Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 1, 153–158

On Realizations of the Conformal Algebra and Nonlinear Invariant Equations in Low-Dimensional Space-Time

Victor LAHNO

Poltava State Pedagogical University, 2 Ostrograds'kogo Str., 36003 Poltava, Ukraine
E-mail: *lvi@pdpu.septor.net.ua, laggo@poltava.bank.gov.ua*

This paper devoted to description of conformally-invariant PDE in low-dimensional spaces.

1 Introduction

This talk is devoted to the question, "What is the most general partial differential equation invariant under a given local Lie group?"

Principal object of our consideration is the second-order partial differential equation

$$F(x, u, \underset{1}{u}, \underset{2}{u}) = 0 \tag{1}$$

invariant under the conformal group $C(1, n)$ and its subgroups – the Poincaré group $P(1, n)$ and extended Poincaré group $\tilde{P}(1, n)$ acting as Lie transformation groups in the space $V = \mathbb{R}(1, n) \times \mathbb{R}^r$ of independent and dependent variables, where $\mathbb{R}(1, n)$ is the pseudo-Euclidean space with the metric tensor

$$g_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta = 0, \\ 0, & \alpha \neq \beta, \\ -1, & \alpha = \beta = 1, 2, \ldots, n. \end{cases} \tag{2}$$

The Lie algorithm requires using for construction of invariant equations not local transformations groups, but the respective Lie algebras of symmetry operators (realizations of Lie algebras in the class of first-order linear differential operators). In our case these linear operators have the form

$$Q = \xi_\mu(x, u)\partial_{x_\mu} + \eta_k(x, u)\partial_{u_k}, \qquad \mu = 0, 1, \ldots, n, \quad k = 1, 2, \ldots, r. \tag{3}$$

A set of $1 + n + C^2_{1+n}$ differential operators P_μ, $J_{\alpha\beta} = -J_{\beta\alpha}$, $\mu, \alpha, \beta = 0, 1, \ldots, n$ of the form (3) satisfying the commutation relations

$$\begin{aligned} &[P_\alpha, P_\beta] = 0, \qquad [P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta} P_\gamma - g_{\alpha\gamma} P_\beta, \\ &[J_{\alpha\beta}, J_{\mu\nu}] = g_{\alpha\nu} J_{\beta\mu} + g_{\beta\mu} J_{\alpha\nu} - g_{\alpha\mu} J_{\beta\nu} - g_{\beta\nu} J_{\alpha\mu} \end{aligned} \tag{4}$$

is called a realization of the Poincaré algebra $p(1, n)$.

A set of $2 + n + C^2_{n+1}$ differential operators $P_\mu, J_{\alpha\beta}, D$ of the form (3) satisfying the commutation relations (4) and

$$[D, J_{\alpha\beta}] = 0, \qquad [P_\alpha, D] = P_\alpha \tag{5}$$

is called a realization the extended Poincaré algebra $\tilde{p}(n, m)$.

A set of $3+2n+C^2_{1+n}$ differential operators P_μ, $J_{\alpha\beta}$, D, K_μ of the form (3) satisfying the commutation relations (4), (5) and

$$[K_\alpha, K_\beta] = 0, \qquad [K_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}K_\gamma - g_{\alpha\gamma}K_\beta,$$
$$[P_\alpha, K_\beta] = 2(g_{\alpha\beta}D - J_{\alpha\beta}), \qquad [D, K_\alpha] = K_\alpha$$

is called a realization of the conformal algebra $c(1,n)$.

