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On the Algebra of Unharmonic Quantum Oscillator

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In present work we consider C^*-algebras $C^*(\mathcal{A}_f)$ associated with simple unimodal non-bijective dynamical system $(f, \mathbb{R})$ with special requirements. In the case when f is polynomial, $\mathcal{A}_f = \mathbb{C}\langle X, X^* \mid XX^* = f(X^*X)\rangle$ and $C^*(\mathcal{A}_f)$ is its enveloping C^*-algebra. As typical examples we consider one-parameter family $f_\mu(x) = \mu x(1-x)$ and two-parameter family called Unharmonic Quantum Oscillator $f_{p,q}(x) = 1 + px - qx^2$. The crossed product structure of $C^*(\mathcal{A}_f)$ is investigated. As a consequence we describe complete isomorphism invariant in terms of corresponding dynamical systems.

1 Introduction

C^*-algebras associated with dynamical systems arise naturally in pure mathematics as well as in applications to physics (see [1] and bibliography for more details) in particular to quantum optics (see [4]). For example Heisenberg algebra generated by operator X such that $XX^* - X^*X = \hbar I$ associated with linear dynamical system $x \to \hbar - x$ on $\mathbb{R}$, q-CCR algebra also associated with linear dynamics $x \to \hbar - qx$. More complicated dynamics appear in algebra of Quantum unit Disk (see [5]), more precisely this algebra is associated with one dimensional dynamical system $x \to \frac{(q+\mu)x+1-q-\mu}{\mu x+1-\mu}$ where μ and q are parameters of deformation. In this article we are concerned with non-linear deformation of q-CCR which we call algebra of Unharmonic Quantum Oscillator. It is given by generator X obeying the following relation $XX^* = \hbar + pX^*X - q(X^*X)^2$ where $q > 0,\ p > 0$.

The representation theory of C^*-algebras given by "dynamical relations" is extensively studied and well known (see [1]). Its connection with many concurrent approaches to associate C^*-algebra to a dynamical systems, for example groupoid approach and cross-product by partial actions of a group or semigroup (see [13, 12]) is very intriguing. In the paper we use recent work [3] to establish connection of algebra of unharmonic quantum oscillator with cross product like algebras.

2 Cross-product like structure of C^*-algebras associated with dynamical systems

Here we present some recent results on cross-product like structure of C^*-algebras associated with dynamical systems developed in [3] which are necessary for the last section of the paper.

Let $\mathcal{A}$ be some unital C^*-subalgebra of $B(H)$ and $U \in B(H)$ be a partial isometry such that the mapping $\mathcal{A} \ni a \mapsto UaU^*$ is an endomorphism of $\mathcal{A}$. If in addition pair $\mathcal{A}$ and U satisfies $Ua = UaU^*U$ and $U^*aU \in \mathcal{A}$ for all $a \in \mathcal{A}$ then $\mathcal{A}$ is called coefficient algebra for the C^*-subalgebra $\mathcal{B}$ generated by $\mathcal{A}$ and U.

Let us fix some notations: $d(x) = UxU^*$, $d_*(x) = U^*xU$. Then the condition that $\mathcal{A}$ is an algebra of coefficients for $\mathcal{B}$ will be reformulated in the following form

$$Ua = d(a)U^*, \quad a \in \mathcal{A}, \qquad d : \mathcal{A} \to \mathcal{A}, \qquad d_* : \mathcal{A} \to \mathcal{A}.$$

If $\mathcal{A}$ is an algebra of coefficients for $\mathcal{B}$ then $\mathcal{B}$ is a uniform closure of the finite combinations of the form

$$x = U^{*N} a_{\overline{N}} + \cdots + U^* a_{\overline{1}} + a_0 + a_1 U + \cdots + a_N U^N. \tag{1}$$

Where $a_{\overline{j}}$, $a_j \in \mathcal{A}$ and satisfy the following property for all k:

$$a_k U^k U^{*k} = a_k, \qquad a_{\overline{k}} U^k U^{*k} = a_{\overline{k}}.$$

In order to guarantee the very important property of uniqueness of representation in the form (1) one needs to impose the following $(*)$-property for all x of the form (1):

$$||a_0|| \leq ||x||. \tag{$*$}$$

We will need the following central result from [3, 2.13]:

Theorem 1. *Let $\mathcal{A}_j$ be an algebra of coefficients for $\mathcal{B}_j$ generated by $\mathcal{A}_j$ and U_j where $j = 1, 2$. Assume that for both algebras property $(*)$ is satisfied. And assume that a mapping $\vartheta : \mathcal{A}_1 \to \mathcal{A}_2$ is an isomorphism such that $\vartheta \circ d_1 = d_2 \circ \vartheta$. Then the mapping $\Psi(x) = \vartheta(x)$ for $x \in \mathcal{A}_1$ and $\Psi(U_1) = U_2$ can be extended to isomorphism of C^*-algebras $\mathcal{B}_1$ and $\mathcal{B}_2$.*

