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Towards the Group Classification of Control Systems

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We consider the group classification problem for the class of control systems which is described by systems of ordinary differential equations, and we give the full group classification for the second-order control systems.

1 Introduction

Recently there has been quite a few publications dealing with the symmetries of control systems. Nevertheless, most of them deal with particular symmetry problems for concrete control systems. Others in some, perhaps hidden, form attempt to solve the problem of finding symmetries for the "general case". It means that the problem is formulated for control systems of the form

$$\dot{x}^i = f^i\left(t, x, u\right), \qquad i = \overline{1, n}, \quad t \in \mathbb{T} \subset \mathbb{R}_+, \quad u \in \mathbb{U}^* \subset \mathbb{R}^r, \quad x \in \mathbb{X}^* \subset \mathbb{R}^n \tag{1}$$

for arbitrary $\{f^i(\cdot), n, r\}$. Symmetry conditions for the coefficients τ, ξ^i, φ^j of the infinitesimal generator of the form

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi^i(t, x, u)\frac{\partial}{\partial x^i} + \varphi^j(t, x, u)\frac{\partial}{\partial u^j} \tag{2}$$

read as

$$Xf^i - X_0\xi^i + f^i X_0\tau = 0, \qquad U_j\xi^i + f^i U_j\tau = 0, \qquad i = \overline{1, n}, \quad j = \overline{1, r}, \tag{3}$$

where

$$X_0 = \frac{\partial}{\partial t} + f^i\frac{\partial}{\partial x^i}, \qquad U_j = \frac{\partial}{\partial u^j}. \tag{4}$$

Up to now there is no recipe for solving system (3) with arbitrary functions $f^i(\cdot)$ and arbitrary numbers (n, r). So, we need to fix something. Following L. Ovsiannikov's approach, we will describe our systems by fixing (n, r) for arbitrary $f^i(\cdot)$ and, hence, we will consider a classical group classification problem for some class of control systems. In the present paper we consider the case $n = 2$, but before that we give some remarks on a definition of equivalence of control systems.

2 Feedback equivalence and essential controls of control systems

Definition 1. System (1) and any system which can be derived from (1) via a diffeomorphism $\hat{t} = \hat{t}(t, x), \hat{x} = \hat{x}(t, x), \hat{u} = \hat{u}(t, x, u)$ are called feedback equivalent control systems.

Based on this definition we can define also the notion of so called "essential controls", a natural generalization of the notion of "essential parameters" (see, for example, [2]).

Definition 2. Controls u^j are called essential ones if there are no controls v^k $(k < r)$, for which $f^i(t, x, u) = \hat{f}^i(t, x, v)$.

Using Definitions 1 and 2 we can give the following proposition.

Proposition 1. *Any control system with r essential controls is feedback equivalent to the system of the (canonical) form*

$$\begin{aligned}
&\dot{x}^1 = f^1(t, x, u),\\
&\dot{x}^2 = f^2(t, x, u),\\
&\cdots\cdots\cdots\\
&\dot{x}^{n-r} = f^{n-r}(t, x, u),\\
&\dot{x}^{n-r+1} = u^1,\\
&\dot{x}^{n-r+2} = u^2,\\
&\cdots\cdots\cdots\\
&\dot{x}^n = u^r.
\end{aligned} \tag{5}$$

It follows from Proposition 1 that there is only one canonical one-dimensional ($n = 1$) control system $\dot{x} = u$, so the group classification of this system is trivial. Thus, we start from the case $n = 2$.

3 Canonical forms for second-order control systems

Obviously (see Proposition 1), there exist only two canonical forms for second-order control systems. One of them (with 2 controls) reads

$$\begin{aligned}
&\dot{x}^1 = u^1,\\
&\dot{x}^2 = u^2,
\end{aligned} \tag{6}$$

and the second one (with a single control) looks like

$$\begin{aligned}
&\dot{x}^1 = F\left(t, x^1, x^2, u\right),\\
&\dot{x}^2 = u.
\end{aligned} \tag{7}$$

Of course, we ignore the case $u^j = 0$. System (6) does not contain any arbitrary elements and its symmetry generator is given by the direct sum $Y = Y_1 \oplus Y_2 \oplus Y_3$ with

$$\begin{aligned}
&Y_1 = \tau\partial_t - Y_0\tau\left(u^1\partial_{u^1} + u^2\partial_{u^2}\right),\\
&Y_2 = \xi^1\partial_{x^1} + \left(Y_0\xi^1\right)\partial_{u^1}, \qquad Y_3 = \xi^2\partial_{x^2} + \left(Y_0\xi^2\right)\partial_{u^2},
\end{aligned} \tag{8}$$

with $\tau = \tau(t, x)$, $\xi^i = \xi^i(t, x)$ arbitrary functions and $Y_0 = \partial_t + u^1\partial_{u^1} + u^2\partial_{u^2}$.

