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Poisson Structure of Rational Reductions of the 2D dToda Hierarchy

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The paper concerns the Hamiltonian structure of the finite-dimensional reductions 2D dispersionless Toda hierarchy constrained by the string equation. We derive the Hamiltonian structure of the reduced dynamics and show connections of integrals of "multi-finger" solutions of the Laplacian growth problem with the "Toda–Krichever" flows of the 2dToda hierarchy constrained by a string equation. The Poisson structure of the rationally reduced 1dToda hierarchy is also derived.

1 Hamiltonians and action-angle variables

In this paper we study the Poisson structure of rational reductions of the 2dToda hierarchy, introduced in the previous paper of the present volume [1]. These reductions, as seen from the above considerations, form a finite-dimensional completely integrable system. We find explicit expressions leading to a canonical Hamiltonian structure on the phase space of rational reductions of the 2D dToda system. Such reductions are interesting from the point of view of the applications mentioned in the Introduction.

A reader interested in the Poisson structure of rational reductions of the 1dToda hierarchy (which are infinite-dimensional, unrestricted by string equation systems) will find it in the second part of the paper. This structure, which turns out to be a quadratic algebra, is interesting by itself.

Since the Toda flows commute, the Toda times are actually the action-angle variables of the system.

Thus we have only to express the times through coordinates of the phase space. For the logarithmic ansatz [1, Proposition 4] the following result holds[1]:

Proposition 1. *The times τ_i, $\bar{\tau}_i$ of equation* ([1].25) *are expressed in terms of z and $\bar{z}$* ([1].23) *as*

$$\tau_0 = \frac{1}{2\pi\sqrt{-1}} \oint_{\infty} \bar{z} d\ln z + \mathrm{const}, \qquad \bar{\tau}_0 = \frac{1}{2\pi\sqrt{-1}} \oint_{0} z d\ln \bar{z} + \mathrm{const},$$
$$\tau_i = \frac{1}{2\pi\sqrt{-1}} \oint_{w_i} \bar{z} dz + \mathrm{const}, \qquad \bar{\tau}_i = \frac{1}{2\pi\sqrt{-1}} \oint_{1/\bar{w}_i} z d\bar{z} + \mathrm{const}, \qquad i = 1, \dots, n+1$$

or, in other words, the following values

$$I_0 = \bar{z}(1/w = 0) = \bar{u} + \sum_{i=1}^{n+1} \bar{a}_i \ln(\bar{w}_i), \qquad \bar{I}_0 = z(0) = u + \sum_{i=1}^{n+1} a_i \ln(w_i),$$

[1]Here and below equations from [1] are cited in the form ([1].N), where N is the number of corresponding equation.

$$I_i = a_i \bar{z}(w_i^{-1}) = a_i \left(r w_i^{-1} + \bar{u} + \sum_{j=1}^{n+1} \bar{a}_j \ln(\bar{w}_j - w_i^{-1}) \right),$$

$$\bar{I}_i = \bar{a}_i z(\bar{w}_i^{-1}) = \bar{a}_i \left(r \bar{w}_i^{-1} + u + \sum_{j=1}^{n+1} a_j \ln(w_j - \bar{w}_i^{-1}) \right), \qquad i = 1, \ldots, n+1, \tag{1}$$

are the action-angle variables and

$$\begin{aligned} Q &= \frac{1}{4\pi\sqrt{-1}} \sum_{i=0}^{n+1} \oint_{1/\bar{w}_i} z d\bar{z} + \oint_{w_i} \bar{z} dz \\ &= \frac{1}{2} r \left(\left(\frac{\partial z(w)}{\partial w} \right)_{w=0} + \left(\frac{\partial \bar{z}(1/w)}{\partial (1/w)} \right)_{1/w=0} \right) - \frac{1}{2} \sum_{i=1}^{n+1} (I_i + \bar{I}_i) \\ &= r^2 - \frac{1}{2} \sum_{i=1}^{n+1} \left(r \left(\frac{a_i}{w_i} + \frac{\bar{a}_i}{\bar{w}_i} \right) + I_i + \bar{I}_i \right) \end{aligned} \tag{2}$$

is a Casimir for ([1].25).

