigma(k)}) = \phi(y_{\sigma(1)}) \cdots \phi(y_{\sigma(k)}),$$

where $y_s \in \{a_s, a_s^*, a_s^* a_s, a_s a_s^*\}$.

Then the assumptions of the central limit theorem hold. Moreover, if we suppose that operators a_i, a_i^*, $i \in \mathbb{N}$ are the creation and annihilation operators acting on the interacting Fock space and ϕ is the Fock functional, then we additionally have the property $\phi(b_{\sigma(1)} \cdots b_{\sigma(2k)}) \neq 0$, where $\sigma \in S_{2k}$, only if $\sigma^{-1}(2i) < \sigma^{-1}(2i-1)$ for any $i = 1, \ldots, k$.

In particular, if a_i, a_i^* are the creation and annihilation operators on the twisted product of copies of the one-mode interacting Fock space, see next Section, and ϕ is the Fock state, the conditions above are satisfied.

Example 2. Suppose that X_i, $i \in \mathbb{N}$ are anti-monotone independent centered symmetric identically distributed random variables with variance 1. Then one can realize them as the field operators acting on the monotone product of the one-mode interacting Fock spaces, i.e. suppose that $X_i = a_i + a_i^*$, a_i the creation and a_i^* the annihilation operators and ϕ is the Fock state. Let us find the measure given by the central limit theorem. To do it we note that $\phi(b_{\sigma(1)} \cdots b_{\sigma(2m)})$ is either 1 or 0. Let us find the number of the non-zero summands in the sum

$$\sum_{\sigma \in S_{2m}} \phi(b_{\sigma(1)} \cdots b_{\sigma(2m)}) \tag{1}$$

Firstly note that $\phi(Y_1 a_i Y_2 a_i^* Y_3) = 0$ if $Y_2 \neq 1$, here

$$Y_1 = \prod_{j < \sigma^{-1}(2i-1)} b_{\sigma(j)}, \qquad Y_2 = \prod_{\sigma^{-1}(2i-1) < j < \sigma^{-1}(2i)} b_{\sigma(j)}, \qquad Y_3 = \prod_{\sigma^{-1}(2i) < j} b_{\sigma(j)}.$$

Further, by definition of the anti-monotone independence if $Y_2 \neq 1$ we have $\phi(Y_1 a_1^* Y_2 a_1 Y_3) = \phi(a_1^*)\phi(Y_1 Y_2 a_1^* Y_3) = 0$, here

$$Y_1 = \prod_{j < \sigma^{-1}(2)} b_{\sigma(j)}, \qquad Y_2 = \prod_{\sigma^{-1}(2) < j < \sigma^{-1}(1)} b_{\sigma(j)}, \qquad Y_3 = \prod_{\sigma^{-1}(2i) < j} b_{\sigma(j)}.$$

Hence if $\phi(b_{\sigma(1)} \cdots \sigma(n)) \neq 0$ we have $Y_2 = 1$ and

$$\phi(b_{\sigma(1)} \cdots \sigma(n)) = \phi(Y_1 a_1^* a_1 Y_3) = \phi(a_1^* a_1)\phi(Y_1 Y_3),$$

where $Y_1 Y_3$ is any product of $a_2, a_2^*, \ldots, a_m, a_m^*$ where each term appears only once. Let k_m be the number of the non-zero summands in (1), then arguments presented above imply the following recurrent formula

$$k_m = (2m-1)k_{m-1}.$$

Indeed, we have $2m-1$ different positions for $a_1^* a_1$ in our permutation. Evidently, $k_1 = 1$, hence

$$k_m = \lim_{N \to \infty} \frac{1}{m!}\phi(S_N^{2m}) = \frac{(2m-1)!!}{m!} = \frac{C_{2m}^m}{2^m},$$

and these moments correspond to the arcsin distribution with density

$$d\mu(x) = 1/\pi\ \chi\big(-\sqrt{2}, \sqrt{2}\big)\frac{dx}{\sqrt{2 - x^2}}.$$

Example 3. Boolean central limit theorem. Let $\{a_i, a_i^*\}$ are boolean independent family of the creation and annihilation operators acting on the boolean Fock space. As above suppose that ϕ is the Fock state and $\phi(a_i^* a_i) = 1$, $i = 1, \dots, d$. Then all conditions of the central limit theorem are satisfied. Since for any i

