# The Structure of Lie Algebras and the Classification Problem for Partial Differential Equations

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We have solved completely the problem of the description of quasi-linear evolution differential equations in two independent variables that are invariant under $so(3)$ and $sl(2,\mathbb{R})$ Lie algebras.

## 1 Introduction

The basic idea is to combine the standard Lie algorithm for point symmetries with the equivalence group of the given type of equation in order to give a classification of evolution equations in some canonical form [1].

We consider third-order evolution equations of the type

$$u_t = F(t,x,u,u_x,u_{xx})u_{xxx} + G(t,x,u,u_x,u_{xx}). \tag{1}$$

Hereafter $u = u(t,x)$, $F$, $G$ are sufficiently smooth functions of the corresponding arguments, $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_{xxx} = \frac{\partial^3 u}{\partial x^3}$, $F \neq 0$.

The first step of the algorithm is the determination of the most general form of the infinitesimal symmetry operator admitted by the PDE (1). To this end, we use Lie's method and look for a symmetry generator in the form

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u, \tag{2}$$

where $\tau$, $\xi$, $\eta$ are arbitrary, real-valued smooth functions defined in some subspace of the space $V = X \otimes \mathbb{R}^1$ of the independent variables $X = \langle t, x\rangle$ and the dependent variable $\mathbb{R}^1 = \langle u\rangle$.

As a result, we find that the operator (2) generates a one-parameter symmetry group of equation (1) if

$$\begin{aligned}&\varphi^t - [\tau F_t + \xi F_x + \eta F_u + \varphi^x F_{u_x} + \varphi^{xx} F_{u_{xx}}]u_{xxx}\\ &\qquad - \varphi^{xxx}F - \tau G_t - \xi G_x - \eta G_u - \varphi^x G_{u_x} - \varphi^{xx} G_{u_{xx}}\Big|_{u_t = Fu_{xxx}+G} = 0,\end{aligned}$$

where

$$\begin{aligned}&\varphi^t = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\ &\varphi^x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\ &\varphi^{xx} = D_x(\varphi^x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),\\ &\varphi^{xxx} = D_x(\varphi^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi)\end{aligned}$$

and $D_t$, $D_x$ are operators of total differentiation in $t$ and $x$ respectively.

We then find the following results:

**Theorem 1.** *The symmetry group of the equation* (1) *is generated by the infinitesimal operators of the form*

$$Q = \tau(t)\,\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\,\partial_u, \tag{3}$$

*where* $\tau$, $\xi$, $\eta$ *are real-valued functions that satisfy the system*

$$\begin{aligned}
&(\tau_t - 3\xi_x - 3u_x\xi_u)F + \tau F_t + \xi F_x + \eta F_{u_x} + (\eta_x + u_x(\eta_u - \xi_x) - u_x^2\xi_u)F_{u_x} + (\eta_{xx} \\
&\quad + u_x(2\eta_{xu} - \xi_{xx}) + u_x^2(\eta_{uu} - 2\xi_{xu}) - u_x^3\xi_{uu} + u_{xx}(\eta_u - 2\xi_x) - 3u_xu_{xx}\xi_u)F_{u_{xx}} = 0,\\
&-\eta_t + u_x\xi_t - (\eta_u - \tau_t + u_x\xi_u)G + F(\eta_{xxx} + u_x(3\eta_{xxu} - \xi_{xxx}) + 3u_x^2(\eta_{xuu} - \xi_{xxu})\\
&\quad + u_x^3(\eta_{uuu} - 3\xi_{xuu}) - u_x^4\xi_{uuu} + 3u_{xx}(\eta_{xu} - \xi_{xx}) + 3u_xu_{xx}(\eta_{uu} - 3\xi_{xu})\\
&\quad - 6u_{xx}u_x^2\xi_{uu} - 3u_{xx}^2\xi_u) + \tau G_t + \xi G_x + \eta G_u + (\eta_x + u_x(\eta_u - \xi_x) - u_x^2\xi_u)G_{u_x}\\
&\quad + (\eta_{xx} + u_x(2\eta_{xu} - \xi_{xx}) + u_x^2(\eta_{uu} - 2\xi_{xu})\\
&\quad - u_x^3\xi_{uu} + u_{xx}(\eta_u - 2\xi_x) - 3u_xu_{xx}\xi_u)G_{u_{xx}} = 0.
\end{aligned} \tag{4}$$