Proof. We first calculate the derivatives of each I_i with respect to the times τ_i, $\bar{\tau}_i$, $i = 1, \ldots, n$. Using integration by parts, we get

$$2\pi\sqrt{-1} \frac{\partial I_i}{\partial \tau_j} = \frac{\partial}{\partial \tau_j} \oint_{w_i} \bar{z} \frac{\partial z}{\partial w} dw = \oint_{w_i} \frac{\partial}{\partial w} \left(\bar{z} \frac{\partial z}{\partial \tau_j} \right) dw + \oint_{w_i} \left(\frac{\partial \bar{z}}{\partial \tau_j} \frac{\partial z}{\partial w} - \frac{\partial z}{\partial \tau_j} \frac{\partial \bar{z}}{\partial w} \right) dw.$$

By equations of motion ([1].25), definition of the Lax–Poisson brackets ([1].5) and by the string equation ([1].6) this value reduces to:

$$\begin{aligned} &\oint_{w_i} \frac{\partial}{\partial w} \left(\bar{z} \frac{\partial z}{\partial \tau_j} \right) dw + \oint_{w_i} \left(\{\mathcal{H}_j, \bar{z}\} \frac{\partial z}{\partial w} - \{\mathcal{H}_j, z\} \frac{\partial \bar{z}}{\partial w} \right) dw \\ &\quad = \oint_{w_i} \frac{\partial}{\partial w} \left(\bar{z} \frac{\partial z}{\partial \tau_j} \right) dw + \oint_{w_i} \frac{\partial \mathcal{H}_j}{\partial w} w \left(\frac{\partial z}{\partial w} \frac{\partial \bar{z}}{\partial x} - \frac{\partial z}{\partial x} \frac{\partial \bar{z}}{\partial w} \right) dw \\ &\quad = \oint_{w_i} \frac{\partial}{\partial w} \left(\bar{z} \frac{\partial z}{\partial \tau_j} \right) dw + \oint_{w_i} \frac{\partial \mathcal{H}_j}{\partial w} dw = 2\pi\sqrt{-1}\delta_{ij}. \end{aligned}$$

We get the Kronecker δ-symbol as the final answer, since the first term in the last line vanishes. Indeed, this occurs because $\bar{z}\frac{\partial z}{\partial \tau_j}$ is a meromorphic in w in the vicinity of any singularity w_i of z and the second term gives the δ_{ij} contribution by definition ([1].24) of $\mathcal{H}_j$.

Using similar reasoning we prove the rest of the proposition. ∎

The values (1) are similar to those appearing in the work [8] in connection with the Laplacian Growth problem as integrals of the Laplacian Growth (string) equation. The following proposition (although not assuming complex structure) confirms this fact.

Proposition 2. *The values* (1) *form a set of integrals of the string equation* ([1].6)*, i.e., provided the string equation holds.*

$$dI_i/dx = 0$$

while for the Casimir (2) *we have*

$$dQ/dx = 1.$$

Proof. This is similar to the proof of proposition (1); one has to take the same steps substituting x for τ_i. ■

As was mentioned in the Introduction the Laplacian growth problem is endowed with a complex structure, with $\bar{z}$ being the complex conjugate of z (and bar denoting complex conjugation). Then, as may be seen from the proposition, the Casimir Q plays the role of the area (which grows with the unit speed), while I are functions of the harmonic moments of the boundary curve.

Since the equations of motion ([1].25) have the following form in I, $\bar{I}$

$$\begin{aligned} &\partial_{\tau_j} Q = 0, && \partial_{\bar{\tau}_j} Q = 0, && \partial_{\tau_j} I_i = \delta_{ij}, \\ &\partial_{\bar{\tau}_j} I_i = 0, && \partial_{\tau_j} \bar{I}_i = 0, && \partial_{\bar{\tau}_j} \bar{I}_i = -\delta_{ij} \end{aligned} \tag{3}$$

these variables are canonical. For instance, we may choose the following Poisson structure

$$\{I_i, \bar{I}_j\}_p = \delta_{ij}, \qquad \{I_i, I_j\}_p = \{\bar{I}_i, \bar{I}_j\}_p = 0, \qquad \{Q, \bar{I}_j\}_p = \{Q, I_j\}_p = 0, \tag{4}$$

$$\mathrm{H}_i = \bar{I}_i, \qquad \bar{\mathrm{H}}_i = I_i. \tag{5}$$

It is important to note that the Poisson brackets $\{\,,\,\}_p$ in (4) are different from $\{\,,\,\}$ in ([1].5): The former defines the Poisson structure on the space of rational solutions of the 2dToda hierarchy, while the latter (Lax–Poisson bracket) is a dispersionless limit of the commutator.

To get similar results for the rational case we have to take a limit as in Corollary 1 of [1].