In order to construct an algebra of coefficients we need an additional piece of notations: if $X \subset \mathcal{B}$ then $E(X)$ will denote the C^*-algebra generated by $\{X, d(X), d^2(X), \ldots, d^n(x), \ldots\}$ and analogously $E_*(X)$ will denote the C^*-algebra generated by $\{X, d_*(X), d_*^2(X), \ldots, d_*^n(x), \ldots\}$. The following theorem (see [3, Theorem 3.11]) gives conditions for existence of algebra of coefficients:

Theorem 2. *Let $d : \mathcal{A}_0 \to B(H)$ is a morphism.*

1. *The following statements are equivalent:*
   a) *There exists an algebra of coefficients $\mathcal{A} \supseteq \mathcal{A}_0$.*
   b) *$U^*U \in \cap_{n=0}^{\infty} d^n(\mathcal{A}_0)'$.*
2. *If the above condition is satisfied then $E_*(E(\mathcal{A}_0))$ is the minimal algebra of coefficients containing $\mathcal{A}_0$ and d is an endomorphism of $E_*(E(\mathcal{A}_0))$. Moreover, each element $\beta \in E_*(E(\mathcal{A}_0))$ can be written as*

$$\beta = \alpha_0 + d_8(\alpha_1) + \cdots + d_*^N(\alpha_N).$$

$U^{*k}U^k$ and U^kU^{*k} are the decreasing sequences of commuting projections.

3 One-dimensional dynamical systems

For convenience of the reader we repeat the relevant material from [2, 1] without proofs, thus making our exposition self-contained. By the dynamical system we mean a continuous map $f : \mathbb{R} \to \mathbb{R}$ or $f : I \to I$, where $I \subset \mathbb{R}$ is a closed bounded interval. By the orbit of dynamical system $(f, \mathbb{R})$ we mean a sequence $\delta = (x_k)_{k \in P}$, where P is one of the sets $\mathbb{Z}$, $\mathbb{N}$ or $-\mathbb{N} = \{-1, -2, \ldots\}$ such that $f(x_k) = x_{k+1}$. But sometimes we will consider orbit as the set $\{x_k |\ k \in P\}$. The set of all orbits will be denoted by $\mathrm{Orb}\,(f)$. For $x \in \mathbb{R}$ denote by $\mathcal{O}_+(x)$ the forward orbit, i.e. $(f^k(x))_{k \geq 0}$. For every orbit $\delta \in \mathrm{Orb}\,(f)$ define $\omega(\delta)$ be the set of accumulation points of forward half-orbit and $\alpha(\delta)$ be the set of accumulation points of backward half-orbit.

By the positive orbit of a dynamical system $(f(), \mathbb{R})$ we mean a sequence $\omega = (x_k)_{k \in \mathbb{Z}}$ such that $f(x_k) = x_{k+1}$ and $x_k > 0$ for all integer k. Unilateral positive orbit is a sequence $\omega = (x_k)_{k \in \mathbb{N}}$ (Fock-orbit) such that $x_1 = 0$ and $f(x_k) = x_{k+1}$, $x_k > 0$ for $k > 1$ or $\omega = (x_{-k})_{k \in \mathbb{N}}$ (anti-Fock-orbit) such that $x_{-1} = 0$ and $f(x_k) = x_{k+1}$, $x_k > 0$ for $k < -1$. Define $\mathrm{Orb}_+(f)$

be the set of all positive non-cyclic orbits. Note that $\omega(\delta) = \varnothing$ for any anti-Fock orbit δ and $\alpha(\delta_1) = \varnothing$ for the Fock orbit δ_1.

Cycle $\beta = \{\beta_1, \ldots, \beta_m\}$ is called attractive if there is a neighborhood U of β such that $f(U) \subseteq U$ and $\cap_{i>0} f^i(U) = \beta$.

Point $x \in \mathbb{R}$ is called non-wandering if for every its neighborhood U there exists a positive integer m such that $f^m(U) \cap U \neq \varnothing$.

Since we will consider only bounded from above functions f and positive orbits we can always consider our dynamical system defined on a closed interval $[0, \sup f]$.

In this article we will deal with simple dynamical system, which possesses one of the equivalent properties listed in the following theorem (see [2, Theorem 3.14]):

Theorem 3. *Let $(f(), I)$ be continuous dynamical system, ($I \subset \mathbb{R}$ is closed bounded interval). The following conditions are equivalent:*

1. *For every $x \in I$ $\omega(x) = \omega(\mathcal{O}_+(x))$ is cycle.*
2. $\mathrm{Per}\,(f)$ *is closed.*
3. *Every non-wandering point is periodic.*

f is called partially monotone if I decomposes into a finite union of sub intervals, on which f is monotone.