The following realization of the conformal algebra $c(1,n)$ is well-known in many problems of mathematical physics:

$$P_\mu = \partial_{x_\mu}, \qquad J_{\mu\nu} = g_{\mu\gamma}x_\gamma\partial_{x_\nu} - g_{\nu\gamma}x_\gamma\partial_{x_\mu}, \qquad D = x_\mu\partial_{x_\mu} + \epsilon u_k\partial_{u_k}, \quad \epsilon = 0,1,$$
$$K_\mu = 2g_{\mu\nu}x_\nu D - (g_{\alpha\beta}x_\alpha x_\beta)\partial_{x_\mu}, \qquad \alpha,\beta,\mu,\nu,\gamma = 0,1,\dots,n, \quad k = 1,\dots,r. \tag{6}$$

In the following we call this realization the standard realization of the conformal algebra.

For the realization (6) in the case of arbitrary n and r the problem of description of equations of the form (1) invariant with respect to the Poincaré group, the extended Poincaré group and the conformal group was completely solved by W.I. Fushchych and I.A. Yehorchenko in [1].

However, the following question will be natural: do other realizations of the algebras $p(1,n)$, $\tilde{p}(1,n)$ and $c(1,n)$ in the class of operators (3) exist that would be different from the standard? In this respect the problem of description of non-equivalent realizations for the algebras $p(1,n)$, $\tilde{p}(1,n)$, $c(1,n)$ in the class of operators (3) will be relevant. W. Fushchych, R. Zhdanov and V. Lahno [2] showed that in the case when translation operators have the form (6)

$$P_\mu = \partial_{x_\mu}, \qquad \mu = 0,1,\dots,n,$$

and $V = \mathbb{R}(1,n)\times\mathbb{R}^1$, $n \geq 3$, realizations of the Poincaré algebra $p(1,n)$ and of the extended Poincaré algebra $\tilde{p}(1,n)$ are equivalent to the standard realization (6). The conformal algebra $c(1,n)$, besides the standard realization (6), has also the following one:

$$P_\mu = \partial_{x_\mu}, \qquad J_{\mu\nu} = g_{\mu\gamma}x_\gamma\partial_{x_\nu} - g_{\nu\gamma}x_\gamma\partial_{x_\mu},$$
$$D = x_\mu\partial_{x_\mu} + u\partial_u, \qquad K_\alpha = 2g_{\alpha\beta}x_\beta D - \left(g_{\mu\nu}x_\mu x_\nu \pm u^2\right)\partial_{x_\alpha}.$$

Note, this realization is realized on the set of solutions of the eikonal equation

$$g_{\mu\nu}u_{x_\mu}u_{x_\nu} \pm 1 = 0,$$

and on the set of solutions of d'Alembert–eikonal system

$$g_{\mu\nu}u_{x_\mu}u_{x_\nu} \pm 1 = 0, \qquad g_{\mu\nu}u_{x_\mu x_\nu} \pm nu^{-1} = 0.$$

Realizations for the spaces $V = \mathbb{R}(1,1)\times\mathbb{R}^1$ and $V = \mathbb{R}(1,2)\times\mathbb{R}^1$ were studied in more detail. Below we will consider these results in particular.

2 $\tilde{P}(1,1)$-, $\tilde{P}(1,1)$- and $C(1,1)$-invariant equations

Here $V = \mathbb{R}(1,1)\times\mathbb{R}^1$, $\mathbb{R}(1,1) = \langle t,x\rangle$, $\mathbb{R}^1 = \langle u\rangle$, $u = u(t,x)$. G. Rideau and P. Winternitz [3] considered the issue of description of equations

$$F(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx}) = 0, \tag{7}$$

that are invariant with respect to the groups $P(1,1)$, $\tilde{P}(1,1)$ and $C(1,1)$. For this purpose they first described realizations of the Lie algebras of these groups assuming that

$$P_0 = \partial_t, \qquad P_1 = \partial_x. \tag{8}$$

Their results can be summarized by the following lists:

1. Inequivalent realizations of the Poincaré algebra:

$$p^1(1,1) : \{P_0 = \partial_t,\ P_1 = \partial_x,\ J_{01} = x\partial_x + t\partial_x\};$$
$$p^2(1,1) : \{P_0 = \partial_t,\ P_1 = \partial_x,\ J_{01} = x\partial_t + t\partial_x + u\partial_u\}.$$

