Diagonal Representation of Density Matrix Using q-Coherent States

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A q-analogue of the diagonal representation of the density matrix is derived using q-boson coherent states. The q-generalization of the self-reproducing property of $\rho(z', z)$ and a self consistent self-reproducing kernel $K(z', z)$ are obtained. Some applications to non-linear issues in optics are indicated

1 Introduction

A diagonal representation of the quantum mechanical density matrix by means of standard bosonic oscillator coherent states was obtained by Sudarshan [1] and Glauber [2]. A remarkable feature of this representation is that the average expectation value of normal ordered operators becomes the same as that of a classical function for a probability distribution over complex plane, thereby bringing one-to-one correspondence between classical complex representation and quantum mechanical density matrices. The study of Quantum Groups has led to a non-linear realization of Lie algebras and this resulted in the construction of $SU_q(2)$ algebra using q-bosonic oscillators by Macfarlane [3] and Biedenharn [4]. A corresponding q-fermion oscillator and $SU_q(2)$ algebra had been proposed by Parthasarathy and Viswanathan [5]. Such q-oscillators effectively deal with non-ideal systems which have interactions and provide a method to study non-linear excitations of EM fields. The use of q-bosonic oscillator to describe density matrix has been made by Nelson and Fields [6]. In this contribution, we reconsider this issue in more detail and present some new results regarding the self-reproducing property and relation between the expansion coefficients of the density matrix in the Fock space description and q-coherent state description.

2 q-boson coherent states

The q-boson oscillator algebra for q-annihilation, q-creation and q-number operators $\{a, a^\dagger, N\}$ is [3, 4]

$$aa^\dagger - qaa^\dagger = 1, \qquad [N, a] = -a, \qquad [N, a^\dagger] = a^\dagger, \tag{1}$$

and the deformation parameter q is real and $0 < q \leq 1$. Further $N \neq a^\dagger a$. The Fock space $\mathcal{F}$ is built upon a vacuum state $|0\rangle$ with $a|0\rangle = 0$. The n q-boson normalized Fock space state is given by

$$|n\rangle = \frac{1}{\sqrt{[n]!}}(a^\dagger)^n|0\rangle, \tag{2}$$

where $\{|n\rangle \mid n \in \mathbb{N} \cup \{0\}\}$ and $[n]$ is the q-number

$$[n] = \frac{1 - q^n}{1 - q}. \tag{3}$$

Further it follows,

$$a|n\rangle = \sqrt{[n]}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{[n+1]}|n+1\rangle,$$
$$N|n\rangle = n|n\rangle, \qquad a^\dagger a|n\rangle = [n]|n\rangle. \tag{4}$$

And $[n]! = [n][n-1][n-2]\cdots[2][1]$. A q-boson coherent state is defined as

$$|z\rangle = \left\{e_q^{|z|^2}\right\}^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}}|n\rangle, \tag{5}$$

where z is a complex variable and e_q^x is the q-exponential function. This is a meromorphic function [7] with no zeroes and simple poles at $x_k = q^{-k}/(1-q)$. The first pole occurs at $x = 1/(1-q)$. Further it can be seen that $|z\rangle$ is an eigenstate of a with eigenvalue z. The most important property of $|z\rangle$ is that if $|z'\rangle$ is another such q-coherent state, then

$$\langle z|z'\rangle = \left\{e_q^{|z|^2} e_q^{|z'|^2}\right\}^{-\frac{1}{2}} e_q^{\bar{z}z'}, \tag{6}$$

which means that such states are not orthogonal. But $\langle z|z\rangle = 1$. Further, by writing $z = re^{i\theta}$ and treating the integration over the bounded variable θ as ordinary integration while that over r^2 as q-integration, it follows from (5)

$$\frac{1}{\pi}\int d^2z|z\rangle\langle z| = \frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^{R_q^2} \frac{1}{2}dr^2|z\rangle\langle z| = 1, \tag{7}$$

where $R_q^2 = 1/(1-q)$.

3 Diagonal representation of density matrix

The density matrix has the expansion

$$\rho = \sum_{n,m=0}^{\infty} \rho(n,m)|n\rangle\langle m|, \tag{8}$$

in terms of Fock space states. From the definition of the q-coherent state (5), it can be shown [8] that

$$|n\rangle\langle m| = \frac{\sqrt{[n]![m]!}}{[n+m]!}\left\{\left(\frac{d}{dr}\right)_q^{(n+m)} \int_0^{2\pi} \frac{d\theta}{2\pi} e_q^{r^2} e^{i(m-n)\theta} |re^{i\theta}\rangle\langle re^{i\theta}|\right\}_{r=0}, \tag{9}$$

where $(\frac{d}{dr})_q$ stands for q-differentiation. Substituting this in (8), we obtain the diagonal representation of the density matrix in terms of q-boson coherent states as

