On Symmetry Group Properties
of the Benney Equations

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In the present study, we investigate the symmetry groups of Benney equations that are the system of nonlinear integro-differential equations. We first investigate the symmetry groups of the Benney equations by using the method. Then we obtain all reduced forms of the system of integro-differential equations with fewer variables based on symmetry groups; and lastly, we seek a similarity solution to the reduced system of the equations.

1   Introduction

Symmetry group analysis deals with applications of continuous symmetry groups to the system of differential equations in engineering, mathematics, and physics. However, in the case of integro-differential equations (IDE), there is no general method of investigating the Lie symmetry groups of these equations based on the solution of their determining equations. The main difficulty in applying Lie's infinitesimal techniques to these systems is their nonlocality, and the approach used in the classical group theory cannot be applied for the investigation of symmetry groups of IDE.

There are some studies about symmetry groups of integro-differential equations in the literature. Meleshko [1] searches the Lie point symmetries of the one-dimensional visco-elastic equation and gets a classification with respect to the free term and the kernel function. The method is based on the fact that one considers the determining equations on any solution at any point and any time, for example, at initial time t_0. Here, it is very important to have an existence of solution of the Cauchy problem, which allows splitting the determining equations. It is worth to note that the same problem stays in an application of the group analysis to differential equations. For differential equations the Cauchy–Kovalewskaya theorem treats this. One-dimensional nonlocal elasticity and one dimensional visco-elasticity equations seem a similar structure due the type of their integral equations, it is clear that these equations have different characteristics with respect to the mathematical and kinematics aspects. First, one-dimensional nonlocal elasticity is in the form of Fredholm integro-differential equations by Özer [2] and one dimensional visco-elasticity equation is in the form of Volterra integro-differential equation [1]. In addition, these two equations describe two different mechanics behaviors of the solids. It is important to define symmetry group properties of nonlocal elasticity equations in order to understand these different characteristics of the equations based on their symmetry groups. In addition, Özer [3] investigated symmetry group properties of two-dimensional elastodynamics problem of nonlocal continuum mechanics and obtained a classification due to the kernel function and the free term.

Bobylev [4] is another author who studied the symmetry groups of the integro-differential equations. He obtained all symmetry groups of the Boltzmann and examined some invariant solutions of the Boltzmann equation using its Lie point symmetries. This approach in this study is based on the assumption that we restrict the determining equations to the subset of the solution of integro-differential equation determined by the initial conditions. Firstly, the coefficients of the infinitesimal operator are assumed to be locally analytic functions, and then

these coefficients are represented by Taylor series with respect to the independent variable of the equation. The determining equation is decomposed with respect to the different powers of an arbitrary parameter that exists in the initial functions. Finally the determining equations are split to the series of equations by equating the coefficient of every different power of the independent parameter to zero. Symmetry groups are found by the so-obtained equations.

Chetverikov and Kudryavtsev [5] have written an important article on the subject. The method in their study consists in reducing an integro-differential equation to a system of boundary differential equation and in computing symmetries and conservation laws for the system of integro-differential equations. In the study symmetry is considered as a geometric concept. Therefore, in order to define the symmetry of something it is first represented by some geometric model. Then the symmetry is found by considering the transformations of the model. In the case of integro-differential equations, the analogy with differential equations can also be used. In the method the first step is the introduction of the nonlocal variables. In the second step the initial system is transformed to the functional differential equations. Later the generalized jet spaces are defined so that boundary differential equations can be expanded as submanifolds of these spaces. Finally, a geometric theory of boundary differential equations is constructed similar to the geometry of nonlinear differential equations. This method is fundamentally different from our method that we use in the study.

The last work to be mentioned here is done by Taranov [6]. The first step in this approach is to find the symmetries of the equation dependent on finite number of variables. The symmetries of the system of integro-differential equations are obtained by approaching the number of variables to infinity.

2 Symmetry groups of the Benney equations

In this part, we present briefly the general characteristics of the method of investigating the symmetry groups of the Benney equations introduced in the study [7]. The method presents an important opportunity for investigation of symmetry groups and similarity solutions, as well as solutions of some boundary value problems related to other problems, including IDE in engineering and science. We consider a scalar kth-order IDE represented by

$$\prod(x, u, \underset{1}{u}, \ldots, \underset{k}{u}) + \int dx \hat{\prod}(x, u, \underset{1}{u}, \ldots, \underset{k}{u}) = 0. \tag{1}$$

Then, the infinitesimal transformation which is the invariance criterion of invariance of an integro-differential equation introduced in [8, 9] is written as

$$\underset{k}{X} \prod(x, u, \underset{1}{u}, \ldots, \underset{k}{u}) + \int dx \left[\underset{k}{X} \hat{\prod}(x, u, \underset{1}{u}, \ldots, \underset{k}{u}) + \hat{\prod} \sum_i \partial_i \xi_i \right] = 0. \tag{2}$$

Roberts [8] and Zawistowski [9] studied the symmetry group properties of the one-dimensional Vlasov–Maxwell equation based on the infinitesimal criterion (2) of integro-differential equations.