2. Inequivalent realizations of the extended Poincaré algebra:

$$\tilde{p}^1(1,1) : \left\{p^1(1,1), D = t\partial_t + x\partial_x\right\};$$
$$\tilde{p}^2(1,1) : \left\{p^1(1,1), D = t\partial_t + x\partial_x + u\partial_u\right\};$$
$$\tilde{p}^3(1,1) : \left\{p^2(1,1), D = \left(t + au + bu^{-1}\right)\partial_t + (x + au - bu^{-1})\partial_x + \lambda u\partial_u\right\},$$

where $\lambda \in \mathbb{R}$, $(a,b) = (0,0)$ or $\lambda = 1$, $(a,b) = (1,0)$ or $\lambda = -1$, $(a,b) = (0,1)$.

3. Inequivalent realizations of the conformal algebra:

$$c^1(1,1) : \left\{\tilde{p}^1(1,1),\ K_0 = \left(t^2 + x^2\right)\partial_t + 2tx\partial_x,\ K_1 = -\left(t^2 + x^2\right)\partial_x - 2tx\partial_t\right\};$$
$$c^2(1,1) : \left\{\tilde{p}^2(1,1),\ K_0 = \left(t^2 + x^2 + au^2\right)\partial_t + 2tx\partial_x + 2tu\partial_u,\right.$$
$$\left. K_1 = -\left(t^2 + x^2 + au^2\right)\partial_x - 2tx\partial_t - 2xu\partial_u,\ a = 0, 1, -1\right\};$$
$$c^3(1,1) : \left\{\tilde{p}^3(1,1)\ (\lambda \in \mathbb{R},\ a = b = 0),\ K_0 = \left(t^2 + x^2 + cu^2 + du^{-2}\right)\partial_t\right.$$
$$\left. + \left(2tx + cu^2 - du^{-2}\right)\partial_x + \left(2u(x + \lambda t) + eu^2 + k\right)\partial_u,\ K_1 = -[J_{01}, K_0]\right\},$$

where if $\lambda \in \mathbb{R}$, $\lambda \neq \pm 1$, then $c = d = e = k = 0$; if $\lambda = 1$, then $d = k = 0$, $c = \pm 1$, $e \in \mathbb{R}$ or $c = 0$, $e = 0, \pm 1$; if $\lambda = -1$, then $c = e = 0$, $d = \pm 1$, $k \in \mathbb{R}$ or $d = 0$, $k = 0, 1$.

The realizations $p^2(1,1)$, $\tilde{p}^3(1,1)$, $c^2(1,1)$ $(a \neq 0)$ and $c^3(1,1)$ are non-equivalent to the standard realizations (6).

Use of non-standard realizations allowed G. Rideau and P. Winternitz to add to the list of equations obtained by W. Fushchych and I. Yehorchenko, new invariant equations. We [4] give up the condition (8) and set the problem of description of all inequivalent realizations of the algebras $p(1,1)$, $\tilde{p}(1,1)$, $c(1,1)$. As a result, we obtained one new realization of the algebra $p(1,1)$

$$p^3(1,1) : \left\{P_0 = \partial_t,\ P_1 = x\partial_t,\ J_{01} = xt\partial_t + \left(x^2 - 1\right)\partial_x\right\}$$

and two new realizations of the algebra $\tilde{p}(1,1)$:

$$p^4(1,1) : \left\{p^3(1,1),\ D = t\partial_t\right\};$$
$$\tilde{p}^5(1,1) : \left\{p^3(1,1),\ D = t\partial_t + u\partial_u\right\}.$$

