# A Conservative Numerical Integration Algorithm for Integrable Henon–Heiles System

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A new conservative numerical integration algorithm for the integrable Henon–Heiles system is presented which conserves *all* of the constants of motion.

## 1 Introduction

The Henon–Heiles system in celestial mechanics is a well-known chaotic Hamiltonian system [4]. Since this two dimensional system has only one constant of motion, it is a non-integrable system. By simple changes of the Hamiltonian we obtain some completely integrable Henon–Heiles type systems. They are completely integrable in the sense of Liouville–Arnold. Namely, they are two-dimensional systems having two constants of motion cf. [8]. To solve numerically the equations of motion, one often tries to discretize them. Even for the integrable cases a good numerical integration scheme will be useful for describing orbits for arbitrary given initial data.

The symplectic integrators [3, 9] and the energy preserving methods [2, 1] are well-known as good numerical integration schemes. The symplectic integrators conserve the symplectic form in the phase space, so that the resulting discrete-time evolution is regarded as a canonical transformation. Though the symplectic integrators do not conserve Hamiltonian and other additional constants of motion, for example, the angular momentum and the Runge–Lenz vector for the Kepler motion, in general, they are widely used in numerical simulation for the solar system. This is because the symplectic integrators give a good approximation of the orbits of the Hamiltonian system in the sense in which they conserve a modified (or approximate) Hamiltonian. However, a non-existence theorem of modified constant of motion besides the modified Hamiltonian is proved recently in [10,11]. The energy preserving methods originated by Greenspan [2] keep the value of the Hamiltonian constant. These methods are based on discrete variational derivatives [1, 5]. Generally, they conserve only energy and do not preserve other additional constants of motion. It has not been known for long time how to design an energy preserving method which conserves all of the additional constants of motion of certain class of integrable Hamiltonian systems. Especially, it is impossible to trace an ellipse of the Kepler motion, since the Runge–Lentz vector is not conserved by any known numerical integrators including the symplectic integrators and the usual energy preserving methods.

In the previous paper [6] the authors have established a new numerical integration algorithm which conserves all of the constants of motion of the two-dimensional Kepler motion including the Runge–Lenz vector. In this paper we consider one of the integrable Henon–Heiles type systems discussed in [8]. We discretize the Henon–Heiles system which preserves all constants of motion as an application of the numerical integration algorithm.

It is also shown in this paper that 1) a time step size, which is larger than those of other known integrators, is easily used, 2) a variable time step is naturally introduced. These basic properties underlie the efficiency of the new numerical integration algorithm.

## 2 Integrable Henon–Heiles system and canonical transformation

The integrable Henon–Heiles system is defined by the Hamiltonian

$$\mathcal{H}_{\mathrm{hh}-1}(p_x, p_y, x, y) = p_x^2 + p_y^2 + x^2 + y^2 + \frac{2}{3}x^3 + 2xy^2 - E_{\mathrm{hh}} \tag{1}$$

on a subset of $\mathbf{R}^4 = \{p_x, p_y, x, y\}$, where $E_{\mathrm{hh}}$ is an arbitrary constant. The time variable is $t$. When the positive sign of $\frac{2}{3}x^3$ in (1) becomes changed to negative, (1) is the Hamiltonian of a chaotic dynamical system in celestial mechanics [4]. The equation of motion for (1) are expressed with

$$\frac{dx}{dt} = 2p_x, \qquad \frac{dy}{dt} = 2p_y, \qquad \frac{dp_x}{dt} = -2x^2 - 2x - 2y^2, \qquad \frac{dp_y}{dt} = -4xy - 2y. \tag{2}$$

After a canonical transformation

$$q_1 = \frac{1}{2}(x+y), \qquad q_2 = \frac{1}{2}(x-y), \qquad p_1 = p_x + p_y, \qquad p_2 = p_x - p_y, \tag{3}$$

the Hamiltonian (1) leads to the Hamiltonian

$$\mathcal{H}_{\mathrm{hh}-2}(p_1, p_2, q_1, q_2) = \frac{1}{2}\left(p_1^2 + p_2^2 + U(q_1) + U(q_2)\right) \tag{4}$$

on a subset of $\mathbf{R}^4 = \{p_1, p_2, q_1, q_2\}$, where the potential functions are

$$U(q_j) = \frac{16}{3}q_j^3 + 4q_j^2 - E_{j,\mathrm{hh}}, \qquad j = 1, 2. \tag{5}$$

The variables $p_k$ and $q_k$ are now separated. Note that the canonical transformation (3) is a mapping from $\mathbf{R}^4$ to $\mathbf{R}^4$. The equations of motion are then

$$\frac{dq_k}{dt} = p_k, \qquad \frac{dp_k}{dt} = -8q_k^2 - 4q_k, \qquad k = 1, 2. \tag{6}$$