As one can see from equation (4) unless $F$ and $G$ are functions of one variable, it is well-night impossible to make the usual step of splitting this equation in terms of powers of the derivative $u_x$. This is where it is necessary to add a further element into the argument. The idea is simple: given that we are not able to derive the Lie invariance algebra by first obtaining the defining equations form (4), we then begin by specifying a Lie algebra and then requiring that it be symmetry algebra of (1). Given a Lie algebra, we then look at the possible representations of this Lie algebra within the class of operators of the form given in (2), and it is in this step that we make use of the equivalence group of equation (1). This gives us canonical representations of the symmetry algebra candidates. The final step is to calculate the allowed forms for the function $F$ and $G$ for a given canonical representation of our chosen Lie algebra. This procedure yields canonical forms of evolution equations which are inequivalent under point transformations of the equivalence group of equation (1). Then, for each such canonical evolution equation, one can calculate the maximal symmetry algebra.

The above method requires a list of inequivalent Lie algebra presentations (in terms of commutation relations). All simple (and therefore all semi-simple) finite-dimensional real and complex Lie algebras have been classified [2,3]. However, the list of solvable Lie algebras is far from complete, and as far as we can ascertain, they are given in the work of Morozov and Mubarakzyanov and Turkowski [4–10]. The work of Morozov and Mubarakzyanov are, as far as we are aware, not translated from the original Russian. Here, the solvable Lie algebras up to and including dimension six are classified. Our method is constructive in the sense that we are able to use the low-dimensional Lie algebras up to dimension five in order to give our complete point symmetry classification, and we conclude that no non-linear evolution equation of the form (1) has a point-symmetry invariance algebra of dimension greater that five. We note that a classification in terms of contact symmetries has been given by Magadeev [11].

## 2 Some previous work

The problem of group classification of such equations was discussed by many authors (see for instance [1,12–18]). Work on classification using the equivalence group has been done by Torrisi and his co-workers [20–22]. The methods of Torrisi et al are based on the infinitesimal representation of the equivalence transformations, in contrast to our approach which involves finite forms of the equivalence transformations.

## 3 Results

In this section we give some results and show how the equivalence group of the equation plays its role in the classification of representation of a given Lie algebra. First, we begin with the description of the equivalence group of equation (1).

**Lemma 1.** *The maximal equivalence group* $\mathcal{E}$ *of equation* (1) *reads as*

$$\bar{t} = T(t), \qquad \bar{x} = X(t,x,u), \qquad v = U(t,x,u), \tag{5}$$

*where* $\dot{T} \neq 0$, $\frac{D(X,U)}{D(x,u)} \neq 0$.

The next step is to give a canonical form (linearization) for a vector field, with respect to the allowed transformations (5). This is given in the following result:

**Lemma 2.** *There are changes of variables* (5)*, that reduce an operator* (3) *to one of the operators below:*

$$Q = \partial_t, \tag{6}$$

$$Q = \partial_x. \tag{7}$$

**Proof.** Making the change of variables (5) transforms operator (3) to the following one:

$$Q \to Q' = \tau\dot{T}\partial_{\bar{t}} + (\tau X_t + \xi X_x + \eta X_u)\partial_{\bar{x}} + (\tau U_t + \xi U_x + \eta U_u)\partial_v. \tag{8}$$

Suppose $\tau \neq 0$. Then, choosing in (5) the function $T$ to be a solution of the equation $\dot{T} = \tau^{-1}$ and the functions $X$ and $U$ to be independent fundamental solutions of the first-order PDE

$$\tau Y_t + \xi Y_x + \eta Y_u = 0, \qquad Y = Y(t,x,u),$$

we find that the operator (8) takes the form $Q' = \partial_{\bar{t}}$.

Now suppose $a = 0$. Then $\xi^2 + \eta^2 \neq 0$. If $b \neq 0$, then choosing in (5) a particular solution of PDE $\xi X_x + \eta X_u = 1$ as the function $X$ and a fundamental solution of PDE $\xi U_x + \eta U_u = 0$ as the function $U$, we transform (8) to become $Q' = \partial_{\bar{x}}$.

If $\xi = 0$, $\eta \neq 0$, then making the change of variables (5) with $\bar{t} = t$, $\bar{x} = u$, $v = x$, we again get the case $\xi \neq 0$.