The $p^3(1,1)$-invariant equation of the form (7) will have following form:

$$\Phi(\mathcal{J}_1, \ldots, \mathcal{J}_5) = 0, \tag{9}$$

where

$$\mathcal{J}_1 = u, \qquad \mathcal{J}_2 = u_t^2\left(x^2 - 1\right), \qquad \mathcal{J}_3 = u_{tt}\left(x^2 - 1\right),$$
$$\mathcal{J}_4 = \left(x^2 - 1\right)^2(u_x u_{tt} - u_t u_{tx}) - x\left(x^2 - 1\right)u_t^2,$$
$$\mathcal{J}_5 = \left(x^2 - 1\right)\left(u_{tt}u_{xx} - u_{tx}^2\right) + 2x\left(x^2 - 1\right)^2(u_x u_{tt} - u_t u_{tx}) - x^2\left(x^2 - 1\right)u_t^2.$$

$\tilde{p}^4(1,1)$- and $\tilde{p}^5(1,1)$-invariant equations have the form

$$\bar{\Phi}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = 0, \tag{10}$$

where

$$\Sigma_1 = \mathcal{J}_1, \qquad \Sigma_2 = \mathcal{J}_2^{-1}\mathcal{J}_3, \qquad \Sigma_3 = \mathcal{J}_2^{-2}\mathcal{J}_4, \qquad \Sigma_4 = \mathcal{J}_2^{-2}\mathcal{J}_5, \quad \text{for} \quad \tilde{p}^4(1,1),$$

and

$$\Sigma_1 = \mathcal{J}_1\mathcal{J}_3, \qquad \Sigma_2 = \mathcal{J}_2, \qquad \Sigma_3 = \mathcal{J}_4, \qquad \Sigma_4 = \mathcal{J}_5, \quad \text{for} \quad \tilde{p}^5(1,1).$$

Notice that (9) and (10) include equations which are generations of the well-known Monge–Amperé equations.

Summing up the above, we see that for second-order scalar differential equations in the two-dimensional space-time we may consider the problem of construction of the general equation invariant with respect to the groups $P(1,1)$, $\tilde{P}(1,1)$, $C(1,1)$ to be resolved completely.

3 $P(1,2)$-, $\tilde{P}(1,2)$- and $C(1,2)$-invariant equations

Let us consider in more detail the results obtained for the equation

$$F(x_\mu, u, u_{x_\mu}, u_{x_\mu x_\nu}) = 0, \qquad \mu,\nu = 0,1,2. \tag{11}$$

As we said above, W. Fushchych and I. Yegorchenko gave full description of equations of the form (11), that are invariant with respect to the groups $P(1,2)$, $\tilde{P}(1,2)$ and $C(1,2)$, generated by the standard realizations of the algebras $p(1,2)$, $\tilde{p}(1,2)$, $c(1,2)$:

$$\begin{gathered}P_\mu = \partial_{x_\mu}, \qquad J_{\mu\nu} = g_{\mu\gamma}\partial_{x_\nu} - g_{\mu\gamma}x_\gamma\partial_{x_\mu}, \\ D = x_\mu\partial_{x_\mu} + \epsilon u\partial_u, \quad \epsilon = 0,1, \qquad K_\mu = 2g_{\mu\nu}x_\nu D - (g_{\alpha\beta}x_\alpha x_\beta)\partial_{x_\mu}, \\ g_{\mu\nu} = \begin{cases} 1, & \mu=\nu=0, \\ 0, & \mu\neq\nu, \\ -1, & \mu=\nu=1,2, \end{cases} \qquad \alpha,\beta,\gamma,\mu,\nu = 0,1,2.\end{gathered} \tag{12}$$

I. Yehorchenko [5] found a new realization of the Poincaré group $p(1,2)$:

$$\begin{gathered}P_\mu = P_{x_\mu}, \qquad J_{01} = x_0\partial_{x_1} + x_1\partial_{x_0} + \sin u\partial_u, \\ J_{02} = x_0\partial_{x_2} + x_2\partial_{x_0} + \cos u\partial_u, \qquad J_{12} = x_2\partial_{x_1} - x_1\partial_{x_2} + \partial_u.\end{gathered} \tag{13}$$

