Differential Invariants of Transformation Groups on the Real Plane

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Using results of A. Gonzalez-Lopez, N. Kamran and P.J. Olver we describe differential invariants and operators of invariant differentiation of transformation groups acting on the real plane.

1 Introduction

Differential invariants emerged as one of the most important tools in investigation of differential equations in the works by S. Lie. In 1884 [3] he proved that any non-singular invariant system of differential equations can be expressed in terms of differential invariants of the corresponding symmetry group. In this paper he also applied differential invariants to integration of ODEs. Differential invariants for all finite-dimensional local transformation groups on a space of two complex variables was described also by S. Lie [4]. A modern treatment of these results was adduced in [6], namely for all non-equivalent realizations of Lie algebras of transformation groups on the complex plane functional bases of differential invariants, operators of invariant differentiation and Lie determinants were constructed. Basic concepts of the theory of differential invariants and review of main results may be found in [7,8].

In this paper for every realization of Lie algebras of transformation groups acting on the real plane from [2] we construct a functional basis of differential invariants and operators of invariant differentiation. For this purpose we find transformations over complex field that reduce realizations from [2] to realizations from [6]. Such nontrivial transformations are adduced in Table 1 (numbers N_1 and N_2 correspond to numeration of real and complex realizations in [2,6]). Using these transformations and results of [6] we completely describe differential invariants of transformation groups on the real plane. This description is presented in the form of Table 2.

Table 1. Transformations of complecificated real realizations to complex ones.

N_1	Transformations of space variables	Transformations of basis elements	N_2
1	$\tilde{x} = x - iy$, $\tilde{y} = x + iy$	$\tilde{e}_1 = \frac{1+i}{2}(e_1 + e_2)$, $\tilde{e}_2 = \frac{1}{c+i}e_3$, $\tilde{e}_3 = \frac{1-i}{2}(e_1 - e_2)$	2.7, $k = 1$
2	$\tilde{x} = x - iy$, $\tilde{y} = \frac{1}{2iy}$	$\tilde{e}_1 = e_1$, $\tilde{e}_2 = e_2$, $\tilde{e}_3 = e_3$	2.2
3	$\tilde{x} = -\frac{1}{ix+y}$, $\tilde{y} = \frac{ix+y}{1+x^2+y^2}$	$\tilde{e}_1 = \frac{1}{2}(ie_2 + e_3)$, $\tilde{e}_2 = ie_1$, $\tilde{e}_3 = \frac{1}{2}(e_3 - ie_2)$	2.2
4	$\tilde{x} = \frac{y-ix}{2}$, $\tilde{y} = -\frac{y+ix}{2}$	$\tilde{e}_1 = ie_1 - e_2$, $\tilde{e}_2 = ie_1 + e_2$, $\tilde{e}_3 = \frac{e_3+ie_4}{2}$, $\tilde{e}_4 = \frac{e_3-ie_4}{2}$	2.9, $k = 1$
7	$\tilde{x} = y + ix$, $\tilde{y} = y - ix$	$\tilde{e}_1 = \frac{e_1+ie_2}{2i}$, $\tilde{e}_2 = \frac{e_3-ie_4}{2}$, $\tilde{e}_3 = \frac{e_6+ie_5}{2}$, $\tilde{e}_4 = \frac{ie_2-e_1}{2i}$, $\tilde{e}_5 = \frac{e_3+ie_4}{2}$, $\tilde{e}_6 = \frac{e_6-ie_5}{2}$	2.4
17	$\tilde{x} = y$, $\tilde{y} = \frac{1}{x-y}$	$\tilde{e}_1 = e_1$, $\tilde{e}_2 = e_2$, $\tilde{e}_3 = e_3$	2.2
18	$\tilde{x} = x$, $\tilde{y} = \frac{1}{y^2}$	$\tilde{e}_1 = e_1$, $\tilde{e}_2 = \frac{1}{2}e_2$, $\tilde{e}_3 = e_3$	2.1
19	$\tilde{x} = x$, $\tilde{y} = \frac{1}{y}$	$\tilde{e}_1 = e_1$, $\tilde{e}_2 = e_2$, $\tilde{e}_3 = -e_3$, $\tilde{e}_4 = e_4$	2.3

Table 2. Differential invariants of transformation groups on the real plane.

