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Renormalization, Ward Identities, Symmetries, and Quantum Anomalies: Logical Aspect

Valentyn I. KUCHERYAVY

Bogolyubov Institute for Theoretical Physics, 14b Metrologichna Str., 03143 Kyiv, Ukraine
E-mail: *mmtpitp@bitp.kiev.ua*

Logical aspects associated with renormalization methods, Ward identities, symmetries, and quantum anomalies are investigated for ultraviolet divergent VV, AA, AV, and VA mass-anisotropic spinor amplitudes characterizing the most important polarization properties of many-fermion vacuum in any space-time dimension $n = 2r+\delta_n$, $\delta_n = 0, 1$. It is shown clearly that genuine algorithmic cause of perturbation theory anomalies is a logical descrepancy arising in the course of evaluating finite values of UV-divergent Feynman amplitudes (FAs) by means of commonly used renormalization techniques. Some important lessons, which teach us the manner how to construct a logically consistent renormalization technique, i.e. self-consistent renormalization (SCR), are revealed as part of this study.

1 Introduction

This report is a brief exposition of some results by the author on the investigation of a complicated tangle of problems associated on the one hand with renormalization methods and on the other hand with symmetries, their breaking, the Ward identities behavior, the Schwinger terms contributions (STC), and quantum anomalies. The main emphasis of this presentation is made on logical aspects of above problems. It will be shown clearly that genuine algorithmic cause of perturbation theory anomalies is a logical trap arising in the course of evaluating finite values of UV-divergent FAs by means of commonly used renormalization techniques.

The subject is illustrated for some set of quantities appearing in vector and axial vector Ward identities associated with VV, AA, AV, and VA mass-anisotropic UV-divergent spinor FAs characterizing the most important polarization properties of the fermion vacuum for n-dimensional quantum field models in which a mass spectrum of many-fermion sector may be both degenerate and nondegenerate (see reviews [1, 2]). Moreover, these amplitudes have of primary importance for investigating of anomaly problems [3–7][1] as well.

[1]It is appropriate to present Jackiw's point of view (see [6, p. 156]) on this question and on the essence of quantum anomalies at all: "…Thus, in a very precise way, the two-dimensional Abelian anomaly [K. Johnson, PL (1963), see Ref. [3] of our paper] is at the heart of the entire anomaly phenomenon …

…We have learned much from mathematicians about the topological and cohomological necessity of anomalies [A.S. Schwarz, PL (1977); L. Brown, R. Carlitz and C. Lee, PR (1977); R. Jackiw and C. Rebbi, PR (1977); M. Atiyah and I. Singer, PNAS USA (1984); L. Alvarez-Gaumé and P. Ginsparg, NP (1984)] but perhaps physics can, in its turn, advance mathematical concepts by insisting on the fact that the essence of the anomaly lies beyond present topological/cohomological ideas. The latter involve integrated, global quantities, like the Chern–Pontryagin number, yet the anomaly is local. Moreover, anomalies are present even in the absence of obstructions, like in Abelian [$U(1)$] theories, as in the discussed example which, being two-dimensional, hardly possesses any structure, save the anomaly. The $U(1)$ anomaly, on the other hand, appears to be the heart of the matter, not only for the non-Abelian anomalies,but also for the non-perturbative ones [E. Witten, PL (1982); R. Jackiw, "Relativity, groups, and topology II" (1984), see Ref. [5] of the present paper].

Thus, it seems that we are not yet at the end of the physics nor of the mathematics that can emerge from understanding anomalies …".