4 Controllability

Controllability (strong accessibility) is known to be directly related to the existence of invariants (first integrals). As we can see below, controllability plays an important role in the group properties of control systems.

Definition 3. We call system (7) (locally) uncontrollable, if (locally) all its solutions (trajectories) lie in an invariant manifold of the form $\omega\left(t, x^1, x^2\right) = C$.

Thus, the existence of such a manifold means that there exists a function ω depending on (t, x^1, x^2) such that $\dot{\omega} = 0$ (i.e., ω is invariant under the action of the input u).

We will now discuss for which choice of $F\left(t, x^1, x^2, u\right)$ in (7) there exist such invariants. To this end we interpret the invariant $\omega\left(t, x^1, x^2\right) = C$ as a nontrivial solution of the system of partial differential equations

$$\frac{\partial \omega}{\partial t} + F\left(t, x^1, x^2, u\right)\frac{\partial \omega}{\partial x^1} + u\frac{\partial \omega}{\partial x^2} = 0, \qquad \frac{\partial \omega}{\partial u} = 0, \tag{9}$$

where the first equation means that ω must be a first integral of system (7), and the second condition means that this first integral must be independent of u. If we denote

$$X_0 = \frac{\partial}{\partial t} + F\left(t, x^1, x^2, u\right)\frac{\partial}{\partial x^1} + u\frac{\partial}{\partial x^2}, \qquad U = \frac{\partial}{\partial u}, \tag{10}$$

system (9) takes the form

$$X_0\omega = 0, \qquad U\omega = 0. \tag{11}$$

In order to look for the functionally independent solutions of system (11), we need to calculate the involutive closure [3] for a distribution of vector fields given as (10). We get the Lie brackets

$$\begin{aligned} &X_1 = [U, X_0] = F_u\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \qquad X_2 = [U, X_1] = F_{uu}\frac{\partial}{\partial x^1}, \\ &X_3 = [X_0, X_1] = \left(F_{ut} + FF_{ux^1} + uF_{ux^2} - F_uF_{x^1} - F_{x^2}\right)\frac{\partial}{\partial x^1}. \end{aligned} \tag{12}$$

System (7) admits a first integral if for the above vector fields the two conditions $U \wedge X_0 \wedge X_1 \wedge X_2 = 0$ and $U \wedge X_0 \wedge X_1 \wedge X_3 = 0$ are fulfilled simultaneously. The first condition reads

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & F & u \\ 0 & 0 & F_u & 1 \\ 0 & 0 & F_{uu} & 0 \end{vmatrix} = F_{uu} = 0, \tag{13}$$

the second is

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & F & u \\ 0 & 0 & F_u & 1 \\ 0 & 0 & F_{ut} + FF_{ux^1} + uF_{ux^2} - F_uF_{x^1} - F_{x^2} & 0 \end{vmatrix} = 0. \tag{14}$$

Therefore, system (7) is uncontrollable if and only if the function F satisfies the following two conditions:

$$F_{uu} = 0, \qquad F_{ut} + FF_{ux^1} + uF_{ux^2} - F_uF_{x^1} - F_{x^2} = 0. \tag{15}$$

One direct implication of these results is the following.

Corollary 1. *System (7) is uncontrollable if and only if $F = \alpha\left(t, x^1, x^2\right)u + \beta\left(t, x^1, x^2\right)$ and the coefficients α and β satisfy*

$$\alpha\beta_{x^1} - \beta\alpha_{x^1} - \alpha_t + \beta_{x^2} = 0.$$

Example 1. The linear system

$$\dot{x}^1 = a_1 x^1 + a_2 x^2 + bu,$$
$$\dot{x}^2 = u$$

will be uncontrollable if and only if the condition $ba_1 + a_2 = 0$ holds true. Indeed, elimination of u leads us to the Pfaffian equation

$$dx^1 - bdx^2 + a_1 \left(bx^2 - x^1\right) dt = 0,$$

which has solutions of the form

$$\omega(t, x^1, x^2) = \left(bx^2 - x^1\right) e^{-a_1 t} = C.$$

5 Group classification for system (7)

As we analyze a whole class of systems of the form (7), with arbitrary function F, there naturally occurs a problem of group classification [4]: For the class of systems (7) find the kernel of basic groups GE_0 and all specializations of the arbitrary function F which allow us to extend the basic groups.