$$\rho = \sum_{n,m=0}^{\infty} \rho(n,m)\frac{\sqrt{[n]![m]!}}{[n+m]!}\left\{\left(\frac{d}{dr}\right)_q^{(n+m)} \int_0^{2\pi} \frac{d\theta}{2\pi} e_q^{r^2} e^{i(m-n)\theta} |re^{i\theta}\rangle\langle re^{i\theta}|\right\}_{r=0}, \tag{10}$$

a q-analogue of the result of [1,2]. Expanding ρ as

$$\rho = \int d^2z\phi(z)|z\rangle\langle z|,$$

with $\int d^2z\phi(z) = 1$ (to ensure that $\operatorname{Tr}\rho = 1$), the expansion coefficients $\rho(n,m)$ in $\mathcal{F}$ and $\phi(z)$ in q-coherent states, are related by

$$\begin{aligned}\rho(n,m) &= \int d^2z\phi(z)\frac{z^n\bar{z}^m}{\sqrt{[n]![m]!}}\left\{e_q^{|z|^2}\right\}^{-1}\\ &= \pi\int_0^{R_q^2} d\left(r^2\right)\phi(r^2)\frac{r^{2n}}{[n]!}\left\{e_q^{r^2}\right\}^{-1}\delta_{n,m},\end{aligned}\tag{11}$$

where in the last step the θ integration is performed. It follows

$$\sum_n^\infty \rho(n,n) = 1.$$

4 Some applications

As a first application, we consider the expansion of normal ordered operators, $F = \sum_{p,s} C_{p,s}(a^\dagger)^p a^s$. Then using $a|z\rangle = z|z\rangle$, and taking matrix elements of F in between q-coherent states, we find

$$C_{p,s} = \sum_{k=0}^{\min(p,s)} \frac{(-1)^k q^{k(k-1)/2}}{[k]!\sqrt{[s-k]![p-k]!}}\langle p-k|F|s-k\rangle,\tag{12}$$

relating $C_{p,s}$ to the Fock space matrix elements of F.

As a second application, we consider the average expectation value of F.

Using $\rho = \int d^2z\phi(z)|z\rangle\langle z|$, we find,

$$\operatorname{Tr}(\rho F) = \sum_{p,s} C_{p,s}\int d^2z\phi(z)\bar{z}^p z^s.\tag{13}$$

The following results are to be noted. First, $\operatorname{Tr}(\rho F)$ is now the expectation value of complex classical function $\bar{z}^p z^s$ for a distribution $\phi(z)$. Second for $\phi(z) = \phi\big(r^2\big)$, this becomes,

$$\pi\sum_{p,s} C_{p,s}\int_0^{R_q^2} d\left(r^2\right)\phi\left(r^2\right)r^{2p}.$$

For $\phi(r^2) = \frac{1}{\pi}e_q^{-r^2}$, q-Gaussian, $\operatorname{tr}(\rho F) = \sum_p C_{p,p}[p]!$.

As a third application, we consider an operator $O = \sum_n B_n(a^\dagger a)^n$. First of all, it is possible to express

$$(a^\dagger a)^n = \sum_s C_s^n (a^\dagger)^s a^s.\tag{14}$$

The coefficients C_s^n satisfy a recursion relation

$$C_s^{n+1} = C_{s-1}^n q^{s-1} + [s]C_s^n.$$

Then we find,

$$\operatorname{Tr}(\rho O) = \pi\sum_n B_n\sum_{s=0}^n C_s^n\int_0^{R_q^2} d\left(r^2\right)\phi\left(r^2\right)r^{2s}.\tag{15}$$

For $\phi(r^2)$ a q-Gaussian, (15) simplifies to

$$\operatorname{Tr}(\rho O) = \sum_n B_n\sum_{s=0}^n C_s^n[s]!.$$

5 Self reproducing property

Now we take up the important feature of ρ, namely, the self-reproducing property. The matrix elements of ρ in between two q-coherent states is calculated to be

$$\langle z'|\rho|z\rangle = \sum_{n,m} \rho(n,m) \left\{ e_q^{|z|^2} e_q^{|z'|^2} \right\}^{-\frac{1}{2}} \frac{\bar{z}'^m z^n}{\sqrt{[m]![n]!}}. \tag{16}$$

Introduce

$$\rho(z',z) = \sum_{n,m} \rho(n,m) \frac{\bar{z}'^m z^n}{\sqrt{[m]![n]!}} \left\{ e_q^{|z'|^2} e_q^{|z|^2} \right\}. \tag{17}$$