2 VV, AA, AV, and VA mass-anisotropic spinor amplitudes

2.1. Consider VV, AA, AV, and VA mass-anisotropic spinor UV-divergent Feynman amplitudes

$$I^{\alpha\beta}_{JJ'}(m,k)_\epsilon := \int_{-\infty}^{\infty} (d^n p)\,\delta(p,k)\,\frac{\mathcal{I}^{\alpha\beta}_{JJ'}(m,p)}{(\mu_{1\epsilon}-p_1^2)(\mu_{2\epsilon}-p_2^2)}, \quad J,\, J' \in \{V,\, A\},$$
$$\mathcal{I}^{\alpha\beta}_{JJ'}(m,p) := \mathrm{tr}[\gamma^\alpha\gamma(m_1+\hat{p}_1)\gamma^\beta\gamma'(m_2+\hat{p}_2)], \quad \gamma,\gamma' \in \{I_g, \widetilde{\gamma}^5\}, \quad \gamma\gamma' \in \{I_g, \varepsilon_g I_g, \widetilde{\gamma}^5\}, \qquad (1)$$
$$\delta(p,k) := \delta(-k_1+p_2-p_1)\,\delta(-k_2+p_1-p_2), \quad k := (k_1,k_2), \quad p := (p_1,p_2),$$
$$(d^n p) := d^n p_1 d^n p_2, \quad d^n p_l := \prod\nolimits_{\sigma=1}^{n} dp_l^\sigma, \quad \mu_{l\epsilon} := m_l^2 - i\epsilon_l, \quad \hat{p}_l := \gamma^\sigma p_{l\sigma}, \quad l=1,2, \qquad (2)$$
$$\gamma^\sigma\gamma = (-1)^{\pi_1}\gamma\gamma^\sigma, \quad \gamma^\sigma\gamma' = (-1)^{\pi_2}\gamma'\gamma^\sigma, \quad (-1)^{\pi_i} = \begin{cases} 1, & \text{if } \gamma,\gamma' = I_g, \\ (-1)^{n+1}, & \text{if } \gamma,\gamma' = \widetilde{\gamma}^5. \end{cases} \qquad (3)$$

The nondegenerate metric $g^{\mu\nu}$ with (p,q)-signature, $p+q=n=2r+\delta_n$, $\delta_n=0,1$, are used for each n-dimensional p_l-integration in equations (1)–(3). From now on V, A, S, and P used in subscripts denote vector, axial vector, scalar, and pseudoscalar respectively.

The matrices γ^α, γ^β, $\widetilde{\gamma}^5$, I_g, act in the N_g-dimensional space of the faithful representation $\pi(g)$ of the lowest dimension for the Clifford algebra $Cl(g)_{\mathbb{K}}$, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, with $\gamma^\sigma \in \Lambda^1(g)$, $\sigma=1,\ldots,n$, being the generating elements of the $Cl(g)_{\mathbb{K}}$-algebra in its matrix representation $\pi(g)$, i.e. $\gamma^\sigma\gamma^\tau+\gamma^\tau\gamma^\sigma = 2g^{\sigma\tau}I_g$, and I_g is the unit matrix of the dimension N_g. The n-degree element $\widetilde{\gamma}^5 \in \Lambda^n(g)$, i.e. the dual conjugation matrix, with the obvious but important properties: $\widetilde{\gamma}^5 := \gamma^1\gamma^2\cdots\gamma^n$, $\gamma^\sigma\widetilde{\gamma}^5 = (-1)^{n+1}\widetilde{\gamma}^5\gamma^\sigma$, $\sigma=1,\ldots,n$; $(\widetilde{\gamma}^5)^2 = \varepsilon_g I_g$, $\varepsilon_g := (-1)^q(-1)^{n(n-1)/2} = (-1)^{\varkappa(\varkappa+1)/2}$, $\varkappa := (q-p)(\mathrm{mod}\,8)$, is the natural n-dimensional analog of the Dirac γ^5-matrix. For more details on properties of the $\widetilde{\gamma}^5$-matrix and on the self-consistent version of the dimensional regularization with the $\widetilde{\gamma}^5$-matrix see [8].

Between amplitudes of equations (1)–(3) there exist the following relations

$$I^{\alpha\beta}_{AA}(m,k)_\epsilon = \varepsilon_g I^{\alpha\beta}_{VV}(m,k)_\epsilon + \varepsilon_g\,((-1)^{n+1}-1)m_1m_2\,\mathrm{tr}[\gamma^\alpha\gamma^\beta]I(m,k)_\epsilon,$$
$$I^{\alpha\beta}_{AV}(m,k)_\epsilon = I^{\alpha\beta}_{VA}(m,k)_\epsilon + ((-1)^{n+1}-1)m_1m_2\,\mathrm{tr}[\widetilde{\gamma}^5\gamma^\alpha\gamma^\beta]I(m,k)_\epsilon, \quad \forall\, n=2r+\delta_n, \qquad (4)$$

where $I(m,k)_\epsilon$ is given below by equations (9). So, the distinction between either $I^{\alpha\beta}_{AA}(m,k)_\epsilon$ and $I^{\alpha\beta}_{VV}(m,k)_\epsilon$ or $I^{\alpha\beta}_{AV}(m,k)_\epsilon$ and $I^{\alpha\beta}_{VA}(m,k)_\epsilon$ may exist only for even $n=2r$ and if both m_1, $m_2>0$; if so, this distinction is determined by the same quantity, $(-2m_1m_2)I(m,k)_\epsilon$, which distinguishes their functions standing either at tensor $\mathrm{tr}[\gamma^\alpha\gamma^\beta]$ or $\mathrm{tr}[\widetilde{\gamma}^5\gamma^\alpha\gamma^\beta]$ respectively.