By the direct calculation we can verify that there is no transformation from $\mathcal{E}$, that reduce operator (6) to the form (7). ■

**Example 1.** $sl(2,\mathbb{R})$. Now we give an example of the calculations involved. We do this using the Lie algebra $sl(2,\mathbb{R})$ which is the Lie algebra with basis $\langle Q_1, Q_2, Q_3\rangle$ satisfying the commutation relations

$$[Q_1, Q_2] = 2Q_2, \qquad [Q_1, Q_3] = -2Q_3, \qquad [Q_2, Q_3] = Q_1. \tag{9}$$

First, we take one operator from the basis and put it into canonical form. So let us choose $Q_3$ for this purpose (this is done for purely practical reasons.) By Lemma 2, we can choose $Q_3$ in one of two canonical forms: $Q_3 = \partial_t$ or $Q_3 = \partial_x$.

We consider in detail the case $Q_3 = \partial_t$. With this choice of $Q_3$ we proceed to find the allowable forms for $Q_3$ or $Q_3$. So put

$$Q_1 = \tau(t)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u.$$

The second commutation relation in (9) then implies that

$$\dot{\tau} = 2, \qquad \dot{\xi} = \dot{\eta} = 0.$$

Therefore, $\tau(t) = 2t + c$, $c = \text{const}$ and $\xi$, $\eta$ are independent of $t$. So we take

$$Q_1 = 2t\partial_t + \xi(x,u)\partial_x + \eta(x,u)\partial_u.$$

The next step is to find a canonical form for $Q_1$ under the equivalence transformations (5). However, we must now use only those equivalence transformations of (5) which preserve the form $Q_3 \to \bar{Q}_3$ with

$$\bar{Q}_3 = \dot{T}\partial_{\bar{t}} + X_t\partial_{\bar{x}} + U_t\partial_v = \partial_{\bar{t}}$$

which yields $\dot{T} = 1$, $X_t = U_t = 0$. Thus we take $T(t) = t$, $X = X(x,u)$, $U = U(x,u)$. Under this type of transformation we find

$$Q_1 \to \bar{Q}_1 = 2\bar{t}\partial_{\bar{t}} + (\xi X_x + \eta X_u)\partial_{\bar{x}} + (\xi U_x + \eta U_u)\partial_v.$$

We now choose $X$ and $U$ so that

$$\xi X_x + \eta X_u = X, \qquad \xi U_x + \eta U_u = 0.$$

This gives us the canonical form

$$\bar{Q}_1 = 2\bar{t}\partial_{\bar{t}} + \bar{x}\partial_{\bar{x}}.$$

This means that we can, up to an equivalence transformation of equation (1), take

$$Q_3 = \partial_t, \qquad Q_1 = 2t\partial_t + x\partial_x.$$

Finally, we need to determine $Q_2$. We put

$$Q_2 = \alpha(t)\partial_t + \beta(t,x,u)\partial_x + \gamma(t,x,u)\partial_u.$$

The commutation relation $[Q_2, Q_3] = Q_1$ gives

$$\alpha(t) = -t^2 + c, \qquad \beta = -xt + b(x,u), \qquad \gamma = \gamma(x,u).$$

Thus we may take

$$Q_2 = -t^2\partial_t + (b(x,u) - xt)\partial_x + \gamma(x,u)\partial_u.$$

Then we use the relation $[Q_1, Q_2] = 2Q_2$. From this we obtain

$$b(x,u) = m(u)x^3, \qquad \gamma(x,u) = n(u)x^2$$

and we have

$$Q_2 = -t^2\partial_t + \left(m(u)x^3 - xt\right)\partial_x + n(u)x^2\partial_u.$$

All that remains to be done is to find a canonical form for $Q_2$. We do this using equivalence transformations (5) which leave invariant the form of $Q_1$, $Q_3$. These are given by

$$T(t) = t, \qquad X(tx,u) = q(u)x, \qquad U = p(u)$$

with $q(u) \neq 0$ and $\dot{p}(u) \neq 0$. Under this transformation we find

$$\bar{Q}_2 = -\bar{t}^2\partial_{\bar{t}} + \left(\frac{q(u)m(u) + n(u)\dot{q}(u)}{q^3(u)}\bar{x}^3 - \bar{x}\bar{t}\right)\partial_{\bar{x}} + \frac{n(u)\dot{p}(u)}{q^2(u)}\bar{x}^2\partial_v.$$

