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On ∗-Representations of One Deformed Quotient of Affine Temperley–Lieb Algebra

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We consider ∗-algebra generated by orthogonal projections with relations of Temperley–Lieb type. In this article we study all irreducible ∗-representations of this algebra and obtain the set of values of parameters when these representations exist.

1 Introduction

Temperley–Lieb algebras generated by n projections $p_1, \ldots, p_n$ with relations

$$p_i p_j = p_j p_i, \qquad |i-j| > 1, \qquad p_i p_{i\pm 1} p_i = \tau p_i, \qquad \tau \in \mathbb{R}, \tag{1}$$

appeared in [3, 4] in the context of ice-type models but they also play an important role in the analysis of subfactors of II_1 factor and in the knot theory (see, e.g., [5–7]). Jones proved that the chain (1) of orthogonal projections in Hilbert space with adding condition involving the trace can be infinite one if $\tau \in [0; 1/4] \cup \left\{ \frac{1}{4\cos^2 \frac{\pi}{n}} \mid n \geq 3 \right\}$.

In the present paper we consider ∗-algebra $TL_{\vec{\tau},n}$ generated by orthogonal projections $p_0, \ldots, p_{n-1}$ with relations

$$p_i p_j = 0, \quad |i-j| > 1, \quad (i,j) \neq (0, n-1) \quad \text{and} \quad p_i p_{\overline{i+1}} p_i = \tau_i p_i, \quad p_i p_{\overline{i-1}} p_i = \tau_{\overline{i-1}} p_i. \tag{2}$$

In [2] we studied such ∗-algebra for $\tau_i = \tau$. For this more general algebra (2) we have found all irreducible ∗-representations and described the set of values of the parameters when these representations exist.

2 Description of all irreducible ∗-representations of algebra $TL_{\vec{\tau},n}$, their existence in depending on values of parameter $\vec{\tau}$

We study ∗-algebra over complex field generated by n ($n \geqslant 3$) orthogonal projections $p_0, \ldots, p_{n-1}$ with relations of Temperley–Lieb type or orthogonality between any two projections. In other words, $p_i^2 = p_i^* = p_i$ and any projections p_i and p_j fulfil condition $p_i p_j = 0$ or for some $0 < \tau_{i,j} < 1$ relations $p_i p_j p_i = \tau_{i,j} p_i$ and $p_j p_i p_j = \tau_{i,j} p_j$ are correct. Such algebra can be described by a marked graph G with n vertices, where two vertices i, j are joined by a line marked with $\tau_{i,j}$ if and only if orthogonal projections p_i, p_j satisfy relations of Temperley–Lieb type. If $\vec{\tau} = (\tau_0, \ldots, \tau_{n-1})$ with $0 < \tau_i < 1$ is fixed vector we may consider ∗-algebra $TL_{\vec{\tau},n}$ described by a graph (see Fig. 1).

In [1] there were proved that ∗-algebra $TL_{\vec{\tau},n}$ has only finite-dimensional irreducible ∗-representations, so in the following we consider nontrivial irreducible finite-dimensional ∗-representations of this algebra and name them simply 'representations'. If π is a ∗-representation of algebra $TL_{\vec{\tau},n}$ in unitary space H we write P_i for $\pi(p_i)$. Next theorem give a description of ∗-representations of algebra $TL_{\vec{\tau},n}$.

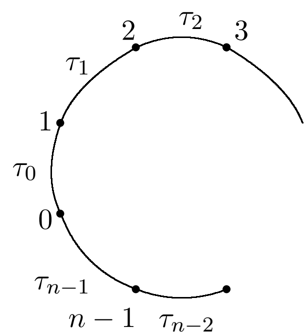

Figure 1.

Theorem 1. *Let irreducible $*$-representation of algebra $TL_{\vec{\tau},n}$ exists in unitary space H. Then we can find the orthonormal basis of H such that in this basis matrices of operators $P_0, \dots, P_{n-1}$ are as follows:*

$$P_0 = \operatorname{diag}(1, 0, \dots, 0),$$

$$P_i = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & t_{i-1} & \sqrt{t_{i-1} - t_{i-1}^2} & 0 & \cdots \\ 0 & \cdots & 0 & \sqrt{t_{i-1} - t_{i-1}^2} & 1 - t_{i-1} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad i = 1, \dots, n-2,$$

where $t_{i-1} = \frac{\tau_{i-1}}{1 - t_{i-2}}$, $t_0 = \tau_0$ and the number of zeroes on the top of diagonal is equal to $i - 1$.