They are just the equations of the two-dimensional anharmonic oscillator with the Hamiltonian (4) on $\mathbf{R}^4 = \{p_1, p_2, q_1, q_2\}$ with the time $t$. Let us set the constant $E_{\mathrm{hh}}$ in $\mathcal{H}_{\mathrm{hh}-2}$ such that

$$\mathcal{H}_{\mathrm{hh}-2}(p_1, p_2, q_1, q_2) = 0. \tag{7}$$

Obviously, $\mathcal{H}_{\mathrm{hh}-1}(p_x, p_y, x, y) = 0$ for such $E_{\mathrm{hh}}$. Since $\mathcal{H}_{\mathrm{hh}-2}$ is a constant of motion of (6), the value of the constant $E_{\mathrm{hh}}$ reflects the initial values of the variables $p_k$ and $q_k$. Namely, $E_{\mathrm{hh}}$ is expressed as

$$E_{\mathrm{hh}} = E_{1,\mathrm{hh}} + E_{2,\mathrm{hh}}, \tag{8}$$

where

$$E_{j,\mathrm{hh}} = \frac{1}{2}\left(p_j(0)^2 + U(q_j(0))\right), \qquad j = 1, 2. \tag{9}$$

We see that the canonical transformation (3) generates a transformation that maps the integrable Henon–Heiles system to a system of anharmonic oscillators.

The integrable Henon–Heiles system has an additional constant of motion. This constant is the angular momentum

$$I_{2,\mathrm{hh}-1}(p_x, p_y, x, y) = 2p_x p_y + 4x^2 y + \frac{4}{3}y^3 + 4xy - E_{1,\mathrm{hh}} + E_{2,\mathrm{hh}}. \tag{10}$$

The constants of motion are expressed by using the canonical variables $p_k$ and $q_k$ as follows. The Hamiltonian $\mathcal{H}_{\mathrm{hh}-2}(p_k, q_k)$ is as (4), where $(p_k, q_k) = (p_1, p_2, q_1, q_2)$, for simplicity. Note that (10) is expressed as

$$I_{2,\mathrm{hh}-2} = \frac{1}{2}\left(p_1^2 - p_2^2\right) + \frac{8}{3}\left(q_1^3 - q_2^3\right) + 2\left(q_1^2 - q_2^2\right). \tag{11}$$

## 3 Discrete integrable Henon–Heiles system

The equations of motion of the two-dimensional oscillator (6) are discretized by the energy preserving method (cf. [2, 5]) as follows,

$$\begin{aligned}
&\frac{Q_k^{(j+1)} - Q_k^{(j)}}{t^{(j+1)} - t^{(j)}} = \frac{P_k^{(j)} + P_k^{(j+1)}}{2},\\
&\frac{P_k^{(j+1)} - P_k^{(j)}}{t^{(j+1)} - t^{(j)}} = -\frac{8\left(\left(Q_k^{(j+1)}\right)^2 + Q_k^{(j+1)}Q_k^{(j)} + \left(Q_k^{(j)}\right)^2\right) + 6\left(Q_k^{(j+1)} + Q_k^{(j)}\right)}{3},\\
&t^{(0)} < \cdots < t^{(j-1)} < t^{(j)} < t^{(j+1)} < \cdots, \qquad (12)
\end{aligned}$$

where $t^{(j)}$ is an arbitrarily increasing sequence which indicates a discrete-time, $P_k^{(j)}$, $Q_k^{(j)}$ are the values of the variables $P_k$, $Q_k$ at the time $t^{(j)}$, respectively. Here $P_k$ and $Q_k$ are discrete analogues of the canonical variables $p_k$ and $q_k$, respectively, where we set

$$P_k^{(0)} = p_k(0), \qquad Q_k^{(0)} = q_k(0), \qquad k = 1, 2. \qquad (13)$$

On the orbit of the two-dimensional discrete-time harmonic oscillator (12) the Hamiltonian (4) is kept constant for any $t^{(j)}$, namely,

$$\mathcal{H}_{\mathrm{hh}-2}\big(P_k^{(j+1)}, Q_k^{(j+1)}\big) = \mathcal{H}_{\mathrm{hh}-2}\big(P_k^{(j)}, Q_k^{(j)}\big), \qquad j = 0, 1, \ldots, \qquad (14)$$

where $\big(P_k^{(j)}, Q_k^{(j)}\big) = \big(P_1^{(j)}, P_2^{(j)}, Q_1^{(j)}, Q_2^{(j)}\big)$. This is a direct consequence of the energy preserving method. Since $\mathcal{H}_{\mathrm{hh}-2}(p_k, q_k) = 0$, $P_k^{(0)} = p_k(0)$ and $Q_k^{(0)} = q_k(0)$, we obtain

$$\begin{aligned}
&\mathcal{H}_{\mathrm{hh}-2}\big(P_k^{(j)}, Q_k^{(j)}\big) = 0, \quad j = 0, 1, \cdots,\\
&E_{k,\mathrm{hh}} = \frac{1}{2}\big(P_k^{(0)}\big)^2 + \frac{8}{3}\big(Q_k^{(0)}\big)^3 + 2\big(Q_k^{(0)}\big)^2, \qquad k = 1, 2. \qquad (15)
\end{aligned}$$