Notice that in accordance with the terminology of the book [4], the kernel of basic groups is the symmetry group admitted by system (7) for all specializations of the function F. Therefore, in our case the kernel of basic groups is empty (i.e., $GE_0 = \{0\}$), because the function F depends on all variables.

Hence, we start our analysis deriving the so-called *determining equations*. In order to determine the coefficients $\tau\left(t, x^1, x^2, u\right)$, $\xi^i\left(t, x^1, x^2, u\right)$, $\varphi\left(t, x^1, x^2, u\right)$ of the infinitesimal symmetry operator

$$X = \tau \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x^1} + \xi^2 \frac{\partial}{\partial x^2} + \varphi \frac{\partial}{\partial u} \tag{16}$$

one needs to solve the system

$$Xf^i - X_0 \xi^i + f^i X_0 \tau = 0, \qquad U\xi^i + f^i U\tau = 0, \qquad i = 1, 2, \tag{17}$$

where $f^1 = F$, $f^2 = u$ (see also [5]). Substituting

$$\xi^i = f^i \tau + \hat{\xi}^i \tag{18}$$

in (17) we get

$$\hat{X} f^i - X_0 \hat{\xi}^i = 0 \qquad U\hat{\xi}^i + U(f^i)\tau = 0, \qquad i = 1, 2, \tag{19}$$

where

$$\hat{X} = \hat{\xi}^1 \frac{\partial}{\partial x^1} + \hat{\xi}^2 \frac{\partial}{\partial x^2} + \varphi \frac{\partial}{\partial u}.$$

If in (19) we replace f^2 by u we obtain

$$\varphi = X_0 \hat{\xi}^2, \qquad \tau = -U\hat{\xi}^2. \tag{20}$$

Now substituting (τ, φ) in (17) and replacing f^1 by F we get

$$\hat{\xi}^1 F_{x^1} + \hat{\xi}^2 F_{x^2} + F_u X_0 \hat{\xi}^2 - X_0 \hat{\xi}^1 = 0, \qquad U\hat{\xi}^1 - F_u U\hat{\xi}^2 = 0. \tag{21}$$

If we introduce

$$\sigma = \hat{\xi}^1 - F_u \hat{\xi}^2, \tag{22}$$

from the second equation of (21) we get

$$F_{uu} \hat{\xi}^2 + \sigma_u = 0. \tag{23}$$

Assuming $F_{uu} \neq 0$, solve (22) and (23) for $(\hat{\xi}^1, \hat{\xi}^2)$:

$$\hat{\xi}^1 = \sigma - F_u F_{uu}^{-1} \sigma_u, \qquad \hat{\xi}^2 = -F_{uu}^{-1} \sigma_u. \tag{24}$$

Substituting (24) in the first equation of the system (21), we finally obtain

$$F_{uu}(\sigma_t + F\sigma_{x^1} + u\sigma_{x^2}) + (F_u F_{x^1} + F_{x^2} - F_{tu} - FF_{ux^1} - uF_{ux^2})\sigma_u = F_{uu} F_{x^1} \sigma. \tag{25}$$

Using some substitutions, we reduced our problem to the single equation (25) for the function σ. In accordance with the terminology of [6], this function is called the generating function of symmetries. Indeed, if we are able to solve equation (25) for a particular specialization of the function F then all coefficients can be calculated via

$$\begin{aligned}
\tau &= U\left(F_{uu}^{-1}\sigma_u\right), \\
\xi^1 &= \sigma + FU\left(F_{uu}^{-1}\sigma_u\right) - F_u F_{uu}^{-1}\sigma_u, \\
\xi^2 &= uU\left(F_{uu}^{-1}\sigma_u\right) - F_{uu}^{-1}\sigma_u, \\
\varphi &= -X_0\left(F_{uu}^{-1}\sigma_u\right),
\end{aligned}$$

where $U = \partial_u$. Equation (25) is a quasi-linear inhomogeneous partial differential equation. The corresponding system of characteristics for this equation reads as

$$\begin{aligned}
\dot{x}^1 &= F(t, x^1, x^2, u), \\
\dot{x}^2 &= u, \\
\dot{u} &= F_{uu}^{-1}(F_u F_{x^1} + F_{x^2} - F_{tu} - FF_{ux^1} - uF_{ux^2}), \\
\dot{\sigma} &= F_{x^1}\sigma.
\end{aligned} \tag{26}$$

