

Orbit Functions of Compact Semisimple Lie Groups as Special Functions

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An orbit function is the contribution to an irreducible character of a compact semisimple Lie group G of rank n from one of its Weyl group orbits. Properties of such functions will be described for compact simple Lie groups of all types. In particular, products of the functions decompose into their sums, the functions are periodic on copies of the fundamental region F of G , they are solutions of corresponding Laplace equation in n dimensions, satisfying Neumann condition at the boundary of F , ... Uncommon applications to image enhancements and a new approach to n -dimensional data compression are shown, when orbit functions are evaluated at suitable sets of conjugacy classes of elements of finite order in G .

1 Introduction

In this talk I will make some general claims, which are to be justified elsewhere [1]. Here the claims are illustrated by examples. More precisely, my aim is twofold: First I want to introduce some mathematical background to [2–4], and then I want to show that the orbit functions, the main subject here, are eigenfunctions of corresponding Laplace operators in \mathbb{R}^n and that they satisfy Neumann condition at the boundary of the fundamental region of the corresponding compact semisimple Lie group.

An orbit function is the contribution to the character of an irreducible representation from a single Weyl group orbit [5–9]. Thus in the term ‘orbit function of the compact semisimple Lie group G ’ the adjective ‘orbit’ refers to the orbit of the Weyl group W of G rather than to the orbit of G . The term was introduced in [10].

Generally one may argue in favor of the fact that a large family of functions, defined on conjugacy classes of any compact semisimple Lie group G , called ‘orbit functions’, belong among the special functions of mathematical physics. The family is quite rich in the sense that it contains not only a basis of the space of class functions on G , but it contains such bases for semisimple Lie groups of all types and ranks.

Although there is no generally accepted definition of special function [11–13], most of the practitioners would probably agree that there are two related requirements which need to be satisfied before a family of functions would be called special.

Firstly, the functions should find an extensive use in some part of science and engineering. Secondly, the functions should have some ‘nice’ mathematical properties. The later often means that they should form a basis in a functional space with some kind of orthogonality property, which almost always implies that they are solutions of a meaningful differential equations, and that products of two functions should simply decompose into the sums of them. In most cases it is assured by the presence of a group underlying the special functions.

So far a claim of practical applicability of orbit functions, would find only a feeble justification, and that only in the work related to us in some way and dealing with ‘abstract’ although rather challenging problems [10, 14–17].

Actual motivation for this article is the recent recognition of the fact that the orbit functions are indispensable for a vast generalization of the discrete cosine transform [2–4], which turned up to be extremely useful in recent years [18]. In addition, it appears that it is difficult to overestimate the future role of orbit functions in some applications to image enhancements and/or requiring data compression [2–4].

The goal in here is to bring together diverse facts about orbit functions, many of which are not found in the literature, although they often are straightforward consequences of known facts. In general, for a given compact simple Lie group G of rank n , most of the properties of orbit functions, which are described in this paper, are implications of properties of either the orbits of the Weyl group, or of the irreducible characters [5–9, 11].

Only examples of dimension 2 and 3 are shown here because they are likely to be used more often. Hopefully they can be understood without general arguments by directly verifying their properties through explicit calculation. Nevertheless, since our conclusions are to be general as to the type of the simple Lie group G , it is imperative to profit from the uniformity of the pertinent parts of Lie theory: Many general facts of the theory are recalled without explanation and with only a few references to the literature.

2 Preliminaries

We are interested in complex valued functions $f(g)$ on any semisimple compact Lie group G , i.e. $g \in G$, which are constant on the conjugacy classes of G (called *class functions*)

$$f(g) = f(hgh^{-1}) \quad \text{for all } g, h \in G.$$

Suppose that a maximal torus $T \subset G$ has been chosen and fixed. Every element of G has conjugates in T . One can choose in T a region \tilde{F} in which every conjugacy class of elements of G is represented by precisely one element of G . For any group, \tilde{F} would be called the fundamental region. In Lie theory it is conventional to call the fundamental region F of G , the region where the parameters x of the elements $g(x) \in \tilde{F}$ are found in a standard parametrization of G . Thus for a group of rank n , the parameters of $g(x)$ are components of a vector $x \in \mathbb{R}^n$.