$$P_{n-1} = \begin{pmatrix} \tau_{n-1} & b_1 & \cdots & b_{n-3} & \lambda & \mu \\ b_1 & \frac{b_1^2}{\tau_{n-1}} & \cdots & \frac{b_1 b_{n-3}}{\tau_{n-1}} & \frac{b_1 \lambda}{\tau_{n-1}} & \frac{b_1 \mu}{\tau_{n-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b_{n-3} & \frac{b_1 b_{n-3}}{\tau_{n-1}} & \cdots & \frac{b_{n-3}^2}{\tau_{n-1}} & \frac{b_{n-3} \lambda}{\tau_{n-1}} & \frac{b_{n-3} \mu}{\tau_{n-1}} \\ \bar{\lambda} & \frac{b_1 \bar{\lambda}}{\tau_{n-1}} & \cdots & \frac{b_{n-3} \bar{\lambda}}{\tau_{n-1}} & \frac{|\lambda|^2}{\tau_{n-1}} & \frac{\bar{\lambda} \mu}{\tau_{n-1}} \\ \mu & \frac{b_1 \mu}{\tau_{n-1}} & \cdots & \frac{b_{n-3} \mu}{\tau_{n-1}} & \frac{\mu \lambda}{\tau_{n-1}} & \frac{\mu^2}{\tau_{n-1}} \end{pmatrix},$$

where $b_i = (-1)^i \tau_{n-1} \prod_{j=0}^{i-1} \frac{t_j}{\sqrt{t_j - t_j^2}}$. Entry $\lambda \in \mathbb{C}$ that 'number' the representations is such that

$$\left| t_{n-3} b_{n-3} + \lambda \sqrt{t_{n-3} - t_{n-3}^2} \right|^2 = \tau_{n-2} \tau_{n-1} t_{n-3}$$

and $\mu = \sqrt{\tau_{n-1} - \tau_{n-1}^2 - \sum_{j=1}^{n-3} b_j^2 - |\lambda|^2}$.

Remark 1. If parameter $\vec{\tau}$ is such that $t_{n-3} = 1$ the matrix of operator P_{n-1} differs from the one pointed out in the Theorem 1, more precisely, first $n-2$-nd rows and columns are the same but $n-1$-st (or even $n-1$-st and n-th) row and column are absent, b_{n-3} satisfies additional condition $b_{n-3}^2 = \tau_{n-2}\tau_{n-1}$ and $\mu^2 = \tau_{n-1} - \tau_{n-1}^2 - \sum_{i=1}^{n-3} b_i^2$, $\left(\tau_{n-1} - \tau_{n-1}^2 - \sum_{i=1}^{n-3} b_i^2 = 0\right)$.

Remark 2. The rank of all orthogonal projections P_i is 1 and dimension of irreducible $*$-representation may be equal to n, $n-1$, or to $n-2$.

Remark 3. If parameter $\vec{\tau}$ is fixed then different permissible λ's define inequivalent irreducible $*$-representations. So, we may say that each irreducible $*$-representation of algebra $TL_{\vec{\tau},n,}$ is given by the number λ.

Now our goal is to produce the set of values of parameter $\vec{\tau}$ for that the $*$-representations exist. Let $F_i^{(k)}$, $i \geq 0$, $0 \leq k \leq n-1$ be the collection of numbers given by recurrent formulas

$$F_0^{(k)} = F_1^{(k)} = 1, \qquad F_{i+2}^{(k)} = F_{i+1}^{(k)} - \tau_{\overline{i+k}} F_i^{(k)}.$$

Proposition 1. *The irreducible $*$-representations of algebra $TL_{\vec{\tau},n,}$ exist if and only if one of following two cases takes place:*

1) $F_i^{(0)} > 0$, $i = 2, \ldots, n-1$ *and at least one of the following inequalities is true*

$$\frac{\left|(-1)^n \sqrt{\tau_0 \cdots \tau_{n-3}\tau_{n-1}} \pm \sqrt{\tau_{n-2}} F_{n-2}^{(0)}\right|}{\sqrt{F_{n-1}^{(0)}}} \leq \sqrt{(1-\tau_{n-1}) F_{n-2}^{(0)} - \tau_0 \tau_{n-1} F_{n-4}^{(2)}},$$

2) $F_i^{(0)} > 0$, $i = 2, \ldots, n-2$, $F_{n-1}^{(0)} = 0$, $F_{n-2}^{(0)} = \sqrt{\frac{\tau_0 \cdots \tau_{n-3}\tau_{n-1}}{\tau_{n-2}}}$ *and*

$$1 - \tau_{n-1} - \tau_0 \tau_{n-1} \frac{F_{n-4}^{(2)}}{F_{n-2}^{(0)}} \geq 0.$$

Note that for $n = 3$ the expressions in the proposition 1 will be correct if $P_{-1}^{(2)} := 0$.

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[1] Popova N. and Vlasenko M., On representations of sets of projections with fixed angles between them, *Ukr. Math. J.*, to appear (in Russian).

[2] Popova N., On one algebra of Temperley–Lieb type, in Proceedings of Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–15 July, 2001, Kyiv), Editors A.G. Nikitin, V.M. Boyko and R.O. Popovych, Kyiv, Insitute of Mathematics, 2002, V.43, Part 2, 486–489.

[3] Temperley H. and Lieb E., Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular plane lattices: some exact results for 'percolation' problem, *J. Proc. Roy. Soc. (London)*, 1971, V.322, 251–280.

[4] Baxter R.J., Exactly solved models in statistical mechanics, London, Acad. Press., 1982.

[5] Jones V., Index for subfactors *J. Invent. Math.*, 1983, V.72, 1–25.

[6] Jones V., Hecke algebra representations of braid groups and link polynomials, *Annals of Math.*, 1987, V.126, 335–388.

[7] Wenzl H., On sequences of projections *C.R. Math. Rep. Acad. Sci. Canada*, 1987, V.9, N 1, 5–9.