$$\phi\left(Y_1 a_i^{(\varepsilon_i)} Y_2 a_i^{(-\varepsilon_i)} Y_3\right) = \phi(Y_1)\phi\left(a_i^{(\varepsilon_i)}\right)\phi(Y_2)\phi\left(a_i^{(-\varepsilon_i)}\right)\phi(Y_3) = 0, \quad \text{if } \ Y_2 \neq 1 \text{ or } \varepsilon_i = 1,$$

where $\varepsilon_i \in \{1, -1\}$ and $a_s^{(1)} = a_s$, $a_s^{(-1)} = a_s^*$. Hence, as in the monotone case, we have $\phi(b_{\sigma(1)} \cdots b_{\sigma(2n)}) \neq 0$ only if

$$b_{\sigma(1)} \cdots b_{\sigma(2n)} = Y_1 a_1^* a_1 Y_3$$

and

$$\phi(b_{\sigma(1)} \cdots b_{\sigma(2n)}) = \phi(a_1^* a_1)\phi(Y_1)\phi(Y_3),$$

where Y_1 is the word obtained by some permutation of the word $a_{i_1}^* a_{i_1} \cdots a_{i_k}^* a_{i_k}$, $i_1 < i_2 < \cdots < i_k \in \{2, \dots, n\}$ and analogously for Y_3. The arguments presented above shows that $\phi(b_{\sigma(1)} \cdots b_{\sigma(2n)}) \neq 0$ if and only if

$$b_{\sigma(1)} \cdots b_{\sigma(2n)} = a_{\pi(1)}^* a_{\pi(1)} \cdots a_{\pi(n)}^* a_{\pi(n)} = 1,$$

where $\pi \in S_n$ is any permutation. Hence we have

$$\lim_{N\to\infty} \phi(S_n^{2n}) = \frac{1}{n!} n! = 1$$

so, $m_{2n-1} = 0$ and $m_{2n} = 1$ – the moments of the discrete measure concentrated on $\{-1, 1\}$.

2 Twisted product

In this Section we discuss the construction of the twisted product Fock space. This is the special kind of the interacting Fock space.

Let $\mathcal{I}$ be totally ordered set. Consider the collection of the one-mode interacting Fock spaces

$$\{(\mathcal{T}(\mathcal{H}_i), a_i, a_i^* \mid \ i \in \mathcal{I}\}.$$

Let $\Omega_i \in \mathcal{T}(\mathcal{H}_i)$ be vacuum vector, consider the orthonormal system

$$\{e_i^{(n)}, \ n \in \mathbb{Z}_+\},$$

such that $e_i^{(0)} := \Omega_i$ and $a_i e_i^{(n)} = \alpha_i^{(n)} e_{n+1}$, $n \in \mathbb{Z}_+$. Denote by ϕ_i the Fock state on $\mathcal{T}(\mathcal{H}_i)$. Then consider the Hilbert space $\mathcal{T}$ with orthonormal basis

$$\Omega, \quad e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)}, \quad i_1 < \cdots < i_k, \quad k \in \mathbb{N}, \quad i_s \in \mathcal{I}, \quad n_s \in \mathbb{N}, \quad s = 1, \dots, k.$$

Further define the creation operators $\widetilde{a}_j$, $j \in \mathcal{I}$

$$\begin{aligned}
&\widetilde{a}_j \Omega = \alpha_j^{(0)} e_j^{(1)}, \\
&\widetilde{a}_j e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)} = \mu^k \alpha_j^{(0)} e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)} \otimes e_j^{(1)}, \quad j > i_k, \\
&\widetilde{a}_j e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)} \\
&\qquad = \mu^{n_1 + \cdots + n_s} \alpha_j^{(0)} e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s)} \otimes e_j^{(1)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)}, \quad i_s < j < i_{s+1}, \\
&\widetilde{a}_j e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)} \\
&\qquad = \mu^{n_1 + \cdots + n_{s-1}} \alpha_j^{(n_s)} e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s+1)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)}, \quad j = i_s, \\
&\widetilde{a}_j e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)} = \alpha_j^{(0)} e_j^{(1)} \otimes e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)}, \quad j < i_1.
\end{aligned}$$