A well known basis in the space of class functions on G consists of the characters $\chi_\Lambda(g(x))$, of finite dimensional irreducible representations Λ of G . To simplify the notation, we write $\chi(\Lambda, x)$ for $\chi_\Lambda(g(x))$. A standard problem of Fourier analysis on G is to find the coefficients of the expansion of a given function $f(x)$ into the series of irreducible characters,

$$f(x) = \sum_{\Lambda} C_{\Lambda} \chi(\Lambda, x), \quad x \in F. \quad (1)$$

In this talk I am interested in the properties of the orbit functions $\Omega(d, x)$, which form another basis in the functional space spanned by the irreducible characters.

$$f(x) = \sum_d D_d \Omega(d, x), \quad x \in F, \quad (2)$$

where the summation extends over the dominant weights d of G .

Orbit functions are closely related to characters $\chi(\Lambda, x)$ of G . Let Λ be the highest weight labelling an irreducible representation of G of finite dimension, and let $DV(\Lambda)$ be the set of its *distinct* dominant weights in the weight system of Λ . Then

$$\chi(\Lambda, x) = \sum_{d \in DV(\Lambda)} c_{\Lambda, d} \Omega(d, x). \quad (3)$$

Here the coefficients $c_{\Lambda,d} \in \mathbb{N}$ are the multiplicities of the dominant weights d in the weight system of the representation Λ . Hence the sum in (3) is finite.

As one of the reasons, why characters are rarely used in extensive applications, one may bring forward the need to know the multiplicities $c_{\Lambda,d}$, and the fact that the larger a representation Λ is, the larger is the sum (3). The multiplicities are calculated using a laborious recursive algorithm, starting from the highest weight Λ . In many situations it is practical to read off their values from the tables [20]. Standard bases of simple roots (α -basis) and of dominant weights (ω -basis) are used here [9].

Due to (3) a number of important properties of characters carry over to properties of orbit functions.

The sets P and P_+ ,

$$P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n \supset P_+ = \mathbb{Z}^{\geq 0}\omega_1 + \cdots + \mathbb{Z}^{\geq 0}\omega_n$$

are called respectively the weight lattice and the cone of dominant weights. The smallest dominant weights, different from zero, are the fundamental weights ω_j .

The fundamental region $F \in \mathbb{R}^n$ of the simple Lie group G of rank n is the simplex which has for its vertices the origin of \mathbb{R}^n and the fundamental weights appropriately scaled. More precisely, the following $n + 1$ points are the vertices of the simplex F .

$$F : \left\{ 0, \frac{\omega_1}{q_1}, \dots, \frac{\omega_n}{q_n} \right\}, \quad (4)$$

where q_k are the coefficients of the *dual of the highest short root* in α -basis. For A_2 we have $q_1 = q_2 = 1$. In particular, F is an equilateral triangle in the case of A_2 . See Fig. 1 of [21] for instructive examples of F .

3 Orbit functions

An orbit function $\Omega(d, x)$ of a compact simple Lie group G (equivalently of the Lie algebra L) can be written as a sum of exponential functions $e^{2\pi i \langle v | x \rangle}$. Each term of the sum is determined by a weight v of a Weyl group orbit $O(d)$ as function of $x \in \mathbb{R}^n$. The symbol $\langle v | x \rangle$ denotes scalar product in the Euclidean space \mathbb{R}^n .

Let $O(d)$ be the Weyl group orbit containing the (unique) dominant weight d . The orbit function $\Omega(d, x)$, referring to $O(d)$, is the following finite sum of exponential functions,

$$\Omega(d, x) = \sum_{v \in O(d)} e^{2\pi i \langle v | x \rangle}, \quad x \in \mathbb{R}^n. \quad (5)$$

Note that for any $v, d \in O(d)$, we have the equality $\Omega(d, x) = \Omega(v, x)$. It is a convention to label the orbit functions by their dominant weights because those weights are easily recognized in ω -basis. The number of summands is equal to $|O(d)|$, the number of weights in the orbit $O(d)$.

