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Generalized Self-Duality in Superfield Formalism

Anatoli PAVLYUK

Bogolyubov Institute for Theoretical Physics, 14b Metrologichna Str., 03143 Kyiv, Ukraine
E-mail: *a_pavlyuk@ukr.net*

The generalized super self-duality equations for $N = 2$ Super Yang–Mills theory in $N = 1$ superspace, which are first-order equations, in terms of component fields as well as in terms of superfields are presented.

In the pure $SU(2)$ Yang–Mills theory the self-duality equations read

$$iF_{mn} = \widetilde{F}_{mn} \equiv \frac{1}{2}\epsilon_{mnkl}F^{kl}, \tag{1}$$

where

$$F_{mn} = \partial_m V_n - \partial_n V_m + ig[V_m, V_n], \qquad \eta_{mn} = \mathrm{diag}\,(-1, 1, 1, 1).$$

Going from vector indices $(m, n = 0, 1, 2, 3)$ to spinor indices $(\alpha, \beta = 1, 2)$

$$x_m \to x_{\alpha\dot\alpha} = \sigma^m{}_{\alpha\dot\alpha} x_m,$$

where $\sigma^m = (-1, \vec\sigma)$, (1 is a unit 2×2 matrix, $\vec\sigma$ are Pauli matrices), one obtains the Yang–Mills strength in the following form

$$F_{\alpha\dot\alpha,\beta\dot\beta} \equiv \sigma^m{}_{\alpha\dot\alpha}\sigma^n{}_{\beta\dot\beta}F_{mn} = \frac{1}{2}\varepsilon_{\alpha\beta} f_{\dot\alpha\dot\beta} + \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta} f_{\alpha\beta},$$

where

$$f_{\dot\alpha\dot\beta} \equiv \epsilon^{\alpha\gamma} F_{\gamma\dot\alpha,\alpha\dot\beta}, \qquad f_{\alpha\beta} \equiv \epsilon^{\dot\alpha\dot\gamma} F_{\alpha\dot\gamma,\beta\dot\alpha}.$$

Now the self-duality equations (1) take the form

$$f_{\alpha\beta} = 0. \tag{2}$$

The two-spinor indices are raised and lowered by means of the two-dimensional Levi-Civita tensors

$$\varepsilon^{12} = -\varepsilon_{12} = \varepsilon^{\dot1\dot2} = -\varepsilon_{\dot1\dot2} = +1.$$

The raising and lowering of spinor indices are defined by

$$\lambda^\alpha = \varepsilon^{\alpha\beta}\lambda_\beta, \qquad \lambda^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\lambda_{\dot\beta}, \qquad \lambda_\alpha = \varepsilon_{\alpha\beta}\lambda^\beta, \qquad \lambda_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\lambda^{\dot\beta}.$$

In the $SU(2)$ $N = 1$ Super Yang–Mills theory [1]

$$L = \mathrm{Tr}\left\{-\frac{1}{4}F_{mn}F^{mn} - i\bar\lambda\bar\sigma^m\mathcal{D}_m\lambda + \frac{1}{2}D^2\right\}, \tag{3}$$

where with help of supersymmetric transformations of (2) one obtains the system of super self-duality equations in component fields [2]

$$f_{\alpha\beta} = 0, \qquad D = 0, \qquad \mathcal{D}_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha} = 0, \qquad \lambda_\alpha = 0. \tag{4}$$

The system of super self-duality equations (4) is a supersymmetric generalization of self-duality equations (2) and satisfies the equations of motion of theory (3). It is invariant under $N = 1$ supersymmetric transformations.

Let us proceed to the $N = 2$ supersymmetric Yang–Mills theory in $N = 1$ superspace, which is described by vector and scalar multiplets of component fields,

$$V = (V_m; \lambda_\alpha; \bar\lambda_{\dot\alpha}; D), \qquad \Phi = (A, B; \psi_\alpha; \bar\psi_{\dot\alpha}; F, G).$$

