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Operators of Binary Darboux Transformations for Dirac's System

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It is shown that the binary Darboux transformations (BDT) [1–4] for Dirac's operator generate transformations operators obtained by L.P. Nizhnik by considering inverse scattering problem of Dirac's system [5, 6]. We found a wide set of explicit solutions of the space-two-dimensional non-linear Schrödinger equation. It is shown that solutions of those equations obtained by inverse scattering method are contained among those obtained by binary Darboux transformations method.

Let us consider Dirac's operator L of the form

$$L = \begin{pmatrix} \partial_x & u_1 \\ u_2 & \partial_y \end{pmatrix}, \quad u_1 = u_1(x, y), \quad u_2 = u_2(x, y), \quad \partial_x := \frac{\partial}{\partial x}, \quad \partial_y := \frac{\partial}{\partial y}. \tag{1}$$

Let 1. $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ be arbitrary and

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} := \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1K} \\ \varphi_{21} & \cdots & \varphi_{2K} \end{pmatrix}$$

some fixed $(2 \times K)$-matrix solutions of Dirac's system

$$LY = 0. \tag{2}$$

2. $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ be arbitrary and

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \begin{pmatrix} \psi_{11} & \cdots & \psi_{1K} \\ \psi_{21} & \cdots & \psi_{2K} \end{pmatrix}$$

some fixed $(2 \times K)$-matrix solutions of transposed system

$$L^\tau Z = 0, \ L^\tau = \begin{pmatrix} -\partial_x & u_2 \\ u_1 & -\partial_y \end{pmatrix}. \tag{3}$$

3.

$$\Omega[Z, Y, M_0, M] := \Omega[Z, Y] \tag{4}$$

is a matrix potential which satisfies the condition $\Omega[Z, Y, M_0, M_0] = 0$, where $M = (x, y)$, $M_0 = (x_0, y_0) \in \overline{\mathbb{R}^2}$.

Remark 1. From equations (2), (3) for arbitrary solutions Y, Z the relation $(Z_1^\top Y_1)_x = -(Z_2^\top Y_2)_y$ follows, which ensures the existence (up to an arbitrary constant) of the matrix potential

$$\Omega: \ \Omega_x = -Z_2^\top Y_2, \ \ \Omega_y = Z_1^\top Y_1.$$

4. Let C be some $(K \times K)$-constant matrix and the potential $C + \Omega[\psi, \varphi]$ is non-generated in a neighborhood of $M_0 = (x_0, y_0) \in \overline{\mathbb{R}^2}$.

By the direct computation we prove the following proposition.

Proposition 1. *The integral operator W defined on the space of solutions of Dirac's system by the formula*

$$WY := Y - \varphi(C + \Omega[\psi, \varphi])^{-1}\Omega[\psi, Y] \tag{5}$$

transforms Dirac's operator (1) *into Dirac's operator $\hat{L} = WLW^{-1}$ of the form*

$$\hat{L} = \begin{pmatrix} \partial_x & \hat{u}_1 \\ \hat{u}_2 & \partial_y \end{pmatrix},$$

where

$$\begin{aligned} \hat{u}_1 &= u_1 - \varphi_1(C + \Omega[\psi, \varphi])^{-1}\psi_2^\top, \\ \hat{u}_2 &= u_2 + \varphi_2(C + \Omega[\psi, \varphi])^{-1}\psi_1^\top. \end{aligned} \tag{6}$$

The functions $\hat{Y} := WY$ is the general solution of Dirac's system

$$\hat{L}\hat{Y} = 0$$

with the coefficients (potentials) $\hat{u}_1$, $\hat{u}_2$ (6).

Remark 2. The (2×2)-matrix integral operator $W - I$ is of the form

$$W - I := \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \tag{7}$$

where operators W_{ij} are defined by their kernels $W_{ij}(x, y, s)$, $i, j = 1, 2$ depending on concrete realization of potential $\Omega[Z, Y]$, are the Volterra operators of variables x and y. The coefficients $\hat{u}_1$, $\hat{u}_2$ (6) are defined as the diagonals of kernels of the operators W_{12}, W_{21} respectively.