3.1 Examples

Clearly the orbit functions of A_1 , where $x = \theta\omega$, and $d = m\omega$, $m \in \mathbb{Z}^{\geq 0}$, with $\theta \in \mathbb{R}$, are $\Omega(d, x) = 2 \cos(\pi m \theta)$ if $m \in \mathbb{Z}^{> 0}$, because $\langle \omega | \omega \rangle = \frac{1}{2}$, and $\Omega(0, x) = 1$.

Consider the example of the generic orbit functions of A_2 . Putting $d = (a b) := a\omega_1 + b\omega_2$, with $a, b > 0$, we have,

$$\begin{aligned} \Omega((a b), x) &= e^{2\pi i \langle (a b) | x \rangle} + e^{2\pi i \langle (-a a+b) | x \rangle} + e^{2\pi i \langle (a+b -b) | x \rangle} \\ &\quad + e^{2\pi i \langle (b -a-b) | x \rangle} + e^{2\pi i \langle (-a-b a) | x \rangle} + e^{2\pi i \langle (-b -a) | x \rangle}. \end{aligned}$$

Using $x = \theta_1\omega_1 + \theta_2\omega_2$, where $\theta_1, \theta_2 \in \mathbb{R}$, and $(ab) = a\omega_1 + b\omega_2 = \frac{1}{3}(2a+b)\alpha_1 + \frac{1}{3}(a+2b)\alpha_2$, where $a, b \in \mathbb{Z}$, we have

$$\begin{aligned} \Omega((ab), x) &= e^{2\pi i \frac{1}{3}((2a+b)\theta_1 + (a+2b)\theta_2)} + e^{2\pi i \frac{1}{3}((-a+b)\theta_1 + (a+2b)\theta_2)} \\ &\quad + e^{2\pi i \frac{1}{3}((2a+b)\theta_1 + (a-b)\theta_2)} + e^{2\pi i \frac{1}{3}((b-a)\theta_1 + (-2a-b)\theta_2)} \\ &\quad + e^{2\pi i \frac{1}{3}((-a-2b)\theta_1 + (a-b)\theta_2)} + e^{2\pi i \frac{1}{3}((-a-2b)\theta_1 + (-2a-b)\theta_2)}. \end{aligned} \quad (6)$$

Similarly one finds $O((a0), x)$ and $O((0b), x)$ using (5):

$$\Omega((a0), x) = e^{\frac{2\pi i}{3}a(2\theta_1 + \theta_2)} + e^{\frac{2\pi i}{3}a(-\theta_1 + \theta_2)} + e^{\frac{2\pi i}{3}a(-\theta_1 - 2\theta_2)}, \quad (7)$$

$$\Omega((0b), x) = e^{\frac{2\pi i}{3}b(\theta_1 + 2\theta_2)} + e^{\frac{2\pi i}{3}b(\theta_1 - \theta_2)} + e^{\frac{2\pi i}{3}b(-2\theta_1 - \theta_2)}. \quad (8)$$

Note that $\Omega((aa), x)$ are real valued for all $a \in \mathbb{Z}^{>0}$:

$$\begin{aligned} \Omega((aa), x) &= 2 \{ \cos 2\pi a(\phi_1 + \phi_2) + \cos 2\pi a(2\phi_2 - \phi_1) + \cos 2\pi a(2\phi_1 - \phi_2) \} \\ &= 2 \{ \cos 2\pi a(\theta_1 + \theta_2) + \cos 2\pi a\theta_1 + \cos 2\pi a\theta_2 \}. \end{aligned} \quad (9)$$

Note also that the pairs $\Omega((a0), x) + \Omega((0a), x)$ or, more generally, $\Omega((ab), x) + \Omega((ba), x)$ are always real.

3.2 Other properties of orbit functions

A number of properties of the orbit functions can be either inferred directly from (5) or from the properties of characters.

1. An orbit function is a finite sum of exponential functions. Therefore it is continuous and has continuous derivatives of all orders in \mathbb{R}^n .
2. All orbit functions of the following groups are real:

$$A_1, B_n, C_n, D_{2k}, E_7, E_8, F_4, G_2.$$

The orbit functions of the remaining groups are real provided the dominant weights d are invariant with respect to the symmetry transformation of the Dynkin diagram.