There are two cases: $n(u) \neq 0$ and $n(u) = 0$. If $n(u) \neq 0$ then we may choose $p(u)$ and $q(u)$ such that

$$\frac{q(u)m(u) + n(u)\dot{q}(u)}{q^3(u)} = 1, \qquad \frac{n(u)\dot{p}(u)}{q^2(u)} = 1$$

which give us

$$\bar{Q}_2 = -\bar{t}^2\partial_{\bar{t}} + \big(\bar{x}^3 - \bar{x}\bar{t}\big)\partial_{\bar{x}} + \bar{x}^2\partial_v.$$

If, however, $n(u) = 0$ then we have

$$\bar{Q}_2 = -\bar{t}^2\partial_{\bar{t}} + \left(\frac{q(u)m(u) + n(u)\dot{q}(u)}{q^3(u)}\bar{x}^3 - \bar{x}\bar{t}\right)\partial_{\bar{x}}.$$

If now $m(u) > 0$ we may choose $m(u) = q^2(u)$ and we find

$$\bar{Q}_2 = -\bar{t}^2\partial_{\bar{t}} + \big(\bar{x}^3 - \bar{x}\bar{t}\big)\partial_{\bar{x}}.$$

If $m(u) < 0$ then we choose $m(u) = -q^2(u)$ and we find

$$\bar{Q}_2 = -\bar{t}^2\partial_{\bar{t}} + \big(-\bar{x}^3 - \bar{x}\bar{t}\big)\partial_{\bar{x}}.$$

Finally, we note that $t \to -t, \quad x \to x, \quad u \to u$ is an equivalence transformation of equation (1), and the last two canonical forms for $Q_2$ are equivalent under this transformation. Summarizing this calculation, we find that the algebra $sl(2, \mathbb{R})$ has four canonical forms with $Q_3 = \partial_t$:

$$\begin{aligned}
&\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t - tx\partial_x + x^2\partial_u,\ \partial_t\rangle,\\
&\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t + x(x^2 - t)\partial_x,\ \partial_t\rangle,\\
&\langle 2t\partial_t,\ -t^2\partial_t,\ \partial_t\rangle,\\
&\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t - xt\partial_x,\ \partial_t\rangle,
\end{aligned}$$

and these are inequivalent under the equivalence group given by (5). The canonical form $Q_3 = \partial_x$ gives rise to a similar calculation, and three inequivalent representations for the Lie algebra $sl(2, \mathbb{R})$ are found. Other six realizations of the algebra $sl(2, \mathbb{R})$ be able admitted by PDEs of the form (1).

These results are summarized in the following:

**Theorem 2.** *There exist six inequivalent realizations of the algebra $sl(2, \mathbb{R})$ by operators* (3), *which are admitted by PDEs of the form* (1)

$$\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t - tx\partial_x + x^2\partial_u,\ \partial_t\rangle, \tag{10}$$

$$\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t + x\big(x^2 - t\big)\partial_x,\ \partial_t\rangle, \tag{11}$$

$$\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u,\ \partial_x\rangle, \tag{12}$$

$$\langle 2x\partial_x - u\partial_u,\ \big(u^{-4} - x^2\big)\partial_x + xu\partial_u,\ \partial_x\rangle, \tag{13}$$

$$\langle 2x\partial_x - u\partial_u,\ -\big(u^{-4} + x^2\big)\partial_x + xu\partial_u,\ \partial_x\rangle, \tag{14}$$

$$\langle 2x\partial_x,\ -x^2\partial_x,\ \partial_x\rangle. \tag{15}$$

The forms of the functions $F$, $G$ determining the corresponding invariant equations are given in Table 1, where $\tilde{F}$, $\tilde{G}$ are arbitrary functions.