Remark 3. It is not difficult to show that formulas (6) give solutions of space-two-dimensional non-linear Schrödinger equations [6, 7]

$$\begin{aligned} &u_{1t} + u_{1xx} + u_{1yy} - 2(v_1 - v_2)u_1 = 0, \\ &u_{2t} - u_{2xx} - u_{2yy} - 2(v_2 - v_1)u_2 = 0, \\ &v_{1x} = (u_1u_2)_y, \qquad v_{2y} = (u_1u_2)_x, \end{aligned} \tag{8}$$

which has the Lax representation $[L, A] = 0$, where L is Dirac's operator (1), and A is some matrix evolution operator [2]. The functions φ, ψ in solutions (6) depend on parameter t in view of the equations

$$A\varphi = 0, \qquad A^\tau\psi = 0.$$

In this paper we show that certain realization of the operator W (5), which is generated by concrete choice of potential (4), coincides with transformation operators for Dirac's systems which were obtained in the papers [5, 6].

Let us consider the operator W defined by the following formula

$$\begin{aligned} \hat{Y} = WY = Y - \varphi &\left[C + \int_{x_0}^{x} (-\psi_2^\top\varphi_2)\, dx + \int_{y_0}^{y} \psi_1^\top\varphi_1\, dy\right]^{-1} \\ &\times \left(\int_{x_0}^{x} (-\psi_2^\top Y_2)\, dx + \int_{y_0}^{y} \psi_1^\top Y_1\, dy\right). \end{aligned} \tag{9}$$

Respectively, the kernels of the operators W_{12}, W_{21} (7) have the form

$$W_{12}(x,y,s)=\varphi_1(x,y)\left[C-\int_{x_0}^{x}\psi_2^\top\varphi_2(x,y_0)\,dx+\int_{y_0}^{y}\psi_1^\top\varphi_1(x,y)\,dy\right]^{-1}\psi_2^\top(s,y),$$
$$W_{21}(x,y,s)=-\varphi_2(x,y)\left[C-\int_{x_0}^{x}\psi_2^\top\varphi_2(x,y_0)\,dx+\int_{y_0}^{y}\psi_1^\top\varphi_1(x,y)\,dy\right]^{-1}\psi_1^\top(x,s), \quad (10)$$

and the coefficients $\hat{u}_1$, $\hat{u}_2$ can be written

$$\hat{u}_1=u_1(x,y)-W_{12}(x,y,x),$$
$$\hat{u}_2=u_2(x,y)+W_{21}(x,y,y). \quad (11)$$

In the case when Dirac's operator is non-perturbed ($u_1=u_2\equiv 0$), the solutions φ, ψ of equations (2), (3) admit evidently, such forms

$$\varphi=\begin{pmatrix}-f_1(y) & 0\\ 0 & g_2(x)\end{pmatrix},\qquad f_1=(f_{11},\dots,f_{1n}),\qquad g_2=(g_{21},\dots,g_{2n}),$$
$$\psi=\begin{pmatrix}g_1(y) & 0\\ 0 & -f_2(x)\end{pmatrix},\qquad g_1=(g_{11},\dots,g_{1n}),\qquad f_2=(g_{21},\dots,f_{2n}).$$

Let a $(2n\times 2n)$-matrix C be of the form

$$C=\begin{pmatrix}0 & I_n\\ I_n & 0\end{pmatrix},\qquad I_n:=\operatorname{diag}(1,\dots,1),$$
$$M_0:=(x_0,y_0)=(-\infty,+\infty),\qquad Y_1=Y_1(y):=Y_1(-\infty,y),\qquad Y_2=Y_2(x):=Y_2(x,+\infty).$$