3. Matrix $C = (c_{\Lambda, z})$ of the multiplicities of the dominant weights in irreducible representation Λ is triangular with diagonal elements being all equal to 1 (assuming a suitable ordering of the weights). Hence it can be easily inverted. Consequently, every orbit function can be expressed as an integer linear combination of a finite number of irreducible characters. Many examples of C are found in the tables of [20].
4. Symmetries of the orbit functions.

By definition, $\Omega(d, x)$ is invariant under the action of the Weyl group:

$$\Omega(d, x) = \Omega(wd, x), \quad \text{for all } w \in W. \quad (10)$$

Indeed, the weights d and wd belong to the same orbit and the expression for $\Omega(d, x)$ contains summation over all the weights of the orbit.

From the equalities of scalar products, $\langle wa | x \rangle = \langle a | w^{-1}x \rangle$ and $\langle a | x \rangle = \langle wa | wx \rangle$ for all $a \in O(d)$ and $w \in W$, follow the corresponding equalities of orbit functions.

Multidimensional periodicity of $\Omega(d, x)$ follows from such a periodicity of $\chi(\Lambda, x)$. Copies of F tile \mathbb{R}^n . Character values on each tile are the same. Such a symmetry of characters

and orbit functions is naturally expressed as the invariance under the affine Weyl group as illustrated in the last section below. Values of orbit functions are thus repeated on each tile. More precisely, characters, and thus also orbit functions, are symmetric with respect to any common $(n - 1)$ -dimensional face of two adjacent copies of F .

5. The normal derivative of $\Omega(d, x)$ to the boundary of F equals zero. It follows from the continuity of $\Omega(d, x)$ and of its derivatives, together with the symmetry with respect to the reflections in sides of F .
6. Orthogonality of orbit functions on F follows from the orthogonality of characters, and from the fact that a given weight belongs to precisely one orbit function.
7. There are other aspects of orbit functions which could be rather useful to study. Let us just point out the following ones:

Decomposition of products of orbit functions into their sums [17].

Branching rules for orbit functions [22–24].

Recursive construction of orbit functions from a few smallest ones.

Scaling symmetries of orbit functions.

Derivatives of orbit functions.

4 Orbit functions and the Neumann boundary value problem on F

It is shown here that an orbit function is an eigenfunction of the Laplace operator Δ , acting on the class functions of G , and that its eigenvalue is, up to a constant, given by the square length of the weight of the corresponding Weyl group orbit. It was already explained that the normal derivative of an orbit function at the boundary of F equals zero (Neumann condition). Hence any function, expanded into a series of orbit functions, has to satisfy the same condition at the boundary of F .

Let $x = \theta_1\omega_1 + \dots + \theta_2\omega_2$. Denoting by ∂_k partial derivative with respect to θ_k , we have the Laplace operator Δ ,

$$\Delta = \sum_{i,j=1}^n M_{ij} \partial_i \partial_j \tag{11}$$

acting on the orbit functions and, more generally, on the class functions $f(x)$. For simple Lie groups of types A_n , D_n , and E_n , the matrix M coincides with the Cartan matrix. For the other cases some columns of the Cartan matrix need to be rescaled.

Next let us show the following properties of orbit functions $\Omega(d, x)$ of A_2 , equivalently of $SU(3)$:

$$\Delta\Omega(d, x) = \lambda\Omega(d, x), \quad \left. \frac{\partial\Omega(d, x)}{\partial m} \right|_{\partial F} = 0, \quad x \in F \subset \mathbb{R}^n, \quad d \in P_+, \tag{12}$$

where ∂F denotes the boundary of F from (4), formed by its $(n - 1)$ -dimensional faces, and m denotes the normal to the boundary, and where

$$\lambda = -4\pi^2 \langle d | d \rangle. \tag{13}$$

Above it was already explained that the orbit functions satisfy the Neumann condition on the boundary of F , and that F is an equilateral triangle in the case of A_2 .