Gauge-invariant supersymmetric Lagrangian of $N = 2$ Yang–Mills theory in $N = 1$ superspace, whose fields belong to the adjoint representation of $SU(2)$ gauge group, has the form

$$\begin{aligned} L = \mathrm{Tr}\bigg\{ &-\frac{1}{4}F_{mn}F^{mn} - \frac{1}{2}\mathcal{D}_m A\mathcal{D}^m A - \frac{1}{2}\mathcal{D}_m B\mathcal{D}^m B + igD[A,B] \\ &- i\bar\lambda\bar\sigma^m\mathcal{D}_m\lambda - i\bar\psi\bar\sigma^m\mathcal{D}_m\psi + ig(A+iB)\{\lambda^\alpha,\psi_\alpha\} \\ &+ ig(A-iB)\{\bar\lambda_{\dot\alpha},\bar\psi^{\dot\alpha}\} + \frac{1}{2}D^2 + \frac{1}{2}F^2 + \frac{1}{2}G^2\bigg\}. \end{aligned} \tag{5}$$

The equations of motion of theory (5) are as follows

$$\begin{aligned} &\varepsilon^{\dot\alpha\dot\gamma}\mathcal{D}_{\alpha\dot\gamma}f_{\dot\alpha\dot\beta} + \varepsilon^{\beta\gamma}\mathcal{D}_{\gamma\dot\beta}f_{\alpha\beta} + 4g\big(\{\lambda_\alpha,\bar\lambda_{\dot\beta}\} + \{\psi_\alpha,\bar\psi_{\dot\beta}\}\big) \\ &\qquad - ig\big([A-iB,\mathcal{D}_{\alpha\dot\beta}(A+iB)] + [A+iB,\mathcal{D}_{\alpha\dot\beta}(A-iB)]\big) = 0, \\ &\mathcal{D}_{\alpha\dot\beta}\mathcal{D}^{\alpha\dot\beta}(A-iB) - 4ig\{\lambda^\alpha,\psi_\alpha\} - 2g[D,A-iB] = 0, \\ &\mathcal{D}_{\alpha\dot\beta}\mathcal{D}^{\alpha\dot\beta}(A+iB) - 4ig\{\bar\lambda_{\dot\alpha},\bar\psi^{\dot\alpha}\} + 2g[D,A+iB] = 0, \\ &D + ig[A,B] = 0, \qquad F = G = 0, \\ &\mathcal{D}^{\alpha\dot\beta}\lambda_\alpha - g[\bar\psi^{\dot\beta},A-iB] = 0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} - g[\psi_\alpha,A+iB] = 0, \\ &\mathcal{D}^{\alpha\dot\beta}\psi_\alpha + g[\bar\lambda^{\dot\beta},A-iB] = 0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\psi^{\dot\beta} + g[\lambda_\alpha,A+iB] = 0. \end{aligned} \tag{6}$$

Lagrangian (5) and the equations of motion (6) are invariant under the following $N = 1$ supersymmetric transformations

$$\begin{aligned} &\delta_\xi(A-iB) = 2\xi^\alpha\psi_\alpha, \qquad \delta_\xi(A+iB) = 2\bar\xi_{\dot\alpha}\bar\psi^{\dot\alpha}, \\ &\delta_\xi V_{\alpha\dot\alpha} = -2i(\xi_\alpha\bar\lambda_{\dot\alpha} + \bar\xi_{\dot\alpha}\lambda_\alpha), \qquad \delta_\xi D = -\xi^\alpha\mathcal{D}_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha} + \bar\xi_{\dot\alpha}\mathcal{D}^{\alpha\dot\alpha}\lambda_\alpha, \\ &\delta_\xi(F+iG) = 2i\bar\xi_{\dot\beta}\big(\mathcal{D}^{\alpha\dot\beta}\psi_\alpha + g[\bar\lambda^{\dot\beta},A-iB]\big), \\ &\delta_\xi(F-iG) = 2i\xi^\alpha\big(\mathcal{D}_{\alpha\dot\beta}\bar\psi^{\dot\beta} + g[\lambda_\alpha,A+iB]\big), \\ &\delta_\xi\lambda_\alpha = \frac{1}{2}\xi^\beta f_{\alpha\beta} + i\xi_\alpha D, \qquad \delta_\xi\bar\lambda_{\dot\alpha} = \frac{1}{2}\bar\xi^{\dot\beta} f_{\dot\alpha\dot\beta} - i\bar\xi_{\dot\alpha}D, \\ &\delta_\xi\psi_\alpha = i\bar\xi^{\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}(A-iB) + \xi_\alpha(F+iG), \\ &\delta_\xi\bar\psi_{\dot\alpha} = -i\xi^\alpha\mathcal{D}_{\alpha\dot\alpha}(A+iB) + \bar\xi_{\dot\alpha}(F-iG), \end{aligned} \tag{7}$$

where ξ_α, $\bar\xi_{\dot\alpha}$ are the parameters of $N = 1$ supersymmetric transformations.