Here, we investigate the symmetry groups of the system of integro-differential equations corresponding to the Benney kinetic equations below:

$$\frac{\partial f(x,t,v)}{\partial t} + v \frac{\partial f(x,t,v)}{\partial x} - \frac{\partial f(x,t,v)}{\partial v} \frac{\partial A_0(x,t)}{\partial x} = 0, \qquad A_0(x,t) = \int_{-\infty}^{\infty} f(x,v,t) dv, \quad (3)$$

where x and t are the independent variables as spatial coordinate and time, respectively, f is the distribution function, v is the horizontal component of the flow velocity, and A_0 is the function defined as

$$\frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial x} + n A_{n-1} \frac{\partial A_0}{\partial x} = 0, \tag{4}$$

where

$$A_n = \int_0^h u^n(x, y, t)dy, \qquad n = 0, 1, 2, \ldots. \tag{5}$$

For this purpose, we can define the infinitesimal operator for the problem (3) with respect to the definition of the infinitesimal operator for a partial differential equation as

$$X = \xi^x \partial_x + \xi^t \partial_t + \xi^v \partial_v + \eta^f \partial_f + \eta^{A_0} \partial_{A_0}. \tag{6}$$

Here x, v, t are independent variables and f, A_0 are dependent variables of the problem. After the calculations, we obtain a five-parameter Lie group for the Benney equations based on the condition (2) as shown below:

$$\eta^{A_0} = 2a_1 A_0, \qquad \eta^f = a_1 f, \qquad \xi^v = a_1 v + a_3,$$
$$\xi^x = (2a_1 - a_4)x + a_3 t + a_5, \qquad \xi^t = (a_1 - a_4)t + a_2, \tag{7}$$

that are equivalent results obtained by Krasnoslobodtsev [10] and Ibragimov [11], where a_1, a_2, a_3, a_4 and a_5 are constants.

3   Reduced forms of the Benney equations

In this part, we present all reduced forms of the Benney equations by using the corresponding symmetry groups. First, we construct the characteristic equations with respect to the expressions (7) as below:

$$\frac{dx}{(2a_1 - a_4)x + a_3 t + a_5} = \frac{dt}{(a_1 - a_4)t + a_2} = \frac{dv}{a_1 v + a_3} = \frac{df}{a_1 f},$$
$$\frac{dx}{(2a_1 - a_4)x + a_3 t + a_5} = \frac{dt}{(a_1 - a_4)t + a_2} = \frac{dv}{a_1 v + a_3} = \frac{dA_0}{2a_1 A_0}. \tag{8}$$

We obtain the general similarity variables and reduced forms of the Benney equations by using the "integrating factor" technique. Then if we substitute the similarity forms to the original equation (3) by using new similarity variables, then we get the new system of IDE with two new similarity variables, namely ξ_1 and ξ_2, as independent variables; and $\tilde{f}$ and $\tilde{A}_0$ as dependent variables.

If we take the conditions as $a_1 \neq 0$, $a_4 \neq 2a_1$, $a_4 \neq a_1$, then we obtain the general reduced form of equation (3) as

$$(-2a_1 + a_4)\xi_1 \frac{\partial \tilde{f}(\xi_1, \xi_2)}{\partial \xi_1} - a_1 \tilde{f}(\xi_1, \xi_2) + a_1 \xi_2 \frac{\partial \tilde{f}(\xi_1, \xi_2)}{\partial \xi_2}$$
$$+ \xi_2 \frac{\partial \tilde{f}(\xi_1, \xi_2)}{\partial \xi_1} - \frac{d\tilde{A}_0(\xi_1)}{d\xi_1} \frac{\partial \tilde{f}(\xi_1, \xi_2)}{\partial \xi_2} = 0, \qquad \tilde{A}_0 = \int_{-\infty}^{\infty} \tilde{f}(\xi_1, \xi_2)d\xi_2, \tag{9}$$

where the first and the second similarity variables and similarity forms are as follows:

$$\xi_1 = x[(a_1 - a_4)t + a_2]^{(-2a_1 + a_4)(a_1 - a_4)^{-1}}$$
$$\quad - \frac{a_3}{a_1}\left[-\frac{a_1 a_5 + a_2 a_3}{(a_3(2a_1 - a_4)} - t\right][(a_1 - a_4)t + a_2]^{(-2a_1 + a_4)(a_1 - a_4)^{-1}},$$
$$\xi_2 = v[(a_1 - a_4)t + a_2]^{-a_1(a_1 - a_4)^{-1}} + \frac{a_3}{a_1}[(a_1 - a_4)t + a_2](-a_1)(a_1 - a_4)^{-1},$$
$$f(x, v, t) = [(a_1 - a_4)t + a_2]^{a_1(a_1 - a_4)^{-1}} \tilde{f}(\xi_1, \xi_2),$$
$$A_0(x, t) = [(a_1 - a_4)t + a_2]^{2a_1(a_1 - a_4)^{-1}} \tilde{A}_0(\xi_1). \tag{10}$$