N	Realization	Fundamental differential invariants	Operator of invariant differentiation
1	$\partial_x,\, \partial_y,\, (cx+y)\partial_x + (cy-x)\partial_y,\, 0 \le c$	$\frac{y'' e^{-c\arctan y'}}{P^{3/2}}$	$\frac{e^{-c\arctan y'}}{\sqrt{P}} D_x$
2	$\partial_x,\, x\partial_x + y\partial_y,\, (x^2-y^2)\partial_x + 2yx\partial_y$	$\frac{y''y+(y')^2+1}{P^{3/2}}$	$\frac{2y}{\sqrt{P}} D_x$
3	$y\partial_x - x\partial_y,\ (1+x^2-y^2)\partial_x + 2xy\partial_y,$ $2xy\partial_x + (1+y^2-x^2)\partial_y$	$\frac{y''(1+x^2+y^2)}{P^{3/2}} + \frac{2(y-xy')}{\sqrt{P}}$	$\frac{1+x^2+y^2}{\sqrt{P}} D_x$
4	$\partial_x,\, \partial_y,\, x\partial_x + y\partial_y,\, y\partial_x - x\partial_y$	$\frac{y'''P}{(y'')^2} - 3y'$	$\frac{P}{y''} D_x$
5	$\partial_x,\, \partial_y,\, x\partial_x - y\partial_y,\, y\partial_x,\, x\partial_y$	$\frac{3y''y^{\mathrm{IV}}-5(y''')^2}{(y'')^{8/3}}$	$\frac{1}{(y'')^{1/3}} D_x$
6	$\partial_x,\, \partial_y,\, x\partial_x,\, y\partial_y,\, y\partial_x,\, x\partial_y$	$\frac{S_5}{R_4^{3/2}}$	$\frac{y''}{\sqrt{R_4}} D_x$
7	$\partial_x,\ \partial_y,\ x\partial_x + y\partial_y,\ y\partial_x - x\partial_y,$ $(x^2-y^2)\partial_x - 2xy\partial_y,\ 2xy\partial_x - (y^2-x^2)\partial_y$	$\frac{U}{Q^3}$	$\frac{P}{\sqrt{Q}} D_x$
8	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,\ y\partial_x,\ x\partial_y,$ $x^2\partial_x + xy\partial_y,\ xy\partial_x + y^2\partial_y$	$\frac{V_7}{S_5^{8/3}}$	$\frac{y''}{S_5^{1/3}} D_x$
9	∂_x	$F(y)$	D_x
10	$\partial_x,\, x\partial_x$	$F(y),\ \frac{(y')^k}{y^{(\mathrm{k})}}$	D_x^2
11	$\partial_y,\, y\partial_y,\, y^2\partial_y$	$\frac{Q_3}{(y')^2}$	D_x
12	$\partial_x,\, \partial_y,\, x\partial_x + cy\partial_y,\, 0 < \|c\| \le 1$	$c=1,\quad \frac{y'''}{(y'')^2}$ $c \ne 1,\quad y''y'^{\frac{2-c}{c-1}}$	$\frac{1}{y''} D_x$ $(y')^{\frac{1}{c-1}} D_x$
13	$\partial_x,\, \partial_y,\, x\partial_x,\, y\partial_y$	$\frac{y'y'''}{(y'')^2}$	$\frac{y'}{y''} D_x$
14	$\partial_x,\, \partial_y,\, x\partial_x,\, x^2\partial_x$	$\frac{Q_3}{(y')^4}$	$\frac{1}{y'} D_x$
15	$\partial_x,\, \partial_y,\, x\partial_x,\, y\partial_y,\, x^2\partial_x$	$\frac{S_4}{Q_3^{\frac{3}{2}}}$	$\frac{y'}{\sqrt{Q_3}} D_x$
16	$\partial_x,\, \partial_y,\, x\partial_x,\, y\partial_y,\, x^2\partial_x,\, y^2\partial_y$	$\frac{U_5}{Q_3^3}$	$\frac{y'}{\sqrt{Q_3}}$
17	$\partial_x + \partial_y,\, x\partial_x + y\partial_y,\, x^2\partial_x + y^2\partial_y$	$\frac{2y'(1+y')+y''(x-y)}{(y')^{3/2}}$	$\frac{x-y}{\sqrt{y'}} D_x$
18	$\partial_x,\, 2x\partial_x + y\partial_y,\, x^2\partial_x + xy\partial_y$	y^3y''	$y^2 D_x$
19	$\partial_x,\, x\partial_x,\, y\partial_y,\, x^2\partial_x + xy\partial_y$	$\frac{3y'y''+yy'''}{\sqrt{y}(y'')^{3/2}}$	$\sqrt{\frac{y}{y''}} D_x$
20	$\partial_y,\, \xi_1(x)\partial_y,\, \ldots,\, \xi_r(x)\partial_y,\, 1 \le r$	$x,\ \frac{W(y,1,\xi_1,\ldots,\xi_r)}{W(1,\xi_1,\ldots,\xi_r)}$	D_x
21	$\partial_y,\, y\partial_y,\, \xi_1(x)\partial_y,\, \ldots,\, \xi_r(x)\partial_y,\, 1 \le r$	$D_x \ln W(y,\xi_1,\ldots,\xi_r)$	D_x
22	$\partial_x,\, \eta_1(x)\partial_y,\, \ldots,\, \eta_r(x)\partial_y,\, 1 \le r$	$\frac{W(y,\eta_1,\ldots,\eta_r)}{W(\eta_1,\ldots,\eta_r)}$	D_x
23	$\partial_x,\, y\partial_y,\, \eta_1(x)\partial_y,\, \ldots,\, \eta_r(x)\partial_y,\, r \ge 1$	$D_x \ln W(y,\eta_1,\ldots,\eta_r)$	D_x
24	$\partial_x,\ \partial_y,\ x\partial_x + cy\partial_y,$ $x\partial_y,\ldots,x^r\partial_y,\ r \ge 1$	$c \ne r+1,\ (y^{(\mathrm{r+1})})^{\frac{2-c+r}{c-(r+1)}} y^{(\mathrm{r+2})}$ $c = r+1,\ y^{(\mathrm{r+1})},\ \frac{y^{(\mathrm{r+3})}}{(y^{(\mathrm{r+2})})^2}$	$(y^{(\mathrm{r+1})})^{\frac{1}{c-(r+1)}} D_x$ $\frac{1}{y^{(\mathrm{r+2})}} D_x$
25	$\partial_x,\ \partial_y,\ x\partial_y,\ \ldots,\ x^{r-1}\partial_y,$ $x\partial_x + (ry + x^r)\partial_y,\ r \ge 1$	$y^{(\mathrm{r+1})} e^{\frac{y^{(\mathrm{r})}}{r!}}$	$e^{\frac{y^{(\mathrm{r})}}{r!}} D_x$
26	$\partial_x,\ \partial_y,\ x\partial_x,\ x\partial_y,\ y\partial_y,$ $x^2\partial_y,\ \ldots,\ x^r\partial_y,\ r \ge 1$	$\frac{y^{(\mathrm{r+1})}y^{(\mathrm{r+3})}}{(y^{(\mathrm{r+2})})^2}$	$\frac{y^{(\mathrm{r+1})}}{y^{(\mathrm{r+2})}} D_x$
27	$\partial_x,\ \partial_y,\ 2x\partial_x + ry\partial_y,\ x\partial_y,$ $x^2\partial_x + rxy\partial_y,\ x^2\partial_y,\ \ldots,\ x^r\partial_y,\ r \ge 1$	$\frac{Q_{r+3}}{(y^{(\mathrm{r+1})})^{\frac{2(r+4)}{r+2}}}$	$\frac{1}{(y^{(\mathrm{r+1})})^{\frac{2}{r+2}}} D_x$
28	$\partial_x,\ \partial_y,\ x\partial_x,\ x\partial_y,\ y\partial_y,$ $x^2\partial_x + rxy\partial_y,\ x^2\partial_y,\ \ldots,\ x^r\partial_y,\ r \ge 1$	$\frac{S_{r+4}}{Q_{r+3}^{3/2}}$	$\frac{y^{(\mathrm{r+1})}}{\sqrt{Q_{r+3}}} D_x$