Later W. Fushchych, R. Zhdanov and V. Lahno [2] showed that in the case $P_\mu = \partial_{x_\mu}$ ($\mu = 0,1,2$) non-equivalent realizations of the algebra $p(1,2)$ are exhausted by realizations (12) and (13) and found new realizations of the algebras $\tilde{p}(1,2)$ and $c(1,2)$ as extensions of the realizations (13) of the algebra $p(1,2)$:

$$\begin{aligned}&P_\mu, J_{\mu\nu} \ \text{ have the form (13)}, \qquad D = x_\mu\partial_{x_\mu}, \\ &K_0 = 2x_0D - (g_{\mu\nu}x_\mu x_\nu)\partial_{x_0} - 2(x_2\cos u + x_1\sin u)\partial_u, \\ &K_1 = -2x_1D - (g_{\mu\nu}x_\mu x_\nu)\partial_{x_1} - 2(x_0\sin u - x_2)\partial_u, \\ &K_2 = -2x_2D - (g_{\mu\nu}x_\mu x_\nu)\partial_{x_2} - 2(x_0\cos u + x_1)\partial_u.\end{aligned}$$

These results were later re-discovered in a somehow new form by F. Güngor [6].

I. Yehorchenko [7] also obtained a general form of the equation (11), that is invariant with respect to the realization (13) of the algebra $p(1,2)$.

We aim to obtain the complete description of inequivalent realizations of the algebras $p(1,2)$, $\tilde{p}(1,2)$, $c(1,2)$, that might be invariance algebras for equation of the form (11). The list of results of our calculations [8] is given below. There are four new cases.

New realizations of the algebra $p(1,2)$

Case 1. Let the operators P_μ have the form

$$P_0 = \partial_{x_0}, \qquad P_1 = \partial_{x_1}, \qquad P_2 = x_2\partial_{x_0} + u\partial_{x_1}.$$

Then there is only one class of inequivalent realizations of the algebra $p(1,2)$, whose operators $J_{\mu\nu}$ can be taken as follows:

$$J_{01} = x_1\partial_{x_0} + x_0\partial_{x_1} + u\partial_{x_2} + x_2\partial_u, \qquad J_{02} = x_0x_2\partial_{x_0} + ux_0\partial_{x_1} + \left(x_2^2 - 1\right)\partial_{x_2} + ux_2\partial_u,$$
$$J_{12} = -x_1x_2\partial_{x_0} - x_1u\partial_{x_1} - x_2u\partial_{x_2} - \left(1 + u^2\right)\partial_u.$$

Case 2. Let the operators P_μ have the form

$$P_0 = \partial_{x_0}, \qquad P_1 = \partial_{x_1}, \qquad P_2 = x_2\partial_{x_0} + \varphi\partial_{x_1}, \quad \varphi = \sqrt{x_2^2 - 1}, \quad |x_2| > 1.$$

Then there are four classes of inequivalent realizations of the algebra $p(1,2)$. Their operators $J_{\mu\nu}$ can be taken in the following form:

$$J_{01} = x_1\partial_{x_0} + x_0\partial_{x_1} + \varphi\partial_{x_2}, \qquad J_{02} = (x_0x_2 + a)\partial_{x_0} + (x_0\varphi + b)\partial_{x_1} + \varphi^2\partial_{x_2} + q\partial_u,$$
$$J_{12} = (-x_1x_2 + \alpha)\partial_{x_0} + (-x_1\varphi + \beta)\partial_{x_1} - x_2\varphi\partial_{x_2} + p\partial_u.$$

Here a, b, α, β, q, p take the following values:

1. $a = \beta = \lambda = \text{const},\ \alpha = b = -\epsilon e^{-2u},\ \epsilon = 0, 1,\ q = -x_2,\ p = \varphi;$
2. $a = \beta = \lambda_1\left[\dfrac{u}{1-u^2} + \dfrac{1}{2}\ln\left|\dfrac{1+u}{1-u}\right|\right] + \lambda_2,$
   $\alpha = b = \lambda_1[1-u^2]^{-1},\ q = \varphi - x_2u,\ p = \varphi u - x_2,\ \lambda_1, \lambda_2 = \text{const};$
3. $a = -\beta = \lambda x_2\varphi,\ b = \lambda x_2^2,\ \alpha = -\lambda\varphi^2,\ q = p = 0,\ \lambda = \text{const};$
4. $a = -\beta = x_2\varphi u,\ b = x_2^2u,\ \alpha = -\varphi^2u,\ q = p = 0.$

Case 3. Let the operators P_μ have the form

$$P_0 = \partial_{x_0}, \qquad P_1 = x_1\partial_{x_0}, \qquad P_2 = \psi\partial_{x_0}, \qquad \psi = \pm\sqrt{1 - x_1^2}, \quad |x_1| < 1.$$

Then there are three classes of inequivalent realizations of the algebra $p(1,2)$. Their operators $J_{\mu\nu}$ can be taken in the following form:

$$J_{01} = (x_0x_1 + B\psi)\partial_{x_0} - \psi^2\partial_{x_1} + (Cx_1 + D\psi)\partial_{x_2} + A\psi\partial_u,$$
$$J_{02} = (x_0\psi - x_1B)\partial_{x_0} + x_1\psi\partial_{x_1} + (C\psi - x_1D)\partial_{x_2} - Ax_1\partial_u,$$
$$J_{12} = \psi\partial_{x_1}.$$

The variables A, B, C, D take the following values:

1. $A = B = C = D = 0;$
2. $A = \sqrt{|x_2|}g(u),\ B = x_2,\ C = 2x_2,\ D = x_2\sqrt{|x_2|}f(u);$
3. $A = x_2f(u),\ B = 0,\ C = x_2,\ D = x_2^2g(u).$

Here f and g are some sufficiently smooth real–valued functions.

Case 4. Finally, let the operators P_μ have the form

$$P_0 = \partial_{x_0}, \qquad P_1 = x_1\partial_{x_0}, \qquad P_2 = x_2\partial_{x_0}.$$

Then there are two classes of inequivalent realizations of the algebra $p(1,2)$. Their operators $J_{\mu\nu}$ can be taken in the following form:

$$J_{12} = x_2\partial_{x_1} - x_1\partial_{x_2}, \qquad J_{01} = (x_0x_1 + x_2\theta)\partial_{x_0} + \left(x_1^2 - 1\right)\partial_{x_1} + x_1x_2\partial_{x_2} + x_2\rho\partial_u,$$
$$J_{02} = (x_0x_2 - x_1\theta)\partial_{x_0} + x_1x_2\partial_{x_1} + \left(x_2^2 - 1\right)\partial_{x_2} - x_1\rho\partial_u.$$

The variables θ, ρ take the following values:

1. $\theta = f(u)\left(1 - \omega^{-1}\right),\ \omega = x_1^2 + x_2^2,\ \rho = 0;$
2. $\theta = 0,\ \rho = \omega^{-1}\sqrt{|\omega - 1|},\ \omega = x_1^2 + x_2^2.$

Here f is some sufficiently smooth real-valued function.

Notice that the obtained realizations can be extended to the realizations of the algebra $\tilde{p}(1,2)$ and cannot be extended to the realizations of the algebra $c(1,2)$.