Through the inverse of the canonical transformation (3) the discrete-time system (12) is converted to

$$\begin{aligned}
&\frac{X^{(j+1)} - X^{(j)}}{t^{(j+1)} - t^{(j)}} = P_X^{(j+1)} + P_X^{(j)}, \qquad \frac{Y^{(j+1)} - Y^{(j)}}{t^{(j+1)} - t^{(j)}} = P_Y^{(j+1)} + P_Y^{(j)},\\
&\frac{P_X^{(j+1)} - P_X^{(j)}}{t^{(j+1)} - t^{(j)}} = -\frac{2}{3}\Big(\Big(\big(X^{(j+1)}\big)^2 + X^{(j+1)}X^{(j)} + \big(X^{(j)}\big)^2\Big)\\
&\qquad + \Big(\big(Y^{(j+1)}\big)^2 + Y^{(j+1)}Y^{(j)} + \big(Y^{(j)}\big)^2\Big)\Big) - \big(X^{(j+1)} + X^{(j)}\big),\\
&\frac{P_Y^{(j+1)} - P_Y^{(j)}}{t^{(j+1)} - t^{(j)}} = -\frac{2}{3}\Big(2X^{(j+1)}Y^{(j+1)} + 2X^{(j)}Y^{(j)}\\
&\qquad + X^{(j+1)}Y^{(j)} + X^{(j)}Y^{(j+1)}\Big) - \big(Y^{(j+1)} + Y^{(j)}\big). \qquad (16)
\end{aligned}$$

Equations (16) preserve the value of $\mathcal{H}_{\mathrm{hh}-1}\big(P_X^{(j)}, P_Y^{(j)}, X_1^{(j)}, Y^{(j)}\big)$ as zero. We call (16) the *discrete integrable Henon–Heiles system.*

## 4 Discrete integrable Henon–Heiles system as exactly conservative integrator

In the previous section the Hamiltonian $\mathcal{H}_{\mathrm{hh}-1}(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)})$ is shown to be constant under the time evolution of the discrete integrable Henon–Heiles system (16). The other constant of motion (10) of the continuous-time integrable Henon–Heiles is also conserved. Now we are in a position to state the main result.

**Theorem 1.** *The integrable Henon–Heiles system* (16) *has the following two constants of motion. The first is the Hamiltonian*

$$\mathcal{H}_{\mathrm{d-hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big) = \mathcal{H}_{\mathrm{hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big). \tag{17}$$

*The second constant is a discrete analogue of the angular momentum* (10)

$$\begin{aligned} &I_{2,\mathrm{d-hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big) \\ &\quad = 2P_X^{(j)}P_Y^{(j)} + 4\big(X^{(j)}\big)^2 Y^{(j)} + \frac{4}{3}\big(Y^{(j)}\big)^3 + 4X^{(j)}Y^{(j)} - E_{1,\mathrm{hh}} + E_{2,\mathrm{hh}}. \end{aligned} \tag{18}$$

The proof follows by a direct calculation with an alternative expression of the discrete integrable Henon–Heiles system (16). We here give an important remark.

**Remark 1.** The form of the constants (17) and (18) of motion is consistent with that of the continuous-time integrable Henon–Heiles system. Namely,

$$\begin{aligned} &\mathcal{H}_{\mathrm{d-hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big) = \mathcal{H}_{\mathrm{hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big), \\ &I_{2,\mathrm{d-hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big) = I_{2,\mathrm{hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big). \end{aligned} \tag{19}$$

Finally in this section we give a numerical example for the discrete integrable Henon–Heiles system. Fig. 1[1] indicates the orbits of the symplectic Euler scheme with step-size $t^{(j+1)} - t^{(j)} = dt = 0.1,\ 0.2,\ 0.4$, the initial value is $X^{(0)} = 0.0$, $Y^{(0)} = 0.0$, $P_X^{(0)} = 0.35$, $P_Y^{(0)} = -0.15$. Especially, the orbit with step-size $t^{(j+1)} - t^{(j)} = dt = 0.4$ is quite different from that of the continuous-time integrable Henon–Heiles system. In contract to the orbits given by the symplectic Euler scheme, the discrete integrable Henon–Heiles system with the constant discrete step-size $t^{(j+1)} - t^{(j)} = \delta = 0.1,\ 0.2,\ 0.4$ draws orbits which give a better approximation than those of the symplectic Euler scheme in Fig. 2. Moreover, the discrete integrable Henon–Heiles system with variable step-size gives the almost same orbit as the continuous-time integrable Henon–Heiles system has. Fig. 3 displays the orbit of the discrete integrable Henon–Heiles system with a step-size changing periodically from 0.1 to 0.2.