For a simple dynamical system (f, I) for some positive integer m the relation $\mathrm{Fix}\left(f^{2^{m+1}}\right) = \mathrm{Fix}\left(f^{2^m}\right)$ holds (see [1]).

The class of such dynamical system is denoted by $\mathcal{F}_{2^m}$. Let us note that when $\mathrm{Per}\,(f)$ is closed [2, Theorem 3.12] implies that the length of every cycle is a power of 2 and They're no homoclinical orbits (i.e. orbit δ such that $\alpha(\delta) = \omega(\delta)$ is a cycle).

We will need the following lemma (see [10]):

Lemma 1. *Let $(f, \mathbb{R})$ be dynamical system with bounded from above f such that $(f(\cdot), [0, \sup f])$ is simple d.s. And let the set of periodic points which are not the points of attractive cycles, i.e. the set $[0, \sup f] \cap \mathrm{Per}\,(f) \setminus \cup_{\beta \text{ is attractive cycle}} \beta$ be finite then for every orbit $\delta \in \mathrm{Orb}_+(f)$ the α-boundary $\alpha(\delta)$ is cycle, which is not attractive.*

4 Simple unimodal mapping

Definition 1. Let $f \in C^0(I, I)$ where $I = [0, 1]$. Then f is called unimodal mapping if it satisfies the following conditions:

1. $f(0) = f(1) = 0$.
2. There is unique extreme point $c \in \mathrm{int}\, I$ and f is monotonously increasing on $[0, c]$ and is monotonously decreasing on $[c, 1]$.

Definition 2. Let dynamical system (f, I) be as in theorem 4 with minimal possible n.

1. Then $B_0 = s_0$ and $B_1 = s_1$ are two one-dimensional cycles. $B_{2^k} = \{\beta_1, \beta_2, \ldots, \beta_{2^k}\}$ denote the unique cycle of period 2^k where $\beta_i < \beta_j$ whereas $i < j$. $B_{2^k} = B^-_{2^k} \dot{\cup} B^+_{2^k}$ where $B^-_{2^k} = \{\beta_1, \ldots, \beta_{2^{k-1}}\}$ and $B^-_{2^k} = \{\beta_{2^{k-1}}, \ldots, \beta_{2^k}\}$. Denote by $B_{2^k}(f^2)$ the cycle of period 2^k of dynamical system (f^2, I_2).
2. We will say that orbit $\delta = (x_k)_{k \in \mathbb{Z}}$ is glued to point β_i of cycle B_{2^k} if there exists integer k_0 such that $x_{k_0} = \beta_i$ and $x_k \notin B_{2^k}$ for all $k < k_0$. An orbit is glued to cycle if it is glued to some point of this cycle.
3. We will say that an orbit is degenerate if it is glued to a cycle of period less then 2^n.

Definition 3. Let $\beta_i \in B_{2^m}$ and denote $D^{\beta_i}_{B_{2^k}} = \{\delta \in P_{B_{2^k}} | \delta \text{ is glued to } \beta_i\}$. Denote by $D^{\beta_i}_{B_{2^k}}(f^2)$ the set $D^{\beta_j}_{B_{2^{k-1}}(f^2)}$ where $j = i - 2^{m-1}$ and $\beta_j \in B_{2^{k-1}}(f^2)$.

In the following theorem from [11] an analog of measurable section for dynamical system has been constructed.

Theorem 4. *Let (f, I) be $\mathcal{F}_{2^n}$ dynamical system with unimodal mapping f, which has only two fix points $s_0 = 0$, $0 < s_1 < 1$, and assume that for every $m \le n$ there is only one cycle of period 2^m which is repellent for $m < n$ and attractive for $m = n$. Define $P_B = \{\delta \mid \delta \in \mathrm{Orb}_+(f), \alpha(\delta) = B\}$ for every cycle B of period $m < n$. Then*

1. $\mathrm{Orb}_+(f) = \dot\cup_B P_B$, *where union is taken over all repellent cycles.*
2. *for each B there is $I_B = [t_1, t_2)$ and one-to-one mapping $\phi : I_B \to P_B$ such that $t \in \phi(t)$ for every $t \in I_B$. Moreover I_B can be chosen to lie in arbitrary neighborhood of B.*
3. $I_{B_1} \cap I_{B_2} = \varnothing$ *for $B_1 \ne B_2$.*

Corollary 1. *Mapping $\cup_\beta I_\beta \ni t \to \delta_t$ constructed in the proof is bijective correspondence between n-copies of $[0, 1)$ and the set of non-cyclic positive orbits.*

5 Enveloping C^*-algebra

By $C^*(A_f)$ we mean a C^*-algebra obtained from free $*$-algebra $\mathcal{F}(X, X^*)$ generated by X with sub-norm $\|b\| = \sup_\pi \|\pi(b)\|$ where supremum is taken over all $\pi \in \mathrm{Rep}\,(\mathcal{F}(X, X^*))$ such that $\pi(XX^*) = f(\pi(X^*X))$ by standard factorization and completion procedure.