For the adjoint (annihilation) operators $\widetilde{a}_j^*$ one has the following

$$\begin{aligned}
&\widetilde{a}_j^*\Omega = 0,\\
&\widetilde{a}_j e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)} = 0, \quad j \neq i_s, \quad s = 1, \ldots, k,\\
&\widetilde{a}_j^* e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)}\\
&\qquad = (\overline{\mu})^{n_1+\cdots+n_{s-1}} (\overline{\alpha}_j)^{(n_s-1)} e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s-1)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)}, \quad j = i_s,
\end{aligned}$$

where we identify $e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(0)} \otimes \cdots \otimes e_{i_k}^{(n_k)}$ with $e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_{s-1}}^{(n_{s-1})} \otimes e_{i_{s+1}}^{(n_{s+1})}$.

We call the interacting Fock space $(\mathcal{T}, \widetilde{a}_i, \widetilde{a}_i^*, i \in \mathcal{I})$ the **twisted product** of the one-mode interacting Fock spaces $(\mathcal{T}(\mathcal{H}_i), a_i, a_i^*)$, $i \in \mathcal{I}$ with the twist parameter $\mu \in \mathbb{C}$. Below we denote by ϕ the Fock state, i.e. the vector state defined by Ω, on $\mathcal{T}$.

It is easy to verify that the operators $\widetilde{a}_i$, $\widetilde{a}_j$ and $\widetilde{a}_i^*$, $\widetilde{a}_j$, $i > j$ satisfy the μ-commutation relations, i.e.

$$\widetilde{a}_i^* \widetilde{a}_j = \mu \widetilde{a}_j \widetilde{a}_i^*, \qquad \widetilde{a}_i \widetilde{a}_j = \mu \widetilde{a}_j \widetilde{a}_j.$$

One can verify also that the joint distributions of a_i, a_i^* with respect to ϕ_i and $\widetilde{a}_i$, $\widetilde{a}_i^*$ with respect to ϕ coincide, i.e. for any non-commutative polynomial $p(x, y)$ one has

$$\phi_i(p(a_i, a_i^*)) = \phi(p(\widetilde{a}_i, \widetilde{a}_i^*)).$$

Finally note that for $\mu = 1$ one has the usual tensor product and for $\mu = 0$ the monotone product of interacting Fock spaces considered by N. Muraki, see [12].

When we have the finite set $\mathcal{I} = \{1, 2, \ldots, d\}$ the twisted product is just the twisted Fock space constructed by W. Pusz and S.L. Woronowicz, see [13]. In this case the orthonormal basis of $\mathcal{T}$ has the form

$$e_1^{(n_1)} \otimes \cdots \otimes e_d^{(n_d)}, \qquad n_s \in \mathbb{Z}_+, \quad s = 1, \ldots, d,$$

where $\Omega := e_1^{(0)} \otimes \cdots \otimes e_d^{(0)}$, i.e. $\mathcal{T} = \bigotimes_{i=1}^d \mathcal{T}(\mathcal{H}_i)$. In this case the operators $\widetilde{a}_i$, $i = 1, \ldots, d$ can be presented as tensor products

$$\widetilde{a}_i = \bigotimes_{j=1}^{i-1} d(\mu) \otimes a_i \otimes \bigotimes_{j=i+1}^{d} 1,$$

where $d(\mu) e_j^{(n)} = \mu^n e_j^{(n)}$, $n \in \mathbb{Z}_+$, $j = 1, \ldots, d$. If we consider the special case $\mu = 0$, we get the realization of the monotone independent non-commutative random variables constructed by U. Franz, see [6]

$$\widetilde{a}_i = \bigotimes_{j=1}^{i-1} P_j \otimes a_i \otimes \bigotimes_{j=i+1}^{d} 1.$$

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