2.2. Owing to the relations $\delta(-k_1+p_2-p_1)(\hat{k}_1-\hat{p}_2+\hat{p}_1)=0$, $\delta(-k_2+p_1-p_2)(\hat{k}_2-\hat{p}_1+\hat{p}_2)=0$, from equations (1)–(3) imply useful decompositions: $\delta(p,k)\hat{k}_1\gamma = \delta(p,k)(\hat{p}_2-\hat{p}_1)\gamma = \delta(p,k)[(-1)^{\pi_1}\gamma(m_1-\hat{p}_1)-(m_2-\hat{p}_2)\gamma+(m_2-(-1)^{\pi_1}m_1)\gamma]$, $\delta(p,k)\hat{k}_2\gamma' = \delta(p,k)(\hat{p}_1-\hat{p}_2)\gamma' = \delta(p,k)[(-1)^{\pi_2}\gamma'(m_2-\hat{p}_2)-(m_1-\hat{p}_1)\gamma'+(m_1-(-1)^{\pi_2}m_2)\gamma']$, which formally produce the vector and axial vector *canonical Ward identities* (CWIs),

$$k_{1\alpha}I^{\alpha\beta}_{JJ'}(m,k)_\epsilon = D^{\cdot\beta}_{JJ'}(m,k)_\epsilon = P^{\cdot\beta}_{JJ'}(m,k)_\epsilon + M^{\cdot\beta}_{JJ'}(m,k)_\epsilon,$$
$$k_{2\beta}I^{\alpha\beta}_{JJ'}(m,k)_\epsilon = D^{\alpha\cdot}_{JJ'}(m,k)_\epsilon = P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon + M^{\alpha\cdot}_{JJ'}(m,k)_\epsilon, \qquad (5)$$

for *main* mass-anisotropic UV-divergent FAs presented by equations (1)–(3); in so doing the vector and axial vector CWIs are given here in the uniform manner. Associated quantities, $D^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $D^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $P^{\cdot\beta}_{JJ'}(m,k)_\epsilon := (-1)^{\pi_1}P^{\cdot\beta}_{JJ';1}(m,k)_\epsilon - P^{\cdot\beta}_{JJ';2}(m,k)_\epsilon$, $P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon := (-1)^{\pi_2}P^{\alpha\cdot}_{JJ';2}(m,k)_\epsilon - P^{\alpha\cdot}_{JJ';1}(m,k)_\epsilon$, $M^{\cdot\beta}_{JJ'}(m,k)_\epsilon := (m_2-(-1)^{\pi_1}m_1)I^{\cdot\beta}_{jJ'}(m,k)_\epsilon$, $M^{\alpha\cdot}_{JJ'}(m,k)_\epsilon := (m_1-(-1)^{\pi_2}m_2)\;I^{\alpha\cdot}_{Jj'}(m,k)_\epsilon$, $j,j'\in\{S,P\}$, entering into the CWIs (5) are similar to the main amplitudes $I^{\alpha\beta}_{JJ'}(m,k)_\epsilon$, and differ from the latest ones only in polynomials of integrands, see Ref. [9], equations (3)–(6).

As is already known from equations (4), the even n are especially important for us. For this reason we shall consider only the case $n = 2r$ in the following. In addition, some simplifications occur since tensors of the type $\mathrm{tr}[\gamma^\alpha\gamma\gamma^\beta\gamma'\gamma^\sigma]$, $\mathrm{tr}[\gamma^\beta\gamma'\gamma]$ are equal zero. So, all amplitudes entering into CWIs (5) can be written in the form ($n = 2r$):