Corollary 1. *Let z, $\bar{z}$ be represented as in* ([1].13)–([1].16). *The following* $2N+1$ *values* $(i = 1, \ldots, N)$

$$I_0 = \bar{u}_0 - \sum_{i=1}^{N} \bar{u}_i/\bar{w}_i, \qquad \bar{I}_0 = u_0 - \sum_{i=1}^{N} u_i/w_i,$$

$$I_{2i-1} = r w_i^{-1} + \bar{u}_0 + \sum_{j=1}^{N} \frac{\bar{u}_j}{1/w_i - \bar{w}_j}, \qquad \bar{I}_{2i-1} = r \bar{w}_i^{-1} + u_0 + \sum_{j=1}^{N} \frac{u_j}{1/\bar{w}_i - w_j},$$

$$I_{2i} = \left(r - \sum_{j=1}^{N} \frac{\bar{u}_j}{(1/w_i - \bar{w}_j)^2} \right) \frac{u_i}{w_i^2}, \qquad \bar{I}_{2i} = \left(r - \sum_{j=1}^{N} \frac{u_j}{(1/\bar{w}_i - w_j)^2} \right) \frac{\bar{u}_i}{\bar{w}_i^2}$$

$$Q = r^2 - \frac{1}{2} \sum_{i=1}^{N} \left(r \left(\frac{\bar{u}_i}{\bar{w}_i^2} + \frac{u_i}{w_i^2} \right) + \bar{I}_{2i} + I_{2i} \right)$$

are the action-angle variables in the rational case [1, Proposition 3], i.e. the variables for which equations in this Proposition have the form (3).

Again, in the rational limit we may choose the Poisson structure as in (4), (5).

2 Poisson structure of the 1dToda hierarchy

In this section we consider the Poisson structure of rational reductions of the 1dToda system.

Recall, that for the 1dToda system one takes into account only a "half" of flows, connected with z (but not $\bar{z}$, or vice verse).

$$\partial_{t_i} z = \{H_i, z\}, \qquad H_i = \left(z(w)^i\right)_+ + \tfrac{1}{2}\left(z(w)^i\right)_0, \qquad i = 0, \ldots, \infty. \tag{6}$$

This system is bi-Hamiltonian (for general information e.g. see [3]) with two (linear and quadratic) compatible Poisson structures. As it was first found in [2] for the generic Toda system ([1].2), the dispersionless linear Poisson brackets for the "field variables" u_i, $i = 1, \dots, \infty$ have the following form

$$\{u_n(x), u_m(y)\}_1 = 2(c_n + c_m - 1)\big[(n+m)u_{n+m}(x)\delta'(x-y) + mu'_{n+m}(x)\delta(x-y)\big], \qquad (7)$$

where

$$c_k = \begin{cases} 1, & \text{if } k > 0, \\ \frac{1}{2}, & \text{if } k = 0, \\ 0, & \text{if } k < 0 \end{cases}$$

while the quadratic brackets are

$$\begin{aligned}\{u_n(x), u_m(y)\}_2 = &\left[\frac{1}{2}(n-m)u_n(x)u'_m(x) + \left(\sum_{k=1}^{1-n}(n-m+k)u_{n+k}(x)u'_{m-k}(x)\right.\right.\\ &\left.\left. + ku'_{n+k}(x)u_{m-k}(x)\right)\right]\delta(x-y)\\ &+\left[\frac{1}{2}(n-m)u_n u_m + \sum_{k-1}^{1-n}(n-m+2k)u_{n+k}u_{m-k}\right]\delta(x-y).\end{aligned} \qquad (8)$$

As seen from the Lemma 1 of [1], the rational functions

$$z(w) = \frac{q_{N+1}(w)}{p_N(w)} = \frac{w^{N+1} + \sum\limits_{i=0}^{N} a_i w^i}{\sum\limits_{i=0}^{N} b_i w^i} \qquad (9)$$

are form-invariant under all of the 1dToda flows ∂_{t_i} (6), without any extra restriction (i.e. the string equation is not needed).

We obtain the corresponding Poisson structures for coefficients a_i, b_i, by using result (8), expressing u_i in terms of a_i, b_i, $i = 0, \dots, N$. Both the linear (7) and quadratic (8) Poisson structures lead to the quadratic brackets for a_i, b_i. Namely, the second Poisson structure for (9) reads as follows

$$\begin{aligned}\{a_k(x), a_l(y)\}_2 = &\left[\sum_{n=1}(l+n-k)a_{k-n}(x)a_{l+n}(y) + na_{k-n}(y)a_{l+n}(x))\right.\\ &\left. + (l-N-1)a_k(x)a_l(y)\right]\delta'(x-y),\end{aligned} \qquad (10)$$