By formula (9) we obtain

$$\begin{pmatrix}\hat{Y}_1\\ \hat{Y}_2\end{pmatrix}:=WY=\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}-\begin{pmatrix}-f_1(y) & 0\\ 0 & g_2(x)\end{pmatrix}$$
$$\times\begin{pmatrix}-\int_{+\infty}^{y}g_1^\top f_1(y)\,dy & I_n\\ I_n & \int_{-\infty}^{x}f_2^\top g_2(x)\,dx\end{pmatrix}^{-1}\left(\int_{-\infty}^{x}(f_2^\top Y_2)\,dx+\int_{+\infty}^{y}g_1^\top Y_1\,dy\right).$$

By using the known formula

$$\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}^{-1}=\begin{pmatrix}A_{11}^{-1}(I+A_{12}T^{-1}A_{21}A_{11}^{-1}) & -A_{11}^{-1}A_{12}T^{-1}\\ -T^{-1}A_{21}A_{11}^{-1} & T^{-1}\end{pmatrix}$$

for a block matrix A, where $T=A_{22}-A_{21}A_{11}^{-1}A_{12}$ and provided that $A_{12}=A_{21}=I_n$, we obtain

$$\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}^{-1}=\begin{pmatrix}-[I_n-A_{22}A_{11}]^{-1}A_{22} & [I_n-A_{22}A_{11}]^{-1}\\ [I_n-A_{11}A_{22}]^{-1} & -[I_n-A_{11}A_{22}]^{-1}A_{11}\end{pmatrix}.$$

Respectively, the operator W is of the form

$$W=I-\begin{pmatrix}-f_1 & 0\\ 0 & g_2\end{pmatrix}\begin{pmatrix}-[I_n-A_{22}A_{11}]^{-1}A_{22} & [I_n-A_{22}A_{11}]^{-1}\\ [I_n-A_{11}A_{22}]^{-1} & -[I_n-A_{11}A_{22}]^{-1}A_{11}\end{pmatrix}$$
$$\times\begin{pmatrix}-\int_{+\infty}^{y}g_1^\top\cdot\,dy & 0\\ 0 & \int_{-\infty}^{x}f_2^\top\cdot\,dx\end{pmatrix}, \quad (12)$$

which coincides with respective transformation operators for Dirac's system obtained in paper [6]. The kernels (10) of the operators W_{12}, W_{21} in formula (12) are

$$W_{12}(x,y,s) = f_1(y)\left[I + \int_{-\infty}^{x} f_2^{\top} g_2(x)\,dx \int_{+\infty}^{y} g_1^{\top} f_1(y)\,dy\right]^{-1} f_2^{\top}(s),$$
$$W_{21}(x,y,s) = g_2(x)\left[I + \int_{+\infty}^{y} g_1^{\top} f_1(y)\,dy \int_{-\infty}^{x} f_2^{\top} g_2(x)\,dx\right]^{-1} g_1^{\top}(s).$$

The solution $\hat{u}_1$, $\hat{u}_2$ (6) of equations (8) by the formulas (11) coincide with the corresponding solution obtained by inverse scattering method [6, 7].

[1] Matveev V.B. and Salle M.A., Darboux transformations and solitons, Berlin Heidelberg, Springer-Verlag, 1991.

[2] Nimmo J.J.C., Darboux transformations for a two-dimensional Zakharov–Shabat/AKNS spectral problem, *Inverse Problems*, 1992, V.8, 219–243.

[3] Guil F. and Manas M., Darboux transformation for the Davey–Stewartson equations, *Phys. Lett. A*, 1996, V.217, 1–6.

[4] Sydorenko Yu., Binary transformation and (2 + 1)-dimensional integrable systems, *Ukr. Math. J.*, 2002, V.52, N 11, 1531–1550 (in Ukrainian).

[5] Nizhnik L.P., The inverse nonstationary scattering problem, Kyiv, Naukova Dumka, 1973 (in Russian).

[6] Nizhnik L.P., Inverse scattering problems for hyperbolic equations, Kyiv, Naukova Dumka, 1991 (in Russian).

[7] Nizhnik L.P. and Pochynayko M.D., Integration of space-two-dimensional nonlinear Schrödinger equation by inverse scattering method, *Func. Analys. and Its Appl.*, 1982, V.16, N 1, 80–82 (in Russian).