$$
\begin{aligned}
&I^{\alpha\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\alpha\gamma\gamma_\sigma\gamma^\beta\gamma'\gamma_\tau]I^{\sigma\tau}_{12}(m,k)_\epsilon + \mathrm{tr}[\gamma^\alpha\gamma\gamma^\beta\gamma']\, m_1m_2I(m,k)_\epsilon,\\
&D^{\cdot\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\beta\gamma'\gamma\gamma_\sigma][I^\sigma_{2;1}(m,k)_\epsilon - I^\sigma_{1;2}(k)_\epsilon + m_1m_2(-1)^{\pi_1}(I^\sigma_2(m,k)_\epsilon - I^\sigma_1(k)_\epsilon)],\\
&D^{\alpha\cdot}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\alpha\gamma\gamma'\gamma_\sigma][I^\sigma_{1;2}(m,k)_\epsilon - I^\sigma_{2;1}(k)_\epsilon + m_1m_2(-1)^{\pi_2}(I^\sigma_1(m,k)_\epsilon - I^\sigma_2(k)_\epsilon)], \qquad (6)\\
&P^{\cdot\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\beta\gamma'\gamma\gamma_\sigma][P^\sigma_{1;2}(m,k)_\epsilon - P^\sigma_{2;1}(m,k)_\epsilon],\\
&P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\alpha\gamma\gamma'\gamma_\sigma][P^\sigma_{2;1}(m,k)_\epsilon - P^\sigma_{1;2}(m,k)_\epsilon], \qquad (7)\\
&M^{\cdot\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\beta\gamma'\gamma\gamma_\sigma][m_{jJ';1}I^\sigma_2(m,k)_\epsilon - m_{jJ';2}I^\sigma_1(m,k)_\epsilon],\\
&M^{\alpha\cdot}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\alpha\gamma\gamma'\gamma_\sigma][m_{Jj';2}I^\sigma_1(m,k)_\epsilon - m_{Jj';1}I^\sigma_2(m,k)_\epsilon],\\
&\text{where } m_{jJ';l} := (m_1m_2(-1)^{\pi_1} - m_l^2), \quad m_{Jj';l} := (m_1m_2(-1)^{\pi_2} - m_l^2), \quad l = 1,2, \qquad (8)
\end{aligned}
$$

in which we make use of the following quantities:

$$
\begin{aligned}
&I(m,k)_\epsilon := \int_{-\infty}^{\infty} \frac{(d^np)\,\delta(p,k)}{(\mu_{1\epsilon} - p_1^2)(\mu_{2\epsilon} - p_2^2)}, \quad I^\sigma_l(m,k)_\epsilon := \int_{-\infty}^{\infty} \frac{(d^np)\,\delta(p,k)p_l^\sigma}{(\mu_{1\epsilon} - p_1^2)(\mu_{2\epsilon} - p_2^2)}, \quad l = 1,2,\\
&I^{\sigma\tau}_{ll'}(m,k)_\epsilon := \int_{-\infty}^{\infty} \frac{(d^np)\,\delta(p,k)p_l^\sigma p_{l'}^\tau}{(\mu_{1\epsilon} - p_1^2)(\mu_{2\epsilon} - p_2^2)}, \quad l,l' \in \{1,2\},\\
&I^\sigma_{l;l'}(m,k)_\epsilon := \int_{-\infty}^{\infty} \frac{(d^np)\,\delta(p,k)p_l^2 p_{l'}^\sigma}{(\mu_{1\epsilon} - p_1^2)(\mu_{2\epsilon} - p_2^2)}, \quad l,l' \in \{1,2\}, \quad l' \neq l, \qquad (9)\\
&P^\sigma_{l;l'}(m,k)_\epsilon := \int_{-\infty}^{\infty} \frac{(d^np)\,\delta(p,k)(m_l^2 - p_l^2)p_{l'}^\sigma}{(\mu_{1\epsilon} - p_1^2)(\mu_{2\epsilon} - p_2^2)} = m_l^2 I^\sigma_{l'}(m,k)_\epsilon - I^\sigma_{l;l'}(m,k)_\epsilon, \quad l' \neq l,\\
&I^\sigma_1(m,k)_\epsilon - I^\sigma_2(m,k)_\epsilon = k_2^\sigma I(m,k)_\epsilon. \qquad (10)
\end{aligned}
$$

It should be particularly emphasized that $I^\sigma_l(m,k)_\epsilon$ and $I^\sigma_{l;l'}(m,k)_\epsilon$ in its turn are constituents of $P^\sigma_{l;l'}(m,k)_\epsilon$. Next, if $m_1, m_2 > 0$, then it follows from equation (10) that $\lim_{\epsilon_l\to 0} P^\sigma_{l;l'}(m,k)_\epsilon = \int_{-\infty}^{\infty}(d^np)\,\delta(p,k)\,p^\sigma_{l'}/(\mu_{l'\epsilon} - p_{l'}^2) = 0$, as a consequence of the "symmetric integration" (skew-symmetric functions are integrated in symmetric limits).