**Table 1.**

| $sl(2,\mathbb{R})$ | $F$ | $G$ |
|---|---|---|
| (10) | $x\tilde{F}(\omega, v)$,<br>$\omega = 2u - xu_1,\ v = 2u - x^2u_{xx}$ | $\frac{u_x^2}{4} + x^{-2}\tilde{G}(\omega, v)$ |
| (11) | $x^{-3}u_x^{-4}\tilde{F}(\sigma, u),\ \sigma = u_{xx}u_x^{-2} + 3x^{-1}u_x^{-1}$ | $-\frac{x^{-2}\omega}{4} + x^{-2}\left(\frac{9\sigma}{\omega^2} - \frac{12}{\omega^3}\right)\tilde{F}(u,\sigma) + \frac{1}{x^2\omega}\tilde{G}(u,\delta)$,<br>$\omega = xu_x,\ \delta = u_{xx}u_x^{-2} - 3x^{-1}u_x^{-1}$,<br>$\sigma = u_{xx}^{-1}u_x^2 - \frac{xu_x}{3}$ |
| (12) | $u^{-6}\tilde{F}(t,\omega),\ \omega = u^{-5}u_{xx} - 2u^{-6}u_x^2$ | $u\tilde{G}(t,\omega) + (12u^{-9}u_x^3 - 9u^{-8}u_xu_{xx})\tilde{F}(t,\omega)$ |
| (13) | $\frac{u^{-6}}{(1+4\omega^2)^{\frac{3}{2}}}\tilde{F}(t,\sigma)$,<br>$\sigma = \frac{2v-10\omega^2-1}{2(1+4\omega^2)^{\frac{3}{2}}},\ \omega = u^{-3}u_x,\ v = u^{-5}u_{xx}$ | $u\tilde{G}(t,\sigma)\sqrt{1+4\omega^2} - u\Big[12\sigma^2\omega\sqrt{1+4\omega^2}$<br>$+ 21\sigma\omega + \frac{15}{2}\frac{\omega+6\omega^3}{(1+4\omega^2)^{\frac{3}{2}}}\Big]\tilde{F}(t,\sigma)$ |
| (14) | $\frac{u^{-6}}{(4\omega^2-1)^{\frac{3}{2}}}\tilde{F}(t,\sigma_1)$ if $4\omega^2 > 1$,<br>$\omega = u^{-3}u_x,\ v = u^{-5}u_{xx},\ \sigma_1 = \frac{2v-10\omega^2-1}{2(4\omega^2-1)^{\frac{3}{2}}}$<br>$\frac{u^{-6}}{(1-4\omega^2)^{\frac{3}{2}}}\tilde{F}(t,\sigma_2)$ if $4\omega^2 < 1$<br>$\omega = u^{-3}u_x,\ v = u^{-5}u_{xx},\ \sigma_2 = \frac{2v-10\omega^2-1}{2(1-4\omega^2)^{\frac{3}{2}}}$ | $u\tilde{G}(t,\sigma_1)\sqrt{4\omega^2-1} - u\Big[12\sigma_1^2\omega\sqrt{4\omega^2-1}$<br>$+ 21\sigma_1\omega + \frac{15}{2}\frac{6\omega^3-\omega}{(1-4\omega^2)^{\frac{3}{2}}}\Big]\tilde{F}(t,\sigma)$<br>$u\tilde{G}(t,\sigma_2)\sqrt{1-4\omega^2} - u\Big[12\sigma_2^2\omega\sqrt{1-4\omega^2}$<br>$+ 21\sigma_2\omega + \frac{15}{2}\frac{6\omega^3-\omega}{(1-4\omega^2)^{\frac{3}{2}}}\Big]\tilde{F}(t,\sigma_2)$ |
| (15) | $u_x^{-3}\tilde{F}(t,u)$ | $-\frac{3}{2}\frac{u_{xx}^2}{u_x^4}\tilde{F}(t,u) + \tilde{G}(t,u)$ |

**Theorem 3.** *There exists only one realization of the algebra so*(3) *by operators of the form* (2) *which is an invariance algebra of* (1)*:*

$$\langle \partial_x,\ \tan u \sin x\partial_x + \cos x\partial_u,\ \tan u \cos x\partial_x - \sin x\partial_u \rangle. \tag{16}$$

*Furthermore, the most general form of the functions $F$, $G$ allowing for PDE* (1) *to be invariant under the above realization is given by*

$$\begin{aligned} F &= \frac{\sec^3 u}{(1+\omega^2)^{3/2}}\tilde{F}(t,\psi), \\ G &= \left[9\omega\psi\tan u - 3\omega\psi^2(1+\omega^2)^{1/2} + \frac{\omega(1+2\omega^2)}{(\omega^2+1)^{3/2}} - \frac{\omega(5+6\omega^2)\tan^2 u}{(\omega^2+1)^{3/2}}\right]\tilde{F}(t,\psi) \\ &\quad + (\omega^2+1)^{1/2}\tilde{G}(t,\psi), \end{aligned}$$

*where we have used the notation*

$$\omega = u_x \sec u, \qquad \psi = \frac{u_{xx}\sec^2 u + (1+2\omega^2)\tan u}{(1+\omega^2)^{3/2}}.$$

*Provided the function $\tilde{G}$ is arbitrary, the realization* (16) *is the maximal symmetry algebra of the corresponding equation* (1).