In Table 2 $W(f_1,\dots,f_k)$ denotes the Wronskian of the functions $f_1(x)$, …, $f_k(x)$, D_x is the operator of total differentiation. We also use the following notations:

$$S_{k+3}=(k+1)^2\big(y^{(k)}\big)^2y^{(k+3)}-3(k+1)(k+3)y^{(k)}y^{(k+1)}y^{(k+2)}+2(k+2)(k+3)\big(y^{(k+1)}\big)^3,$$
$$Q_{k+2}=(k+1)y^{(k)}y^{(k+2)}-(k+2)\big(y^{(k+1)}\big)^2,\qquad R_4=3y''y^{\rm IV}-5(y''')^2,$$
$$P=1+(y')^2,\qquad Q=y'''P-3y'(y'')^2,$$
$$U=4y^{\rm V}P^3Q+10y^{\rm IV}y''P^3\left(4y'''y'+3(y'')^2\right)-5(y^{\rm IV})^2P^4+40(y''')^2(y'')^2\left((y')^2-2\right)P^2$$
$$-40(y''')^3y'P^3-180y'''y'(y'')^4\left((y')^2-1\right)P^2-(y'')^6\left(45(6(y')^2+1)-135(y')^4\right),$$
$$U_5=(y')^2\left(Q_3D_x^2Q_3-\frac{5}{4}(D_xQ_3)^2\right)+y'y''Q_3D_xQ_3-\left(2y'y'''-(y'')^2\right)Q_3^2,$$
$$V_7=(y'')^2\left(S_5D_x^2S_5-\frac{7}{6}(D_xS_5)^2\right)+y''y'''S_5D_xS_5-\frac{1}{2}\left(9y''y^{\rm IV}-7(y''')^2\right)S_5^2.$$

The presented results may be used for group classification of ODEs of any finite order. So, in the near future we plan to review and to generalize results of group classification of third and fourth ODEs that were obtained in [1, 10]. To investigate differential invariants of transformations groups acting in the spaces of more than two variables in similar way we can use classification of realizations of real low-dimensional Lie algebras that was obtained in [9].

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