Thus, the problem of calculation of the point symmetries for (open loop) second-order control systems is reduced to the problem of integration of (closed loop) system (26). Conversely, if we know some symmetries (coefficients τ, ξ^1, ξ^2), we are able to give some first integral of system (26). Indeed, converting formulas (18), (22), (24), we have

$$\sigma = \xi^1 - F_u \xi^2 + (uF_u - F)\tau. \tag{27}$$

Remark 1. With system (7) is associated the Pfaffian system $I = \left\{\omega^1, \omega^2\right\}$, where

$$\omega^1 = dx^1 - Fdt, \qquad \omega^2 = dx^2 - udt \tag{28}$$

on a subset of $\mathbb{R}^4$. With I is associated the so-called derived flag [1]. The first derived system $I^{(1)}$ includes the single one-form ω:

$$I^{(1)} = \left\{\omega = dx^1 + (uF_u - F)dt - F_u dx^2\right\}. \tag{29}$$

With this we are in a position to define the formula (27) in the intrinsic form

$$\sigma = X \rfloor \omega. \tag{30}$$

where ω is the generator of the first derived flag $I^{(1)}$, $\rfloor$ the inner product, and X the symmetry generator (16).

The conditions (15) for uncontrollability play the role of the classification conditions for equation (25), and if they are met the symmetry algebra is wider (see also [5] for further details).

Let us consider some examples.

Example 2. The linear time-varying system

$$\dot{x}^1 = tu + x^2,$$
$$\dot{x}^2 = u$$

is uncontrollable. Condition (15) is fulfilled and the system has a first integral

$$x^1 - tx^2 = C.$$

The symmetry algebra for this system is spanned by the vector fields

$$\begin{aligned} X_1 &= \gamma(x^1 - tx^2)\partial_{x^1}, \\ X_2 &= -\sigma_u\partial_t + \left(t\sigma - \left(tu + x^2\right)\sigma_u\right)\partial_{x^1} + (\sigma - u\sigma_u)\partial_{x^2} \\ &\quad + \left(\sigma_{tu} + \left(tu + x^2\right)\sigma_{ux^1} + u\sigma_{ux^2}\right)\partial_u, \end{aligned}$$

where $\gamma\left(x^1 - tx^2\right)$ and $\sigma\left(t, x^1, x^2, u\right)$ are arbitrary functions of the given arguments.

Example 3. For the system in Brunovský normal form

$$\dot{x}^1 = x^2,$$
$$\dot{x}^2 = u$$

one has $F_{uu} = 0$. Thus, in accordance with (25), $\sigma_u = 0$ and $\sigma = \sigma\left(t, x^1, x^2\right)$. The infinitesimal symmetry operator takes the form

$$X = -\sigma_{x^2}\partial_t + \left(\sigma - x^2\sigma_{x^2}\right)\partial_{x^1} + \left(\sigma_t + x^2\sigma_{x^1}\right)\partial_{x^2} + X_0^2(\sigma)\partial_u,$$

where $X_0 = \partial_t + x^2\partial_{x^1} + u\partial_{x^2}$, $\sigma = \sigma\left(t, x^1, x^2\right)$.

Example 4. For the nonlinear system

$$\dot{x}^1 = u^2,$$
$$\dot{x}^2 = u$$

the equation $F_uF_{x^1} + F_{x^2} - F_{tu} - FF_{ux^1} - uF_{ux^2} = 0$ is fulfilled, so (from (25)) it follows that $X_0\sigma = 0$, and $\sigma = \sigma\left(u, x^1 - tu^2, x^2 - tu\right)$. The symmetry operator is

$$X = \sigma_{uu}\partial_t + \left(u^2\sigma_{uu} - 2u\sigma_u + \sigma\right)\partial_{x^1} + (u\sigma_{uu} - \sigma_u)\partial_{x^2} - \left(\sigma_{tu} + u^2\sigma_{x^1u} + u\sigma_{x^2u}\right)\partial_u.$$

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