As shown in (see [1]) there is a bijective correspondence between representations of C^*-algebras

$$A = C^*\langle X, X^* \mid XX^* = f(X^*X)\rangle$$

with certain orbits of dynamical systems $(f, \mathbb{R}_+)$. In particular, if f partially monotone continuous map and $(f, \mathbb{R})$ is $\mathcal{F}_{2^m}$ dynamical system. Then every positive non-cyclic orbit $\omega(x_k)_{k\in Z}$ corresponds to an irreducible representation π_ω in Hilbert space $l_2(Z)$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k \in Z$ and $X = UC$ is a polar decomposition. For the Fock and anti-Fock representations the similar formulae hold with the exception that space is $l_2(N)(l_2(-N))$ and $Ue_1 = 0$ for the Fock representation. To cyclic positive orbit $\omega = (x_k)_{k\in N}$ of length m there corresponds a family of m-dimensional irreducible representation $\pi_{\omega,\phi}$ in Hilbert space $l_2(\{1, \dots, m\})$ given by the formulae: $Ue_0 = e^{i\phi}e_{m-1}$, $Ue_k = e_{k-1}$ $Ce_k = \sqrt{x_k}e_k$ for $k = 1, \dots, m$; $0 \le \phi \le 2\pi$ and $X = UC$

Let f be bounded from above Hermitian polynomial (hence f is always partially monotone and continuous). Let $A_f = \mathbb{C}\langle X, X^* \mid XX^* = f(X^*X)\rangle$ be $*$-algebra given by generators and relations which has at least one representation. Let $C = \sup f$. Then for any representation π of $*$-algebra A_f we have $\|X\| \le \sqrt{C}$. Thus there is (exists) enveloping C^*-algebra, which we denote by $C^*(A_f)$. Let us note that by Theorem 3.3 [2] for $f \in C^1(I, I)$ simplicity of dynamical system is equivalent to $(f, I) \in \mathcal{F}_{2^m}$ for some integer m.

6 Description of the dual space of $C^*(\mathcal{A}_f)$

Let A be C^*-algebra by its spectrum (sometimes called dual space), denoted by $\hat{A}$ we understand the set of unitary equivalence classes in the set $\mathrm{Irr}\,(A)$ of irreducible representations of A with the Jacobson topology (see [7, Chapter 3] about several equivalent definitions). The closure of the set $S \subseteq \hat{A}$ is $[S] = \{\pi \in \hat{A} \mid \mathrm{Ker}\,\pi \supset \cap_{\rho\in S} \mathrm{Ker}\,\rho\}$ or equivalently $[S] = \{\pi \in \hat{A} \mid$ for all $y \in A$ $\|\pi(y)\| \le \sup_{\rho\in S} \|\rho(y)\|\}$ obviously it is enough to verify last inequality only for elements of a dense subspace of A.

In the following and consequent theorems, in case f is not a polynomial, by $C^*(A_f)$ we mean a C^*-algebra obtained from free $*$-algebra $\mathcal{F}(X, X^*)$ generated by X with prenorm $\|b\| =$

$\sup_\pi \|\pi(b)\|$ where supremum is taken over all $\pi \in \mathrm{Rep}\,(\mathcal{F}(X, X^*))$ such that $\pi(XX^*) = f(\pi(X^*X))$ by standard factorization and completion procedure. This C^*-algebra has obvious universal properties similar to those in case of polynomial map f. Theorem 4 describes the set $\mathrm{Orb}_+(f)$ but in order to describe spectrum of $C^*(\mathcal{A}_f)$ we need finer description. The reason is that some orbit $\delta \in P_\beta$ may be eventually periodic, hence $\omega(\delta)$ could be a cycle of length 2^m for $m < n$ and so $C^*(\pi_\delta)$ would not be isomorphic to $C^*(\pi_\gamma)$ for non-eventually periodic ('generic') orbit γ.

Let us give some definitions.

Definition 4. 1. Let $\delta = (x_k)_{k\in\mathbb{Z}} \in P_{B_{2^k}}$ where $k \geq 0$ be such that $x_0 \in I_2$ then $r(\delta) = (x_{2k})_{k\in\mathbb{Z}}$ is an orbit of (f^2, I_2). If $\delta = (y_k)_{k\in\mathbb{Z}}$ is in $\mathrm{Orb}_+(f^2, I_2)$ then $r^{-1}(\delta)$ where $(r^{-1}(\delta))_{2k} = y_k$, $(r^{-1}(\delta))_{2k+1} = f(y_k)$ is an orbit of (f, I), moreover r^{-1} is inverse to r.