In the theory (5) there is no supersymmetric generalization of self-duality equations (2) to super self-duality equations. We propose the system of first-order equations, which satisfies the second-order equations of motion (6) [3]

$$\begin{aligned} &f_{\alpha\beta} = 2igc_{\alpha\beta}[A,B], \\ &\mathcal{D}_{1\dot\alpha}(A-iB) = k_1\mathcal{D}_{2\dot\alpha}(A-iB), \qquad \mathcal{D}_{1\dot\alpha}(A+iB) = k_2\mathcal{D}_{2\dot\alpha}(A+iB), \\ &D + ig[A,B] = 0, \qquad F = G = 0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} - g[\psi_\alpha,A+iB] = 0, \\ &\mathcal{D}^{\alpha\dot\beta}\psi_\alpha + g[\bar\lambda^{\dot\beta},A-iB] = 0, \qquad \lambda_\alpha = \bar\psi_{\dot\alpha} = 0, \qquad \psi_1 = k_1\psi_2, \end{aligned} \tag{8}$$

where constant complex bispinor coefficients $c_{\alpha\beta}$ are symmetric and unimodular

$$c_{\alpha\beta} = c_{\beta\alpha}, \qquad \det \|c_{\alpha\beta}\| \equiv c_{11} \cdot c_{22} - c_{12}^2 \equiv \frac{1}{2} c^{\alpha\beta} c_{\alpha\beta} = 1.$$

The coefficients k_1, k_2 can be calculated through $c_{\alpha\beta}$:

$$k_1 = \frac{c_{11}}{c_{12} - i} = \frac{c_{12} + i}{c_{22}}, \qquad k_2 = \frac{c_{11}}{c_{12} + i} = \frac{c_{12} - i}{c_{22}}.$$

In turn, the coefficients $c_{\alpha\beta}$ are expressed through k_1, k_2:

$$c_{11} = \frac{2ik_1k_2}{k_1 - k_2}, \qquad c_{12} = \frac{i(k_1 + k_2)}{k_1 - k_2}, \qquad c_{22} = \frac{2i}{k_1 - k_2}.$$

The system of first-order equations (8) is invariant under $N = 1$ supersymmetric transformations (7) in the subspace of the parameters of transformations, which is defined by the following condition

$$\xi_1 = k_2 \xi_2.$$

In the case of vanishing component fields of scalar multiplet

$$A = B = \psi_\alpha = \bar{\psi}_{\dot{\alpha}} = F = G = 0$$

the system (8) reduces to the system of super self-duality equations (4). Because of these properties we call (8) the system of super quasi-self-duality equations. Considering only bosonic sector of theory (5), we obtain from (8) the system of quasi-self-duality equations. The concept of quasi-self-duality was introduced by V.A. Yatsun [4, 5].

The fact that the solutions of the system (8) are the solutions of the equations of motion (6) is immediately proved by taking covariant derivatives from the first three equations of the system (8). To see this we take a covariant derivative $\epsilon^{\beta\gamma} \mathcal{D}_{\gamma\dot{\beta}}$ from the first equation of the system (8):

$$f_{\alpha\beta} - g c_{\alpha\beta} [A - iB, A + iB] = 0.$$

Then by using the second and third equations of the system (8) and taking into account the Bianchi identity $\epsilon^{\dot{\alpha}\dot{\gamma}} \mathcal{D}_{\alpha\dot{\gamma}} f_{\dot{\alpha}\dot{\beta}} = \epsilon^{\beta\gamma} \mathcal{D}_{\gamma\dot{\beta}} f_{\alpha\beta}$ and $\lambda_\alpha = \bar{\psi}_{\dot{\alpha}} = 0$, we obtain the first equation of the system (6). Taking a covariant derivative $\mathcal{D}^{1\dot{\alpha}}$ and $\mathcal{D}^{2\dot{\alpha}}$ from the second and third equations of (8) and using the first equation of this system, we obtain corresponding equations of motion from the system (6).