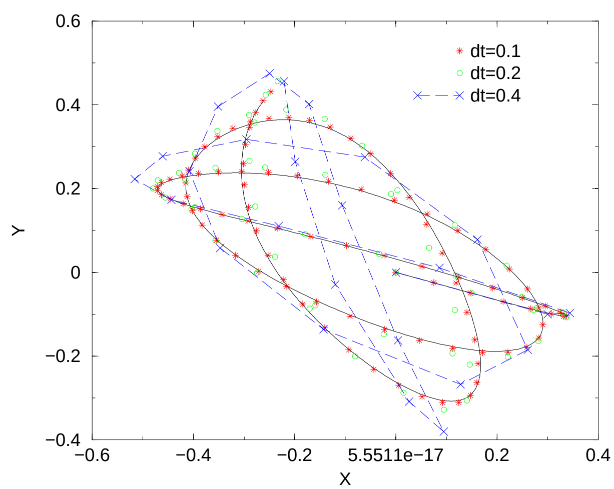


**Figure 1.** Symplectic Euler scheme $\delta = 0.1,\ 0.2,\ 0.4$.

[1]Figures in colour will be available only in electronic version.

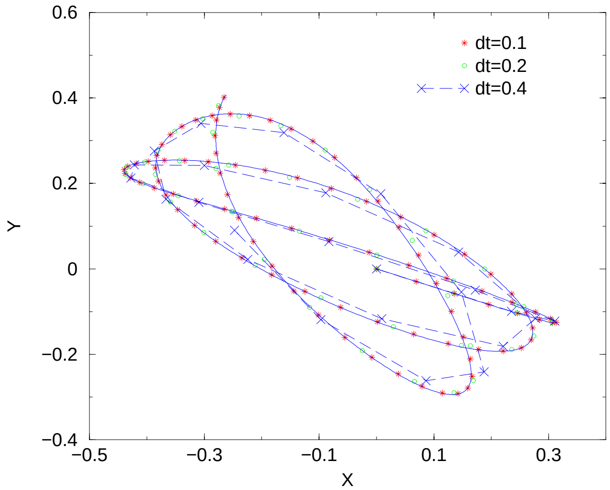


**Figure 2.** Discrete integrable Henon–Heiles $\delta = 0.1,\ 0.2,\ 0.4$.

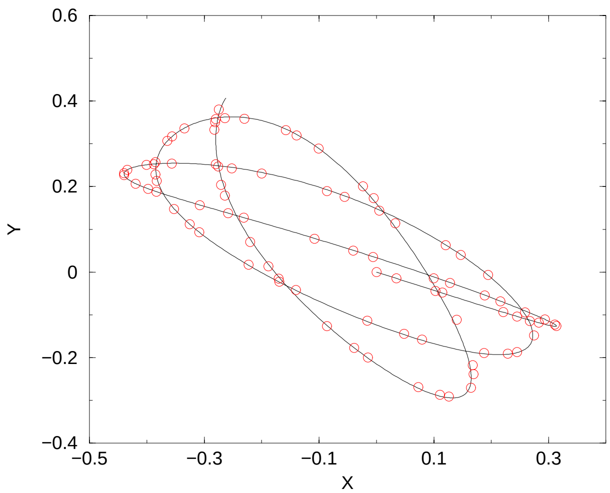


**Figure 3.** Discrete Integrable Henon–Heiles with a variable step-size.

# 5 Conclusion

In this paper a new numerical integration algorithm for the integrable Henon–Heiles system is designed with the help of canonical transformation. All of the constants of motion are then exactly conserved. This property is rather different from the known numerical integrators of the integrable Henon–Heiles system. Other types of the integrable Henon–Heiles systems [8] can be discretized in the same method.

There are some additional advantages of the integrable Henon–Heiles system. As is observed in numerical examples, the orbit of the discrete integrable Henon–Heiles gives a better approximation than the symplectic Euler scheme. Secondly, a variable step time is naturally assigned to the discrete integrable Henon–Heiles system. Using the new stable numerical integration algorithm, the authors wish to discretize some other integrable Hamiltonian systems besides the integrable Henon–Heiles system and the two-dimensional Kepler motion.

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