2. Let $x \in [0, M]$ define $\mu_-(x) = (y_k)_{k\in\mathbb{Z}}$ be in P_{B_0} where $y_k = f^k(x)$ for $k \geq 0$ and $y_{-k} = f_-^{-1}(y_{-k+1})$ for $k > 0$. If $x \in [s_1, M]$ define $\mu_+(x) = (y_k)_{k\in\mathbb{Z}}$ be in P_{B_0} where $y_k = f^k(x)$ for $k \geq 0$ and $y_{-1} = f_+^{-1}(x)$ and $y_{-k} = f_-^{-1}(y_{-k+1})$ for $k > 1$.

Denote $R_{B_{2^k}} = P_{B_{2^k}} \setminus \cup_{k<m<n} D_{B_{2^k}}^{B_{2^m}}$.

Let H be Hilbert space with orthonormal basis $(e_k)_{k\in\mathbb{Z}}$. Let U be unitary operator defined by $Ue_k = e_{k+1}$. For every orbit $\delta = (x_k)_{k\in\mathbb{Z}} \in Orb_+(f)$ there is repellent cycle B such that $\delta \in P_B$ further on we will always assume that $x_0 \in I_B$. Let us define operator C_δ via the rule $C_\delta e_k = x_k e_k$. Let Z denote the set of non-periodic orbit. Define $(\Psi(X))(\delta) = U\sqrt{C_\delta}$ and extend it to $C^*(\mathcal{A}_f)$. We have presentation $\Psi : C^*(\mathcal{A}_f) \to B(H)^Z$ of elements of enveloping algebra as a operator-valid functions on Z. Later on we will see that if Z endowed with topology induced from dual space, $\hat{C}^*(\mathcal{A}_f)$, and R is a subspace of non-degenerate orbits then for all $y \in C^*(\mathcal{A}_f)$ $\Psi(y)$ is continuous on R in norm topology on $B(H)$ and continuous on Z in strong topology.

In the following theorem we denote by $[X]$ the closure of X in the topology of $\hat{C}^*(\mathcal{A}_f)$ where subset $X \subset \mathrm{Orb}_+(f)$ is identified with the corresponding set of irreducible representations. If $Y \subset \mathbb{R}$ then $\overline{Y}$ denote closure in topology of $\mathbb{R}$. The set of cyclic orbits is $\mathrm{Per}(f)/_\sim$ where $x \sim y$ iff x and y belong to the same orbit. The following theorem from [11] gives the complete description of the dual space.

Theorem 5. *Let dynamical system (f, I) be as in Theorem 4 with minimal possible n. The dual space (spectrum) of $C^*(\mathcal{A}_f)$ is homeomorphic to $\mathrm{Orb}_+(f) \sqcup_\theta (\mathrm{Per}\,(f)/_\sim \times \mathbb{T})$ where $\theta : \mathrm{Per}\,(f)/_\sim \times \mathbb{T} \to \mathrm{Orb}_+(f)$ via the rule $\theta((x, \phi))$ is the cyclic orbit containing x. Topology on $\mathrm{Orb}_+(f)$ is given by the following family of closed sets $\{\overline{\Sigma} = \{\gamma \in \mathrm{Orb}_+(f) \,|\, \gamma \in \overline{\cup_{\delta\in\Sigma}\delta}\}; \Sigma \subseteq \mathrm{Orb}_+(f)\}$ and the space $\mathrm{Orb}_+(f) \bigsqcup_\theta (\mathrm{Per}\,(f)/_\sim \times \mathbb{T})$ is factor-set of disjoint union $\mathrm{Orb}_+(f) \bigsqcup (\mathrm{Per}\,(f)/_\sim \times \mathbb{T})$ with equivalence relation which identifies cyclic orbit containing x with $(x, 1) \in \mathrm{Per}\,(f)/_\sim \times \mathbb{T}$ where topology is given by the following family of closed sets: S is closed in $\mathrm{Per}\,(f)/_\sim \times \mathbb{T}$ and $\overline{\Sigma} \cup \theta^{-1}(\overline{\Sigma})$ where $\Sigma \subset \mathrm{Orb}_+(f)$. Moreover it is determined up to homeomorphism by integer n.*

Corollary 2. *Representation $\oplus_{\delta\in R_{B_0}} \pi_\delta$ of $C^*(\mathcal{A}_f)$ is faithful. Let $\delta \in R_{B_0}$. Then $X = \overline{\{x | x \in \delta\}}$ is a compact subspace in $\mathbb{R}$ and R_{B_0} is locally compact space. $C^*(\mathcal{A}_f)$ is a C^*-subalgebra in the C^*-algebra $C(R_{B_0}, \mathbb{Z} \times_\psi C(X))$ of all continuous mappings from R_{B_0} to cross-product C^*-algebra $\mathbb{Z} \times_\psi C(X)$.*

Note that in case $n = 0$ there systems that are not orbit-equivalent but as we saw they have isomorphic enveloping C^*-algebras.