Our approach is, as we have demonstrated, a combination of the Lie point symmetry analysis and an exploitation of the equivalence group of the equation to give possible canonical forms for the various Lie algebras. Each representation of the Lie algebra is then tested as a symmetry algebra, and is discarded if it gives no result (by which we mean, amongst other things, that the equation admitting a given representation as symmetry algebra must have $F \neq 0$ in (1)).

The present method gives a complete point-symmetry classification of (1), so that any evolution equation of the form is necessarily point-equivalent under the transformations (5) to one of the canonical forms for equation (1).

[1] Zhdanov R.Z. and Lahno V.I., Group classification of heat conductivity equations with a nonlinear source, *J. Phys. A: Math. Gen.*, 1999, V.32, 7405–7418.

[2] Goto M. and Grosshans F.D., Semisimple Lie algebras, Marcel Dekker, 1978.

[3] Helgason S., Differential geometry, Lie groups, and symmetric spaces, New York, Academic Press, 1978.

[4] Morozov V.V., Classification of six-dimensional nilpotent Lie algebras, *Izv. Vys. Ucheb. Zaved., Matematika*, 1958, N 4 (5), 161–171 (in Russian).

[5] Mubarakzyanov G.M., On solvable Lie algebras, *Izv. Vys. Ucheb. Zaved., Matematika*, 1963, N 1 (32), 114–123 (in Russian).

[6] Mubarakzyanov G.M., The classification of the real structure of five-dimensional Lie algebras, *Izv. Vys. Ucheb. Zaved., Matematika*, 1963, N 3 (34), 99–106 (in Russian).

[7] Mubarakzyanov G.M., The classification of six-dimensional Lie algebras with one nilpotent basis element, *Izv. Vys. Ucheb. Zaved., Matematika*, 1963, N 4 (35), 104–116 (in Russian).

[8] Mubarakzyanov G.M., Some theorems on solvable Lie algebras, *Izv. Vys. Ucheb. Zaved., Matematika*, 1966, N 3 (55), 95–98 (in Russian).

[9] Turkowski P., Solvable Lie algebras of dimensional six, *J. Math. Phys.*, 1990, V.31, 1344–1350.

[10] Turkowski P., Low-dimensional real Lie algebras, *J. Math. Phys.*, 1988, V.29, 2139–2144.

[11] Magadeev B.A., On group classification of nonlinear evolution equations, *Algebra i Analiz*, 1993, V.5, 141–156 (in Russian).

[12] Ovsiannikov L.V., Group properties of nonlinear heat equation, *Dokl. AN SSSR*, 1959, V.125, N 3, 492–495 (in Russian).

[13] Dorodnitsyn V.A., On invariant solutions of non-linear heat equation with a source, *Zhurn. Vych. Matemat. Matem. Fiziki*, 1982, V.22, 1393–1400 (in Russian).

[14] Oron A. and Rosenau P., Some symmetries of the nonlinear heat and wave equations, *Phys. Lett. A*, 1986, V.118, 172–176.

[15] Edwards M.P., Classical symmetry reductions of nonlinear diffusion-convection equations, *Phys. Lett. A*, 1994, V.190, 149–154.

[16] Cherniha R. and Serov M., Symmetries, ansätze and exact solutions of nonlinear second-order evolution equations with convection terms, *Euro. J. Appl. Math.*, 1998, V.9, 527–542.

[17] Gandarias M.L., Classical point symmetries of a porous medium equation, *J. Phys. A: Math. Gen.*, 1996, V.29, 607–633.

[18] Akhatov I.S., Gazizov R.K. and Ibragimov N.K., Group classification of equations of nonlinear filtration, *Proc. Acad. Sci. USSR*, 1987, V.293, 1033–1035 (in Russian).

[19] Torrisi M., Tracina R. and Valenti A., A group analysis approach for a nonlinear differential system arising in diffusion phenomena, *J. Math. Phys.*, 1996, V.37, 4758–4767.

[20] Torrisi M. and Tracina R., Equivalence transformations and symmetries for a heat conduction model, *Int. J. Non-Linear Mech.*, 1998, V.33, 473–487.

[21] Ibragimov N.H., Torrisi M. and Valenti A., Preliminary group classification of equation $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, *J. Math. Phys.*, 1991, V.32, 2988–2995.

[22] Ibragimov N.K. and Torrisi M., A simple method for group analysis and its applications to a model of detonation, *J. Math. Phys.*, 1992, V.33, 3931–3937.