4 The relations between Benney equations and shallow-water equations and similarity solutions

It may possible to transform the solutions of Benney equations into solutions of nonlinear-shallow water equations

$$\frac{\partial h(x,t)}{\partial t} + u(x,t)\frac{\partial h(x,t)}{\partial x} + h(x,t)\frac{\partial u(x,t)}{\partial x} = 0,$$
$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial h(x,t)}{\partial x} = 0. \tag{11}$$

by using the following expression given by Gibbons [12]:

$$f(x,t,v) = h(x,t)\delta(u(x,t) - v), \tag{12}$$

where $h(x,t)$ and $u(x,t)$ are solutions of the nonlinear shallow-water equations (11) and δ is Dirac's delta function. In addition to this relation, expression (12) gives us the relationships between Benney equations and both shallow-water equations. In the case of reduced Benney equations, we can use the following relation:

$$\tilde{f}(\xi_1, \xi_2) = \tilde{h}(\xi_1)\delta(\tilde{u}(\xi_1) - \xi_2) \tag{13}$$

which have similar forms with relations (12). Then one may obtain a system of ordinary differential equations (ODE) by using the relations (12). In this section, we give an example of the application of symmetry groups.

For this purpose, we seek the solution for the reduced form of the Benney equations below:

$$(\xi_1 + \xi_2)\frac{\partial \tilde{f}(\xi_1, \xi_2)}{\partial \xi_1} - \frac{\partial \tilde{f}(\xi_1, \xi_2)}{\partial \xi_2}\int_{-\infty}^{\infty}\frac{\partial \tilde{f}(\xi_1, \xi_2)}{\partial \xi_1}d\xi_2 = 0, \tag{14}$$

which is a special form of the reduced form. Using the form (13) we get the following system of ODE:

$$\xi_1\tilde{h}'(\xi_1) + \tilde{h}'(\xi_1)\tilde{u}(\xi_1) + \tilde{h}(\xi_1)\tilde{u}'(\xi_1) = 0 \quad \text{and} \quad \tilde{u}(\xi_1)\tilde{u}'(\xi_1) + \xi_1\tilde{u}'(\xi_1) + \tilde{h}'(\xi_1) = 0 \tag{15}$$

from the solutions to the system of ODE in the similarity form,

$$\tilde{h}(\xi_1) = \frac{1}{9}\xi_1^2, \qquad \tilde{u}(\xi_1) = -\frac{2}{3}\xi_1. \tag{16}$$

These solutions correspond to the similarity solution of the classical case (11). One can write the equation (3) due to the solutions (16) in the variables x, t, and v as

$$f(x,t,v) = \frac{1}{9}(1-t)^{-2}(x-1)^2\delta\left(v + \frac{2}{3}(t-1)^{-1}(x-1)\right). \tag{17}$$

5 Conclusions

In this study, we obtained symmetry groups of Benney equations that are in the form of the system of IDE. As mentioned in the introduction, although the symmetry group analysis has many applications for the ODE and PDE in the literature, there are relatively few applications on symmetry group of IDE since there is no general method for solving determining equations. Several solution methods may be offered for investigating the determining equations for symmetry groups of problems including IDE. In the case of Benney equations investigated in the study by Krasnoslobodtsev [10] and Ibragimov et al. [11], symmetry groups of these equations

were investigated based on the canonical multiplication law and the methods of moments, respectively. In our study, we calculated the symmetry groups of Benney equations by using a different method, explained in the study. After calculations, it was found a five-parameter Lie group of transformations of Benney equations. These results are parallel to the results presented in Ibragimov's study based on using the canonical Lie–Bäcklund operators. These results are parallel to the results presented in Ibragimov's study based on using the canonical Lie–Bäcklund operators. We obtained the general reduced forms of Benney equations by using the characteristics based on symmetry groups. Then we obtained the general reduced forms of Benney equations by using the characteristics based on symmetry groups. One of the important advantages of investigating solutions of Benney equations is based on the fact that one may transform these solutions into solutions of the shallow-water equations and Benney equations derived form the two-dimensional Euler equations for inviscid, incompressible fluid, and from the reduced form of the equation. In the study, we showed a similarity solution for a reduced equation, and proved that each solution obtained from solutions of each reduced form of the systems of IDE can be used to obtain similarity solutions of Benney equations.

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