Before considering examples let us make some remarks

Remark 1. Then class of $\mathcal{F}_{2^n}$ unimodal dynamical systems with negative Swartzian, i.e. $\mathcal{F}_{2^n} \cap SU$ gives examples of d.s. satisfying conditions imposed in this section.

Now let us consider some examples. Consider the family of quadratic maps $f_\mu(x) = \mu x(1-x)$ where $\mu > 0$. It is known that $f_\mu \in SU$. Then for $\mu < \mu^* \approx 3.57$ dynamical system $(f_\mu, \mathbb{R})$ is $\mathcal{F}_{2^n}$ for some n. Denote by $\mu_n > 0$ the greatest value for which $(f_\mu, \mathbb{R})$ is $\mathcal{F}_{2^n}$. Then for $0 < \mu \le \mu_1$ dynamical system $(f_\mu, \mathbb{R})$ has two stable points and has no other cycles, for $\mu_1 < \mu \le \mu_2$ dynamical system $(f_\mu, \mathbb{R})$ has two stable points and one cycle of period 2, for $\mu_n < \mu \le \mu_{n+1}$ dynamical system $(f_\mu, \mathbb{R})$ has two stable points and one repellent cycle of period 2^m for $m < n+1$ one attractive cycle of period 2^{n+1} and no other cycles. The following proposition is a consequence of more general result from [14].

Proposition 1. *There is $\nu_{n+1} : \mu_n < \nu_{n+1} \le \mu_{n+1}$ such that dynamical systems (f_{μ_1}, I) and (f_{μ_2}, I) are conjugate iff μ_1 and μ_2 belongs to the same set (μ_n, ν_{n+1}) or $\{\nu_{n+1}\}$ or $(\nu_{n+1}, \mu_{n+1}]$.*

7 Cross-product structure of $C^*(\mathcal{A}_f)$

Let X be a compact topological Hausdorff space and (f, X) a continuous dynamical system. Let $\Xi = \{\delta \subseteq X | \delta -$ minimal invariant subset such that for all $x \in \delta$ exists $y \in \delta : \ f(y) = x\}$. For each non eventually periodic orbit $\delta \in \Xi$ for all $x \in \delta$ exists unique $y \in \delta$ such that $f(y) = x$. Denote such y by $f_\delta^{(-1)}(x)$. And for arbitrary integer $l > 0$ define $f_\delta^{(-l)}(x)$ recursively as $f_\delta^{(-l)}(x) = f_\delta^{(-1)}\big(f_\delta^{(-l+1)}(x)\big)$. If δ is eventually periodic then $f_\delta^{(-1)}(x)$ is defined as such an $y \in \delta$ that $f(y) = x$ and y is non-cyclic. Define topological space $\Omega = \{(\delta, y) \,|\, \delta \in \Xi,\ y \in \delta\}$. The family of sets of the form $B_{n,U}(\delta, y) = \{(\tau, z) \in \Omega \,|\, f_\tau^n(z) \in U\}$ where $n \in \mathbb{Z}$, U is an open neighborhood of $f_\delta^n(y)$ constitute a basis of open neighborhoods of $(\delta, y) \in \Omega$.

In terms of convergent sequences the topology on Ω could be defined as follows (δ_k, y_k) converges to (δ, y) if and only if for any integer $n \in \mathbb{Z}$

$$\lim_{k\to+\infty} f_{\delta_k}^{(n)}(y_k) = f_\delta^{(n)}(y).$$

Proposition 2. *Ω is a compact Hausdorff space. And mapping $\sigma : \Omega \to \Omega$, $\sigma((\delta, y)) = (\delta, f(y))$ is a homeomorphism of Ω.*

Theorem 6. *Let (f, I) be $\mathcal{F}_{2^n}$ and (g, I) be $\mathcal{F}_{2^k}$ dynamical systems with unimodal mapping f, which has only two fixed points $s_0 = 0$, $0 < s_1 < 1$, and assume that for every $m \le n$ ($m \le k$ for g correspondingly) there is only one cycle of period 2^m which is repellent for $m < n$ ($m < k$ for g correspondingly) and attractive for $m = n$ ($m = k$ for g correspondingly). Then $\Omega(f)$ is homeomorphic to $\Omega(g)$ if and only if $m = n$. In this case dynamical systems $(\Omega(f), \sigma)$ and $(\Omega(g), \sigma)$ are conjugated.*

