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On Integrable Systems of ODEs

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We classify normal systems of three and four first-order ordinary differential equations (ODEs) that are invariant with respect to three- and four-dimensional solvable real Lie algebras respectively, and therefore can be integrated by the Lie method.

Description of the most general form of systems of differential equations of some type that are invariant with respect to the algebras of a fixed dimension is one of the main problems of group analysis. The standard Lie algorithm leads to classification of such systems if all the realizations of these algebras in Lie vector fields are known [5]. For instance, in [1] the second-order partial differential equations that are invariant under the standard realizations of the Euclid, Poincaré, Galilei, conformal and projective algebras were obtained. New realizations of the above algebras was constructed and a partial differential equations invariant with respect to these realizations were described in [2, 3, 7] (see also references therein).

A necessary step for searching realizations of Lie algebras is classification of these algebras, i.e. classification of possible commutative relations between basic elements. As for solvable Lie algebras, classification was completely made for only low-dimensional algebras. All possible non-isomorphic complex Lie algebras of dimension $N \leq 4$ were listed by S. Lie himself. G.M. Mubarakzyanov was the first who got the complete and suitable for application classification of the real Lie algebras of dimension $N \leq 4$ [4]. In paper [6] that are based on his classification realizations of these algebras in the spaces of the arbitrary finite number of the variables have been classified.

In this paper we construct the systems of the ODEs that are invariant with respect to three- and four-dimensional solvable real Lie algebras using results of [6]. The numeration of algebras and realizations corresponds to that paper: $R(A, N)$ denotes the Nth realization of the algebra A (see Tables 2–4) in [6].

Consider systems of ODEs of the form

$$F_k(t, x, \dot{x}) = 0, \tag{1}$$

where F_k are smooth functions, $k = 1, 2, 3$ for three-dimensional Lie algebras and $k = 1, \ldots, 4$ for four-dimensional ones. We will imply that the variable t is independent, and $x = (x_1, x_2, \ldots, x_k)$ variables are dependent. We denote derivatives with dots over the symbols $\dot{x}_i = \frac{dx_i}{dt}$, $\dot{x} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_k)$.

Let us note that invariant systems obtained by various choices of the independent variable in basis elements of realizations are equivalent to each other. The variables are chosen to simplify calculations in a such way that the canonic forms of realizations would not contain the operator of differentiation with respect to the independent variable. A simplest general form of invariant systems corresponds to this choice. An arbitrary realization of an m-dimensional Lie algebra ($m \leq 4$) is equivalent to a realization into which no more than $m+1$ first variables explicitly enter. (There is exception is only for the realizations of algebra $4A_1$ with rank 2 and 3, namely $R(4A_1, 2)$, $R(4A_1, 5)$). Under the choice of the respective coordinates and the independent variable ($t = x_{m+1}$), all the variables x_i and $\dot{x}_i$, $i \in \{m+2, \ldots, n\}$ will be invariants. Therefore, we search the differential invariants of realizations of three- and four-dimensional solvable Lie algebras in the space with three (four) dependent variables.

At first we construct the systems of three ODEs that are invariant with respect to realizations of three-dimensional solvable Lie algebras (see Table 2 [6]). Let $x_4 = t$ be an independent variable and x_1, x_2, x_3 be functions of t.

We find the functional bases of the first-order differential invariants for every realization of three-dimensional solvable Lie algebras. The corresponding calculations are not given here as there are quite cumbersome. Using the constructed sets of differential invariants of Lie algebras we obtain the exhaustive list of inequivalent (with respect to arbitrary regular changes of variables) systems of first-order ODEs that are invariant under three-dimensional solvable real Lie algebras.

By the direct verification we make sure that such systems may be reduced to the normal forms if the rank of the corresponding realization equals three.

Theorem 1. *Let system of the three ordinary first-order ODEs be invariant with respect to a realization of a real three-dimensional solvable Lie algebra with rank* 3*. Then the system is integrable in quadratures by means of the Lie method and using the same transformation of variables, which reduce the realization to one of the realizations given in Table* 2 *of* [6]*, we can transform the system to one of the following normal forms:*

$$R(3A_1,1) \qquad \dot{x}_1 = f_1(t), \quad \dot{x}_2 = f_2(t), \quad \dot{x}_3 = f_3(t);$$

$$R(A_{2.1}\oplus A_1,1) \qquad \dot{x}_1 = f_1(t)e^{x_3}, \quad \dot{x}_2 = f_2(t), \quad \dot{x}_3 = f_3(t);$$

$$R(A_{3.1},1) \qquad \dot{x}_1 = f_1(t) + x_3\, f_2(t), \quad \dot{x}_2 = f_2(t), \quad \dot{x}_3 = f_3(t);$$