Consider the $SU(3)$ -Laplace operator Δ acting on a single term $e^{2\pi i\langle z|x\rangle}$, and suppose that both, the fixed vector $z = (a, b)$ and the variable vector $x = (\theta_1, \theta_2)$, are given in ω -basis. Their scalar product is calculated using the inverse of the Cartan matrix of A_2 .

$$\langle z|x\rangle = z^T Q x = \frac{1}{3}(a \ b) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \frac{1}{3}(2a + b)\theta_1 + \frac{1}{3}(a + 2b)\theta_2. \quad (14)$$

In particular, for a general $z \in P$ and $\xi = \omega_1 + \omega_2$,

$$\langle z|z\rangle = \frac{2}{3}(a^2 + ab + b^2), \quad \text{and} \quad \langle z|\xi\rangle = a + b.$$

Application of the Laplace operator to the exponential function is straightforward,

$$\begin{aligned} \Delta e^{2\pi i\langle z|x\rangle} &= (\partial_1 \ \partial_2) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix} e^{2\pi i\langle z|x\rangle} = 2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2) e^{\frac{2\pi i}{3}\{(2a+b)\theta_1+(a+2b)\theta_2\}} \\ &= 6\left(\frac{2\pi i}{3}\right)^2 (a^2 + ab + b^2) e^{\frac{2\pi i}{3}\{(2a+b)\theta_1+(a+2b)\theta_2\}} = -4\pi^2 \langle z|z\rangle e^{2\pi i\langle z|x\rangle}. \end{aligned}$$

Weights of an orbit $O(d)$ are equidistant from the origin, i.e. for any $z \in O(d)$ we have $\langle z|z\rangle = \langle d|d\rangle$. Therefore the eigenvalues of Δ are the same on all the exponential terms of an orbit function. Consequently, the eigenvalues of the orbit functions and of each of its exponential terms are the same.

4.1 Symmetry under the affine Weyl group

The invariance of the orbit functions under the Weyl group is built into their definition through the summation over the elements of an W -orbit. Indeed, W merely permutes the terms in that sum. More spectacular is the symmetry of orbit functions under the affine Weyl group W^{aff} . It is a larger group than W . It has one more generator, namely the affine reflection r_0 ,

$$r_0 x = x + \left(1 - \frac{2\langle x|\xi\rangle}{\langle \xi|\xi\rangle}\right) \xi = \xi + r_\xi x.$$

Here $r_\xi \in W$ is the reflection in the plane orthogonal to the highest short root ξ . In case of A_2 one has $\xi = \alpha_1 + \alpha_2 = \omega_1 + \omega_2$, and $\langle \xi|\xi\rangle = 2$. The action of W^{aff} on F results in tiling of \mathbb{R}^n by copies of F . In particular,

$$r_0 F = \{r_0, r_0\omega_1, r_0\omega_2\} = \{\xi, \omega_1, \omega_2\}.$$

Note the translations $r_0 r_\xi x = x - \xi$, and $r_\xi r_0 = x + \xi$. Putting $z = (a, b)$, $a, b \in \mathbb{Z}$ and $x = (\theta_1, \theta_2)$, $\theta_1, \theta_2 \in \mathbb{R}$,

$$\langle z|r_0 x\rangle = \langle z|\xi\rangle + (1 - \langle \xi|x\rangle)\langle z|\xi\rangle = \langle z|\xi\rangle + \langle z|r_\xi x\rangle = a + b + \langle r_\xi z|x\rangle.$$

Consequently,

$$e^{2\pi i\langle z|r_0 x\rangle} = e^{2\pi i\langle r_\xi z|x\rangle}.$$

Here $r_\xi z = -b\omega_1 - a\omega_2$ is another weight from the orbit of z . Therefore $\Omega(z, x) = \Omega(z, r_0 x)$. Thus the orbit functions are invariant with respect to the affine Weyl group W^{aff} .

5 Discretization of the orbit functions

Here I want to recall the main result of [10], which is a discrete orthogonality of a finite (arbitrarily large) set of orbit functions of G , on a grid S of discrete points in F (hence in all \mathbb{R}^n). Equipped with such an orthogonality, one can invert finite (arbitrarily long) Fourier expansions of class functions $f(x)$ on G . This setting of the expansion problem is particularly suitable when the function $f(x)$ is provided digitally, i.e. by its values on the points of a grid.