Proof. We give an internal description of Ω from which this theorem follows. Consider in more detail the case $n = 1$. The general case could be dealt with by induction in n which is a common situation for $\mathcal{F}_{2^n}$ dynamical systems (see [11]). In this case $\Xi = I_0 \dot\cup I_1 \dot\cup \{0, s_1, \beta_1, \beta_2\}$ where 0, s_1 are fixed points and $B_2 = \{\beta_1, \beta_2\}$ is a cycle of period 2. (I_0 and I_1 are I_{B_0} and I_{B_1} correspondingly in our previous notations). Then Ω could be identified with $(I_0 \times \mathbb{Z}) \dot\cup (I_1 \times \mathbb{Z}) \dot\cup \{0, s_1, \beta_1, \beta_2\}$. We use the homeomorphism $\Phi : I_B \to P_B$ to identify the subspace of orbits with α-boundary B with semi-interval I_B. Then point $(x, m) \in I_B \times \mathbb{Z}$ is identified with $(\Phi(x), f_{\Phi(x)}^{(m)}(x)) \in \Omega$. Periodic points 0, s_1, β_1, β_2 correspond to $(\{0\}, 0)$, $(\{0\}, 0)$, (B_2, β_1), (B_2, β_2) respectively. In order to distinguish point $(x, m) \in I_0 \times \mathbb{Z}$ from the point $(x, m) \in I_1 \times \mathbb{Z}$ we will write subscript $(x, m)_0$ and $(x, m)_1$ respectively. From the definition of the topology on Ω one can easily obtain the following *homogeneity* condition: sequence (x_k, m_k) converges to (x, m) if and only if $(x_k, m_k - m)$ converge to $(x, 0)$. We will need some notation, definitions and facts from the proof of Theorem 3 [11]. Let $I_0 = (a, b]$ and $I_1 = (t_1, t_2]$. Then there is a family of semi-intervals $J^k_{B_0}$ where $k \in \mathbb{Z}$ such that $\dot\cup_{k\geq 1} J^k_{B_0} = [b_1, b)$ and $\dot\cup_{k\leq 1} J^k_{B_0} = (a, a_1]$ for some $a_1 < b_1$ and such that $f^{2k} : J^k_{B_0} \to I_1$ is a homeomorphism for each $k \in \mathbb{Z}$. Denote $J^\infty_{B_0} = (a_1, b_1)$. Further on we will identify $J^k_{B_0}$ with $(0, 1]$ by means of a homeomorphism $\phi_k : J^k_{B_0} \to (0, 1]$ and $\phi_\infty : J^\infty_{B_0} \to (0, 1)$. Thus each point $(x, m)_0$ will be coded by a triple $(y, k, m) \in (0, 1] \times (\mathbb{Z} \cup \{\infty\}) \times \mathbb{Z}$ where $x \in J^k_{B_0}$ and $y = \phi_k(x)$. Then using nice descriptions of $\mathcal{F}_{2^n}$ dynamics (see Theorems 1, 2, 3 [11]) the convergent sequences in Ω could be completely described, for example $(x_k, m)_1$ converges to $(x, t)_1$ if and only if x_k converge to $x < 1$ and $m = t$ or x_k converge to 1 and $x = 0$, $t = m + 1$; $(x_m, k, 2 + 2k) \to (x, 0)_1$ if $k \to \infty$, $x_m \to x$; $(x, 2m)_i \to (B_2, \beta_2)$ whenever $m \to +\infty$ and $i = 0, 1$; $(x, m)_0 \to (B_0, 0)$ and $(x, m)_1 \to (B_1, s_1)$ whenever $m \to -\infty$. This description of Ω does not depend on concrete function f but only on the integer n. The rest of the claims are consequences of this fact. ■

Let us proceed with the description of the $C^*(A_f)$. In the case if f is unimodal $C^*(A_f)$ is generated by unitary U and Hermitian C such that $UC^2U^* = f(C^2)$. Let $\mathcal{A}_0$ be unital C^*-algebra generated by C. Obviously, $\mathcal{A}_0 \subset C^*(A_f)$ and letter is generated by $\mathcal{A}_0$ and U. Since each irreducible representation π of $C^*(A_f)$ is associated to some orbit $\delta \in \mathrm{Orb}\,(f)$ and $\pi(C^2)$ is the diagonal operator with points of δ on its main diagonal the universal representation π_u of $C^*(A_f)$ acts on $H = \oplus_{\delta\in \mathrm{Orb}\,(f)} l_2(\mathbb{Z})$ and $\pi_u(C^2)$ is the multiplication operator $\pi_u(C^2)\xi(\delta) = \delta\xi(\delta)$ for $\xi(\delta) \in l_2(\mathbb{Z})$ and $\pi_u(U)\xi(\delta) = \xi(f(\delta))$, $\pi_u(U^*)\xi(\delta) = \xi(f^{(-1)}_\delta(\delta))$. Then $\mathcal{A} = E_*(E(\mathcal{A}_0))$ is a commutative algebra of diagonal operators generated by $\pi_u(C^2)$ and $\pi_u(U)$ and thus is isomorphic to some $C(Y)$ where Y is a space of multiplicative linear functionals on $\mathcal{A}$. From the above description of $\mathcal{A}$ follows that a multiplicative linear functional $\rho : \mathcal{A} \to \mathbb{C}$ is of the form $\rho(g) = (g(\delta))_k$ where $g \in \mathcal{A}$ (considered as a diagonal operator in H), $\delta \in \mathrm{Orb}_+(f)$ and $k \in \mathbb{Z}$. Hence $\hat{\mathcal{A}}$ can be identified with Ω. One can check that the weak topology on $\hat{\mathcal{A}}$ coincides with the topology on Ω under this identification. Thus $\mathcal{A}$ is an algebra of coefficients for $C^*(A_f)$. Consider an element