Further application of the obtained realizations for construction of invariant equations results in considerable technical difficulties. For the realization

$$P_0 = \partial_{x_0}, \qquad P_1 = x_1\partial_{x_0}, \qquad P_2 = x_2\partial_{x_0}, \qquad J_{12} = x_2\partial_{x_1} - x_1\partial_{x_2},$$
$$J_{01} = x_0x_1\partial_{x_0} + \left(x_1^2 - 1\right)\partial_{x_1} + x_1x_2\partial_{x_2}, \qquad J_{02} = x_0x_2\partial_{x_0} + x_1x_2\partial_{x_1} + \left(x_2^2 - 1\right)\partial_{x_2}$$

the search for functions $\mathcal{J}_1$, $\mathcal{J}_2$, …, $\mathcal{J}_7$ in the respective invariant equation

$$F(\mathcal{J}_1, \ldots, \mathcal{J}_7) = 0$$

showed that they have an extremely cumbersome form. Below we adduce four of the seven functions:

$$\mathcal{J}_1 = u, \qquad \mathcal{J}_2 = u_{x_0}^2 u_{x_0x_0}, \qquad \mathcal{J}_3 = (\Sigma_1 - 1)u_{x_0}^2,$$
$$\mathcal{J}_4 = \Sigma_1^{-1}\left[(1 - \Sigma_1)^3(2\Sigma_1\Sigma_2 u_{x_0}^2 + (1 - \Sigma_1)\Sigma_2^2 + \Sigma_3^2) + u_{x_0}^4(\Sigma_1 - 1)^2\Sigma_1^2\right],$$

where

$$\Sigma_1 = x_1^2 + x_2^2, \qquad \Sigma_2 = x_1(u_{x_1}u_{x_0x_0} - u_{x_0}u_{x_0x_1}) + x_2(u_{x_2}u_{x_0x_0} - u_{x_0}u_{x_0x_2}),$$
$$\Sigma_3 = x_2(u_{x_1}u_{x_0x_0} - u_{x_0}u_{x_0x_1}) - x_1(u_{x_2}u_{x_0x_0} - u_{x_0}u_{x_0x_2}).$$

We believe that in this respect a constructive problem may be presented by a problem of description of conformally invariant equations of the considered form using the basis of differential invariants for the standard realization found by W. Fushchych and I. Yehorchenko realization and realization (12) of the algebra $p(1,2)$.

[1] Fushchych W.I. and Yehorchenko I.A., Second-order differential invariants of the rotations group $O(n)$ and its extensions: $E(n)$, $P(1,n)$, *Acta Appl. Math.*, 1992, V.28, N 1, 69–92.

[2] Fushchych W., Zhdanov R. and Lahno V., On linear and non-linear representations of the generalized Poincaré groups in the class of Lie vector fields, *J. Nonlinear Math. Phys.*, 1994, V.1, N 3, 295–308.

[3] Rideau G. and Winternitz P., Nonlinear equations invariant under the Poincaré, similitude and conformal groups in two-dimensional space-time, *J. Math. Phys.*, 1990, V.31, N 5, 1095–1105.

[4] Lahno V.I., On new relativistically invariant nonlinear equations in two-dimensional space-time, *Rep. Math. Phys.*, 1998, V.41, N 3, 271–277.

[5] Yehorchenko I.A., Nonlinear representations of the Poincaré algebra and invariant equations, in Symmetry Analysis of Equations of Mathematical Physics, Kyiv, Institute of Mathematics, 1992, 62–66.

[6] Güngör F., Nonlinear equations invariant under Poincaré, similitude and conformal groups in three-dimensional space-time, *J. Phys. A: Math. Gen.*, 1998, V.31, 697–706.

[7] Yehorchenko I.A., Differential invariants for a nonlinear representation of the Poincaré algebra. Invariant equations, in Proceedings of the Second International Conference "Symmetry in Nonlinear Mathematical Physics" (7–13 July, 1997, Kyiv), Editors M.I. Shkil, A.G. Nikitin and V.M. Boyko, Kyiv, Institute of Mathematics, V.1, 1997, 200–205.

[8] Lahno V.I., Realizations of the Poincaré algebra and Poincaré-invariant equations in three-dimensional space-time, *Rep. Math. Phys.*, 2000, V.46, N 2, 137–142.