$$R(A_{3.2},1) \qquad \dot{x}_1 = [f_1(t) + x_3 f_2(t)]e^{x_3}, \ \ \dot{x}_2 = f_2(t)e^{x_3}, \ \ \dot{x}_3 = f_3(t);$$

$$R(A_{3.3},1) \qquad \dot{x}_1 = f_1(t)e^{x_3}, \quad \dot{x}_2 = f_2(t)e^{x_3}, \quad \dot{x}_3 = f_3(t);$$

$$R(A_{3.4},1) \qquad \dot{x}_1 = f_1(t)e^{x_3}, \quad \dot{x}_2 = f_2(t)e^{ax_3}, \quad \dot{x}_3 = f_3(t);$$

$$R(A_{3.5},1) \qquad \dot{x}_1 = f_1(t)e^{bx_3}\sin(x_3+f_2(t)), \ \dot{x}_2 = f_1(t)e^{bx_3}\cos(x_3+f_2(t)), \ \dot{x}_3 = f_3(t).$$

Further we carry out the similar construction of differential invariants of realizations of four-dimensional solvable real Lie algebras, assuming $x_5 = t$ is the independent variable and x_1, x_2, x_3, x_4 are functions of t. We choose only the realizations that exist in the space of variables x_1, x_2, x_3, x_4, $x_5 = t$.

As to systems of four ODEs which are invariant with respect to the four-dimensional solvable real Lie algebras, in a similar way we obtain the following theorem.

Theorem 2. *Let system of the four ordinary first-order ODEs be invariant with respect to a realization of a real four-dimensional solvable Lie algebra with rank* 4*. Then the system is integrable in quadratures with Lie's method and using the same transformation of variables which reduce the realization to one of the realizations given in Tables* 3 *and* 4 *of* [6] *we can transform the system to one of the following normal forms:*

$$R(A_{4.1},1) \qquad \dot{x}_1 = f_1(t) + x_4 f_2(t) + \frac{1}{2} f_3(t) {x_4}^2, \ \dot{x}_2 = f_2(t) + x_4\, f_1(t),$$
$$\dot{x}_3 = f_3(t), \quad \dot{x}_4 = f_4(t);$$

$$R(A_{4.2},1) \qquad \dot{x}_1 = f_1(t)e^{qx_4}, \ \ \dot{x}_2 = [x_4 + f_2(t)]f_3(t)e^{x_4}, \ \ \dot{x}_3 = f_3(t)e^{x_4}, \ \ \dot{x}_4 = f_4(t);$$

$$R(A_{4.3},1) \qquad \dot{x}_1 = f_1(t)e^{x_4}, \quad \dot{x}_2 = x_4\, f_3(t) + f_2(t), \ \ \dot{x}_3 = f_3(t), \quad \dot{x}_4 = f_4(t);$$

$$R(A_{4.4},1) \qquad \dot{x}_1 = [\frac{1}{2}{x_4}^2 + x_4\, f_2(t) + f_1(t)]f_3(t)e^{x_4},$$
$$\dot{x}_2 = [x_4 + f_2(t)]f_3(t)e^{x_4}, \ \ \dot{x}_3 = f_3(t)e^{x_4}, \quad \dot{x}_4 = f_4(t);$$

$R(A_{4.5},1)$ $\quad \dot x_1 = f_1(t)e^{ax_4}, \quad \dot x_2 = f_2(t)f_1{}^{\frac{b}{a}}(t)\,x_4^b, \quad \dot x_3 = f_3(t)f_1{}^{\frac{c}{a}}(t)\,x_4^c, \quad \dot x_4 = f_4(t);$

$R(A_{4.6},1)$ $\quad \dot x_1 = f_1(t)e^{qx_4}, \quad \dot x_2 = -f_3(t)e^{2px_4}\sin(f_2(t)+x_4),$
$\dot x_3 = f_3(t)e^{2px_4}\cos(f_2(t)+x_4), \quad \dot x_4 = f_4(t);$

$R(A_{4.7},1)$ $\quad \dot x_1 = [f_1(t)e^{x_4} + x_3(f_2(t)+x_4)]\,f_3(t)e^{x_4},$
$\dot x_2 = [f_2(t)+x_4]\,f_3(t)e^{x_4}, \quad \dot x_3 = f_3(t)e^{x_4}, \quad \dot x_4 = f_4(t);$

$R(A_{4.8},1)$ $\quad \dot x_1 = [f_1(t)e^{qx_4}+x_3]f_2(t)e^{x_4}, \quad \dot x_2 = f_2(t)e^{x_4}, \quad \dot x_3 = f_3(t)e^{qx_4}, \quad \dot x_4 = f_4(t);$