A superfield formulation of the super quasi-self-duality equations (8) can be obtained in the superspace $(x^m; \theta_\alpha; \bar{\theta}_{\dot{\alpha}})$ with the fixed Grassmann variable $\theta_1 = k_2 \theta_2$. A spinor chiral superfield is defined by [6]

$$W_\alpha = -\frac{1}{8g} \overline{\mathcal{D}\mathcal{D}} \big(e^{-2gV} \mathcal{D}_\alpha e^{2gV} \big),$$

where V is a vector superfield in the Wess–Zumino gauge

$$V = -\theta^\alpha \bar{\theta}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta} D,$$

and the covariant derivatives are

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}}, \qquad \overline{\mathcal{D}}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \partial_{\alpha\dot{\alpha}}.$$

In component fields it is given by

$$W_\alpha = -i\lambda_\alpha - \frac{i}{2}\theta^\beta(f_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}D) + \theta^\beta\bar\theta^{\dot\beta}\partial_{\beta\dot\beta}\lambda_\alpha + \theta\theta\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta}$$
$$- \frac{1}{4}\theta\theta\bar\theta^{\dot\alpha}\epsilon^{\beta\gamma}\partial_{\gamma\dot\alpha}(f_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}D) - \frac{i}{4}\theta\theta\bar\theta\bar\theta\Box\lambda_\alpha, \qquad (9)$$

where

$$\Box = \partial_m\partial^m = -\frac{1}{2}\partial_{\alpha\dot\beta}\partial^{\alpha\dot\beta}.$$

The Hermitian adjoint superfield is

$$\big(W_\alpha\big)^\dagger = \overline{W}_{\dot\alpha} = -\frac{1}{8g}\mathcal{D}\mathcal{D}\big(\overline{\mathcal{D}}_{\dot\alpha}e^{2gV}\cdot e^{-2gV}\big).$$

A scalar chiral superfield is defined by the condition $\overline{\mathcal{D}}_{\dot\alpha}\Phi = 0$ and in components has the form

$$\Phi = \frac{1}{2}(A - iB) + \theta\psi + \frac{i}{2}\theta^\alpha\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}(A - iB) + \frac{i}{2}\theta\theta\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\psi^\alpha$$
$$+ \frac{1}{8}\theta\theta\bar\theta\bar\theta\Box(A - iB) + \frac{1}{2}\theta\theta(F + iG).$$

Consequently, its Hermite conjugate $\Phi^\dagger$ is given by $\mathcal{D}_\alpha\Phi^\dagger = 0$. The gauge-covariant spinor derivatives are

$$\nabla_\alpha = \mathcal{D}_\alpha + i[\mathcal{A}_\alpha, \cdot\,], \qquad \nabla_{\dot\alpha} = \overline{\mathcal{D}}_{\dot\alpha} + i[\mathcal{A}_{\dot\alpha}, \cdot\,],$$

where

$$\mathcal{A}_\alpha = -ie^{-2gV}\mathcal{D}_\alpha e^{2gV}, \qquad \mathcal{A}_{\dot\alpha} = i\overline{\mathcal{D}}_{\dot\alpha}e^{2gV}\cdot e^{-2gV}.$$

The system of super quasi-self-duality equations (8) can be presented in superfield formalism in the following form

$$\nabla_\alpha W_\beta = -2ig(c_{\alpha\beta} - i\varepsilon_{\alpha\beta})[\Phi, \Phi^\dagger - 2g[V, \Phi^\dagger]],$$
$$\nabla_1\Phi = k_1\nabla_2\Phi, \qquad \nabla_{\dot\alpha}\Phi^\dagger = 0, \qquad \theta_1 = k_2\theta_2. \qquad (10)$$

The system of superfield quasi-self-duality equations (10) includes the equation for Grassmann coordinates

$$\theta_1 = k_2\theta_2. \qquad (11)$$

To obtain component system (8) from superfield system (10) we have to calculate first three superfield equations of (10) in component fields, and only after that we have to impose the condition (11) on the Grassmann variables in these superfield equations.

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