Then, $\rho(z',z) = \langle z'|\rho|z\rangle$. Now we use the completeness relation (7) for the q-coherent states, to write

$$\begin{aligned} \rho(z',z) &= \langle z'|\rho|z\rangle \\ &= \int d^2\xi \langle z'|\rho|\xi\rangle\langle\xi|z\rangle \frac{1}{\pi} \\ &= \int d^2\xi K(\xi,z)\rho(z',\xi), \end{aligned} \tag{18}$$

where the reproducing kernel $K(\xi,z)$ is defined as

$$K(\xi,z) = \frac{1}{\pi}\langle\xi|z\rangle. \tag{19}$$

(18) gives the self-reproducing property of $\rho(z',z)$ with $K(\xi,z)$ as the self-reproducing kernel. The theory of self-reproducing kernel has been studied by Aronszajn [9]. The above kernel (19) satisfies the required properties of a self-reproducing kernel. Namely, first, it satisfies the matrix multiplication property,

$$\int d^2\xi\; K(z,\xi)K(\xi,z') = K(z,z'). \tag{20}$$

Second, since ρ is Hermitian, the kernel satisfies the Hermiticity property,

$$K(z,\xi)^* = K(\xi,z). \tag{21}$$

The explicit expression for the kernel is,

$$K(\xi,z) = \frac{1}{\pi} \left\{ e_q^{|\xi|^2} e_q^{|z|^2} \right\}^{-\frac{1}{2}} e_q^{\bar{\xi}z}, \tag{22}$$

from which it follows

$$K(z,z) > 0. \tag{23}$$

6 q-coherent states in physical situations

We now give an attempt to use q-coherent states to some physical situations.

The fluctuations of photon number in the standard Fock space description is zero. However, optic field described by the standard coherent state has large uncertainty in photon number.

This is seen by considering $\langle z|a^\dagger a|z\rangle = |z|^2 = \langle N\rangle$ for an ordinary coherent state. Also, $\langle N^2\rangle = \langle z|a^\dagger aa^\dagger a|z\rangle = |z|^2 + |z|^4$. Then the uncertainty in the photon number is

$$\triangle n = \sqrt{\langle N^2\rangle - \langle N\rangle^2} = |z|. \tag{24}$$

For large $|z|$, the variance of the photon number is large. If we choose to describe the optic field by a q-coherent state, the photon number is taken as photon q-number. Then using q-coherent states, we have $\langle z|a^\dagger a|z\rangle = \langle N\rangle = |z|^2$. On the other hand, $\langle z|a^\dagger aa^\dagger a|z\rangle = \langle N^2\rangle = |z|^2 + q|z|^4$. Then,

$$\triangle n = |z|\sqrt{1 + (q-1)|z|^2}. \tag{25}$$

By comparing this with (24), we see that the uncertainity in photon q-number depends non-linearly on $|z|$. For $q < 1$, the photon q-number has less uncertainity. Further, the probability of finding the optic field in number state is $|\langle n|z\rangle|^2$, which for ordinary coherent state is

$$P_n = e^{-|z|^2}\frac{|z|^{2n}}{n!},$$

which is just a Poisson distribution. If we use q-coherent states, then

$$P_n = \{e_q^{|z|^2}\}^{-1}\frac{|z|^{2n}}{[n]!},$$

which is q-Poisson distribution.

We now consider the propagation of light in a non-linear medium like Kerr medium. This is usually described by the Hamiltonian [10] as

$$H = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\chi(a^\dagger)^2 a^2, \tag{26}$$

where a, $a^\dagger$ are the usual annihilation and creation operators with $aa^\dagger - a^\dagger a = 1$ and χ represents the interaction strength of the light with the non-linear medium. Using photon number states,

$$\langle n|H|n\rangle = \hbar\omega\left\{n + \frac{\chi}{2\omega}n(n-1)\right\}. \tag{27}$$

Instead of describing the propagation by the Hamiltonian (26), we desire to use a Hamiltonian

$$\mathcal{H} = \hbar\omega a^\dagger a, \tag{28}$$

where now a, $a^\dagger$ are q-annihilation and creation operators (1). Then it follows

$$\langle n|\mathcal{H}|n\rangle = \hbar\omega[n]. \tag{29}$$

The important observation is that (29) will describe the content of (27). By setting $q = 1 - \epsilon$, we have from (3),

$$\langle n|\mathcal{H}|n\rangle = \hbar\omega\left\{n + \frac{\epsilon}{2}n(n-1)\right\}, \tag{30}$$

keeping terms up to ϵ. Thus (30) has the same content of (27), for $\epsilon = \frac{\chi}{\omega}$. However $\mathcal{H}$ is simple, like a free Hamiltonian and the interaction is shifted to q-number. As we have kept terms linear in ϵ, such description will be valid for high frequency propagation.

7 Summary

We have obtained an explicit diagonal representation of the quantum mechanical density matrix using q-coherent states. Certain classes of operators are studied for their statistical averages. The expansion coefficients of ρ in Fock space and q-coherent state representation are related. The C_s^n coefficients introduced in (14) with their recursion relation, are q-Stirling numbers of the second kind and they enter in q-coherent state description. The self-reproducing property of $\rho(z', z)$ is established. In this, the completeness relation of q-coherent states (7) plays a crucial role. The self-reproducing kernel is found and it satisfies the required Hermiticity property and matrix multiplication rule. Applications representing the use of q-coherent states in the description of light propagating in non-linear medium are illustrated.

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