$R(A_{4.9},1)$ $\quad \dot x_1 = [(f_1(t)+x_3f_2(t)\sin(x_4+f_3(t))]e^{2qx_4}, \quad \dot x_2 = f_2(t)e^{2qx_4}\sin(x_4+f_3(t)),$
$\dot x_3 = f_3(t)e^{2qx_4}\cos(x_4+f_3(t)), \quad \dot x_4 = f_4(t);$

$R(A_{4.10},1)$ $\quad \dot x_1 = f_1(t)e^{2x_3}\sin(x_4+f_2(t)),$
$\dot x_2 = f_1(t)e^{2x_3}\cos(x_4+f_2(t)), \quad \dot x_3 = f_3(t), \quad \dot x_4 = f_4(t);$

$R(4A_1,1)$ $\quad \dot x_1 = f_1(t), \quad \dot x_2 = f_2(t), \quad \dot x_3 = f_3(t), \quad \dot x_4 = f_4(t);$

$R(A_{2.1}\oplus 2A_1,1)$ $\quad \dot x_1 = f_1(t)e^{x_4}, \quad \dot x_2 = f_2(t), \quad \dot x_3 = f_3(t), \quad \dot x_4 = f_4(t);$

$R(A_{2.1}\oplus A_{2.1},1)$ $\quad \dot x_1 = f_1(t)e^{x_3}, \quad \dot x_2 = f_2(t)e^{x_4}, \quad \dot x_3 = f_3(t), \quad \dot x_4 = f_4(t);$

$R(A_{3.1}\oplus A_1,1)$ $\quad \dot x_1 = f_1(t)+x_4f_3(t), \quad \dot x_2 = f_2(t), \quad \dot x_3 = f_3(t), \quad \dot x_4 = f_4(t);$

$R(A_{3.2}\oplus A_1,1)$ $\quad \dot x_1 = [x_3+f_1(t)]f_2(t)e^{x_3}, \quad \dot x_2 = f_2(t)e^{x_3}, \quad \dot x_3 = f_3(t), \quad \dot x_4 = f_4(t);$

$R(A_{3.3}\oplus A_1,1)$ $\quad \dot x_1 = f_1(t)e^{x_3}, \quad \dot x_2 = f_2(t)e^{x_3}, \quad \dot x_3 = f_3(t), \quad \dot x_4 = f_4(t);$

$R(A_{3.4}\oplus A_1,1)$ $\quad \dot x_1 = f_1(t)e^{x_3}, \quad \dot x_2 = f_2(t)e^{ax_3}, \quad \dot x_3 = f_3(t), \quad \dot x_4 = f_4(t);$

$R(A_{3.5}\oplus A_1,1)$ $\quad \dot x_1 = f_1(t)e^{2bx_3}\sin(x_3+f_1(t)),$
$\dot x_2 = f_1(t)e^{2bx_3}\cos(x_3+f_1(t)), \quad \dot x_3 = f_3(t), \quad \dot x_4 = f_4(t).$

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[1] Fushchych W.I. and Yegorchenko I.A., Second-order differential invariants of the rotations group $O(n)$ and its extensions: $E(n)$, $P(1,n)$, $G(1,n)$, *Acta Appl. Math.*, 1992, V.28, N 1, 69–92.

[2] Fushchych W. and Zhdanov R., Symmetries and exact solutions of nonlinear Dirac equations, Kyiv, Mathematical Ukraina Publisher, 1997.

[3] Lahno V., On realizations of the conformal algebra and nonlinear invariant equations in low-dimensional space-time, Contribution in these Proceedings.

[4] Mubarakzyanov G.M., On solvable Lie algebras, *Izv. Vys. Ucheb. Zaved. Matematika*, 1963, N 1 (32), 114–123 (in Russian).

[5] Olver P.J., Applications of Lie groups to differential equations, New York, Springer, 1986.

[6] Popovych R., Boyko V., Nesterenko M. and Lutfullin M., Realizations of real low-dimensional Lie algebras, *J. Phys. A: Math. Gen.*, 2003, V.36, 7337–7360; math-ph/0301029.

[7] Yehorchenko I., Differential invariants of nonlinear representation of the Poincaré algebra. Invariant equations, in Proceedings of the Second International Conference "Symmetry in Nonlinear Mathematical Physics" (7–13 July, 1997, Kyiv), Editors M.I. Shkil', A.G. Nikitin and V.M. Boyko, Kyiv, Institute of Mathematics, V.1, 1997, 200–205.