Start by fixing a natural number M . Then $S^{[M]}$, the set of points of the grid, consists of all distinct points $s \in S^{[M]} \subset F$ which, in the ω -basis, have rational coordinates with the common denominator equal to M , subjects to the additional restriction $\langle s | \xi \rangle \leq 1$.

Consider $\Omega(d, s)$ for $s \in S^{[M]}$. Then for certain set of dominant weights d , we have

$$\langle \Omega(d) | \Omega(d') \rangle_M := \sum_{s \in S^{[M]}} C_s^{[M]} \Omega(d, s) \overline{\Omega(d', s)} = \delta_{d, d'} \cdot K. \quad (15)$$

Here $C_s^{[M]}$, K are known constants [10], overbar indicates complex conjugation, and the dominant weights d, d' belong to a set which contains comparable number of distinct dominant weights as is the number of elements in $S^{[M]}$.

5.1 Examples

Let $G = A_2$, equivalently $G = SU(3)$. In this case F is an equilateral triangle. Put $M = 2$. The grid $S^{[2]}$ in F contains 6 points s . In ω -basis these are the following ones:

Three vertices of F : (00) , (01) , (10) , and the middle points of the edges of F : $(\frac{1}{2} \frac{1}{2})$, $(0 \frac{1}{2})$, $(\frac{1}{2} 0)$.

Values of a few lowest orbit functions $\Omega(d, s)$ at the points $s \in S^{[2]}$, are shown in Table 1. To simplify the notation, we also use $\omega = e^{2\pi i/6}$. Note that the orbit functions of the first 6 rows of the table are orthogonal on the $S^{[2]}$ according to (15). Orbit functions of the remaining rows (and any higher one we might have shown) are multiples of one of the first six ones.

Table 1. Values of several A_2 -orbit functions at the six points of $S^{[2]}$, and the coefficients $C_s^{[2]}$. Notation: $\omega = e^{2\pi i/6}$.

s	(00)	$(\frac{1}{2} 0)$	$(0 \frac{1}{2})$	(10)	$(\frac{1}{2} \frac{1}{2})$	(01)
$\Omega((00), s)$	1	1	1	1	1	1
$\Omega((10), s)$	3	ω^5	ω	$3\omega^4$	-1	$3\omega^2$
$\Omega((01), s)$	3	ω	ω^5	$3\omega^2$	-1	$3\omega^4$
$\Omega((20), s)$	3	$3\omega^4$	$3\omega^2$	$3\omega^2$	3	$3\omega^4$
$\Omega((11), s)$	6	-2	-2	6	-2	6
$\Omega((02), s)$	3	$3\omega^2$	$3\omega^4$	$3\omega^4$	3	$3\omega^2$
$\Omega((30), s)$	3	-1	-1	3	-1	3
$\Omega((21), s)$	6	$2\omega^5$	2ω	$6\omega^4$	-2	$6\omega^2$
$\Omega((12), s)$	6	2ω	$2\omega^5$	$6\omega^2$	-2	$6\omega^4$
$\Omega((03), s)$	3	-1	-1	3	-1	3
$\Omega((22), s)$	6	6	6	6	6	6
$C_s^{[2]}$	1	3	3	1	3	1

In particular, one finds

$$\langle \Omega(00) | \Omega(00) \rangle_2 = 12, \quad \langle \Omega(10) | \Omega(10) \rangle_2 = 36, \quad \langle \Omega(11) | \Omega(11) \rangle_2 = 144.$$