$$\begin{aligned}\{b_k(x), b_l(y)\}_2 = &\left[\sum_{n=1}(k-l-n)b_{k-n}(x)b_{l+n}(y) - nb_{k-n}(y)b_{l+n}(x))\right.\\ &\left. + \frac{k-l}{2}b_k(x)b_l(y)\right]\delta'(x-y),\end{aligned} \qquad (11)$$

$$\{a_k(x), b_l(y)\}_2 = \frac{k-N-1}{2}a_k(x)b_l(y)\delta'(x-y). \qquad (12)$$

The first Poisson structure can be obtained from (10)–(12) with the help of the linear transformation (shift by a constant)

$$a_i \to a_i + \lambda b_i, \qquad z(w,x) \to z(w,x) + \lambda$$

and using the bi-Hamiltonian nature of (7), (8).

$$\begin{aligned}\{a_k(x), a_l(y)\}_1 = \Bigg[\sum_{n=1} \Big(&(k-l-n)(a_{k-n}(x) b_{l+n}(y) + b_{k-n}(x) a_{l+n}(y)) \\ &- n(a_{k-n}(y) b_{l+n}(x) + b_{k-n}(y) a_{l+n}(x)) \Big) + \frac{N+1-l}{2} b_k(x) a_l(y) \\ &+ \frac{k+N+1-2l}{2} a_k(x) b_l(y) \Bigg] \delta'(x-y), \end{aligned} \tag{13}$$

$$\begin{aligned}\{a_k(x), b_l(y)\}_1 = \Bigg[\sum_{n=1} \Big(&(k-l-n) b_{k-n}(x) b_{l+n}(y) - n b_{k-n}(y) b_{l+n}(x) \Big) \\ &+ \frac{N+1-l}{2} b_k(x) b_l(y) \Bigg] \delta'(x-y), \end{aligned} \tag{14}$$

$$\{b_k(x), b_l(y)\}_1 = 0 \tag{15}$$

in all the above expressions $a_{N+1} = 1$ and $a_i = 0$ if i goes beyond the range $i = 0, \ldots, N+1$ (and $b_j = 0$ if $j \neq 0, \ldots, N$).

These brackets form a bi-Hamiltonian structure for rational reductions of the 1dToda hierarchy:

$$\partial_{t_i} z = \{\mathrm{H}_i, z\}_1 = \{\mathrm{H}_{i-1}, z\}_2 \tag{16}$$

with the following Hamiltonians:

$$\mathrm{H}_i = \frac{1}{i+1} \int (z^{i+1}(x))_0 dx. \tag{17}$$

3 Summary and perspectives

We have established the Hamiltonian structure on the space of rational solutions of the 2dToda hierarchy connected with the problem of ideal interface dynamics. A further application of this result can become quite challenging in dealing with the problem of encountering a surface tension for the Laplacian growth process.

The Laplacian growth equation ([1].1), ([1].6) describes the propagation of the boundary with zero surface tension between the fields. Although, addressing a number of important questions (e.g. finger width selection problem in [7]), such an idealized model does not account for essential physical features, such as fractal formation, stability etc. Thus, the inclusion of tension effects is important for the solution of the problem.

There are two approaches to look for in such a generalization. The first one is to introduce tension terms in the theory, destroying the integrability of the problem. In another approach one might look for integrable deformations of the idealized model, simulating surface effects and stabilizing interface dynamics.

Another feature that gives our result a certain interest lies in the possible investigation of the perturbed (by small surface tension) system given in terms of separated variables (4). In other words, is it possible, in some situations, to approximate the perturbed solution by the multi-finger ansatz ([1].23) with different dependence of coordinates on time x? If the answer is affirmative we will get a finite-dimensional dynamical system, conveniently written in terms of I, $\bar{I}$, Q.

Finally, returning to the idealized problem, one can further elaborate the theory of rational reductions from the point of view of symmetries. An important observation of our study is that once a rational reduction is compatible with 2dToda dynamics, the string condition is satisfied automatically. In other words *the string (Laplacian growth) equation* ([1].6) *turns out to be a consequence of rationality* in the context of the 2dToda hierarchy. On the other hand, it is well known that the string equation is connected with additional (Orlov–Schulman [3, 9]) symmetries of the 2dToda hierarchy [10]. It would be interesting to see how the string constraint can be an element of a set of the symmetry constraints leading to the rationality of solutions.

Amongst the other open questions and future directions is the following problem: to investigate Hamiltonian structure of rational reductions in the context of the two-matrix [4] model, whose partition function is a tau-function of the 2dToda system constrained by the string equation. The applicability of our study to the models of normal matrices is another interesting aspect worthy of further analysis [6].

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