$$x = U^{*N} a_{\overline{N}} + \cdots + U^* a_{\overline{1}} + a_0 + a_1 U + \cdots + a_N U^N.$$

Since $\langle x e_k(\delta), e_k(\delta)\rangle = \langle a_0 e_k(\delta), e_k(\delta)\rangle$ we get that $||x|| \leq ||a_0||$. Thus property $(*)$ is satisfied also. Applying theorem (isomorphism) we get the following

Theorem 7. *Let (f, I) be $\mathcal{F}_{2^n}$ and (g, I) be $\mathcal{F}_{2^k}$ dynamical systems with unimodal mapping f, which has only two fixed points $s_0 = 0$, $0 < s_1 < 1$, and assume that for every $m \leq n$ ($m \leq k$ for g correspondingly) there is only one cycle of period 2^m which is repellent for $m < n$ ($m < k$ for g correspondingly) and attractive for $m = n$ ($m = k$ for g correspondingly). (In particular, if $f \in \mathcal{F}_{2^n} \cap U$ and $g \in \mathcal{F}_{2^k} \cap U$). Then*
1. *$C^*(A_f) \cong C(\Omega) \times_\sigma \mathbb{Z}$.*
2. *$C^*(A_f) \cong C^*(A_g)$ if and only if $n = k$.*

Let $f_{p,q}(x) = 1 + px - qx^2$ with $\{p, q\} \subset \mathbb{R}$ and $q > 0$ to provide boundedness. Since when $p < 0$ dynamical system is one-to-one on $\mathbb{R}_+$ (and so all irreducible representations are one-dimensional) we assume that $p > 0$. This dynamical system is conjugated to $f_\mu(x) = \mu x(1 - x)$ where $\mu = 1 + \sqrt{p^2 - 2p + 1 + 4q}$. The values of parameter μ when bifurcations of cycles of one parametric family $\{f_\mu\}$ occurs are given in [2]. However, conjugacy relation does not preserve positiveness, i.e. $\mathrm{Orb}_+(f_{p,q})$ may not map into $\mathrm{Orb}_+(f_\mu)$. This two-parameter deformation unlike previously considered f_μ give rise to Fock and anti-Fock representations. If (p, q) belong to domain $D = \left\{(p, q) \,|\, q < \frac{1}{2} - \frac{p^2}{4} + \frac{p}{2} + \frac{\sqrt{1+2p}}{2}\right\}$ then for every $x \in [0; \sup f_{p,q}]$ $\mathcal{O}_+(x) \subset [0; \sup f_{p,q}]$. Thus for such (p, q) algebra $C^*(\mathcal{A}_{f_{p,q}})$ has Fock representation and as it easily can be shown has no anti-Fock representations. In the complement of D algebra $C^*(\mathcal{A}_{f_{p,q}})$ has anti-Fock representations. In present paper we consider only the case when there is no anti-Fock representations. Let (p,q) be such that $\mu_{n-1} < 1 + \sqrt{p^2 - 2p + 1 + 4q} \le \mu_n$ and $(p,q) \in D = \left\{(p,q) \,|\, q < \frac{1}{2} - \frac{p^2}{4} + \frac{p}{2} + \frac{\sqrt{1+2p}}{2}\right\}$ then $f_{p,q} \in \mathcal{F}_{2^n}$. Using results cited in Section 1 and above theorem one can prove the following

Theorem 8. *Let $(p,q) \in D$ and $\mu_{n-1} < 1 + \sqrt{p^2 - 2p + 1 + 4q} \le \mu_n$. Let $C^*(\mathcal{A}_{f_{p,q}})'$ is a C^*-algebra generated by elements of polar decomposition of operator X in universal representation of $C^*(\mathcal{A}_{f_{p,q}})$ that is by partial isometry U and positive operator C such that $X = UC$. Then U is an isometry and $C^*(\mathcal{A}_{f_{p,q}})'$ has an algebra of coefficients extending $C^*(C)$. The isomorphism class of $C^*(\mathcal{A}_{f_{p,q}})'$ does not depend on (p,q).*

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