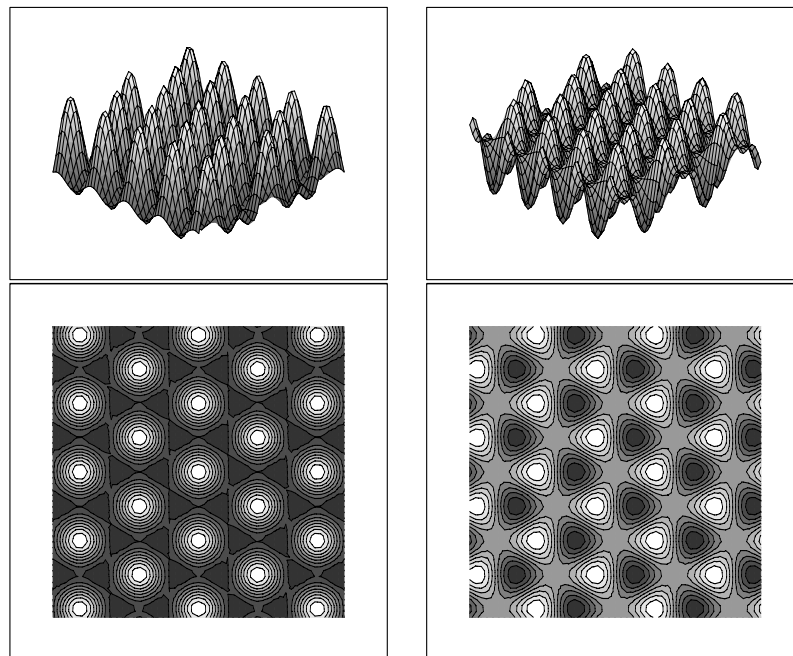


Figure 1. A_2 orbit function $\Omega((20), x)$ real (left) and imaginary (right) parts.

Finally let us illustrate how to calculate the entries of Table 1. Take $\Omega((21), (\frac{1}{2} 0))$. Its value, $2\omega^5$, is found from (6) by putting there $a = 2$, $b = 1$, and $\theta_1 = \frac{1}{2}$, $\theta_2 = 0$, followed by obvious simplifications.

Crystallographic symmetries of orbit functions are illustrated on Fig. 1 and 2. Fig. 1 shows the real and imaginary parts of the A_2 -orbit function $\Omega((20), x)$. Examples of the real orbit functions $\Omega((21), x)$ and $\Omega((11), x)$ of C_2 and G_2 respectively are shown on Fig. 2.

Acknowledgements

I am grateful to A. Klimyk for helpful comments, and to A. Zaratsyan for preparation of the Figures.

I also acknowledge partial support for the work from the National Science and Engineering Research Council of Canada, from NATO, from the Laboratoires Universitaires Bell, and from Lockheed Martin Canada.

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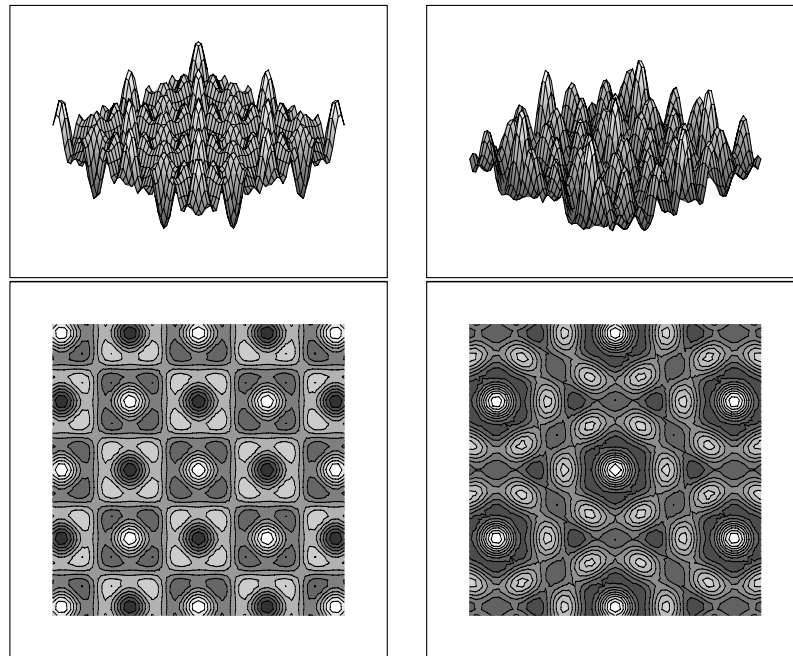


Figure 2. $\Omega((21), x)$ of C_2 (left) and $\Omega((11), x)$ of G_2 (right) orbit functions.

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