2.3. The closely similar to $P^\sigma_{l;l'}(m,k)_\epsilon$, but nevertheless different expressions,

$$
\begin{aligned}
P^\sigma_{l\epsilon;l'}(m,k)_\epsilon &:= \int_{-\infty}^{\infty} \frac{(d^np)\,\delta(p,k)}{(\mu_{1\epsilon} - p_1^2)(\mu_{2\epsilon} - p_2^2)}\,(\mu_{l\epsilon} - p_l^2)p^\sigma_{l'} \qquad (11)\\
&= \int_{-\infty}^{\infty} (d^np)\,\delta(p,k)\,p^\sigma_{l'}/(\mu_{l'\epsilon} - p_{l'}^2) = 0, \quad l,l' \in \{1,2\}, \quad l' \neq l; \quad m_1, m_2 \geq 0,
\end{aligned}
$$

are zero due to the "symmetric integration" as well. Equations (11) represent the simplest examples of the *reduction identities*. From equations (9)–(11) it obvious but the very useful relations follows:

$$
P^\sigma_{l;l'}(m,k)_\epsilon = i\epsilon_l I^\sigma_{l'}(m,k)_\epsilon, \quad I^\sigma_{l;l'}(m,k)_\epsilon = \mu_{l\epsilon}I^\sigma_{l'}(m,k)_\epsilon, \quad l,l' \in \{1,2\}, \ l' \neq l. \qquad (12)
$$

So, owing to equations (11)–(12) the quantities $P^{\cdot\beta}_{JJ'}(m,k) := \lim_{\epsilon_1,\epsilon_2\to 0} P^{\cdot\beta}_{JJ'}(m,k)_\epsilon$ and $P^{\alpha\cdot}_{JJ'}(m,k) := \lim_{\epsilon_1,\epsilon_2\to 0} P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, closely related to the Schwinger terms contributions (STCs) of current commutators to the CWIs (5), are expressed for all $I^{\alpha\beta}_{JJ'}(m,k)_\epsilon$ in terms of the important quantity

$$
\lim_{\epsilon_1,\epsilon_2\to 0}[P^\sigma_{2;1}(m,k)_\epsilon - P^\sigma_{1;2}(m,k)_\epsilon] = \begin{cases} \lim_{\epsilon_1,\epsilon_2\to 0}[i\epsilon_2 I^\sigma_1(m,k)_\epsilon - i\epsilon_1 I^\sigma_2(m,k)_\epsilon]|_{\substack{m_1=0\\ m_2=0}}, \\ 0, \quad \text{if } m_1, m_2 > 0. \end{cases} \qquad (13)
$$

Consequently, the STCs may be nonzero only for the chiral case, $m_1 = m_2 = 0$.

2.4. Consider now the mass-isotropic case, $m_1 = m_2 = m \geq 0$, $\epsilon_1 = \epsilon_2 = \epsilon \geq 0$, for $n = 2r$. Then, due to equations (6)–(10), (13) and (3) the general CWIs (5) take simpler forms:

$$
\begin{aligned}
&k_{1\alpha} I^{\alpha\beta}_{VJ'}(m,k) = 0, \quad k_{1\alpha} I^{\alpha\beta}_{AJ'}(m,k) = \mathrm{tr}[\gamma^\beta \gamma' \widetilde{\gamma}^5(-\hat{k}_2)](-2m^2) I(m,k), \quad {\scriptstyle J'\in\{V,A\}},\\
&k_{2\beta} I^{\alpha\beta}_{JV}(m,k) = 0, \quad k_{2\beta} I^{\alpha\beta}_{JA}(m,k) = \mathrm{tr}[\gamma^\alpha \gamma \widetilde{\gamma}^5 \hat{k}_2](-2m^2) I(m,k), \quad {\scriptstyle J\in\{V,A\}}, \quad m > 0,\\
&k_{1\alpha} I^{\alpha\beta}_{JJ'}(m,k)|_{m=0} = \mathrm{tr}[\gamma^\beta \gamma' \gamma(-\hat{k}_2)] \lim_{\epsilon\to 0}[i\epsilon I(m,k)_\epsilon|_{m=0}],\\
&k_{2\beta} I^{\alpha\beta}_{JJ'}(m,k)|_{m=0} = \mathrm{tr}[\gamma^\alpha \gamma \gamma' \hat{k}_2] \lim_{\epsilon\to 0}[i\epsilon I(m,k)_\epsilon|_{m=0}], \quad {\scriptstyle J,J'\in\{V,A\}},
\end{aligned}
\tag{14}
$$

where $I^{\alpha\beta}_{JJ'}(m,k) := \lim_{\epsilon\to 0} I^{\alpha\beta}_{JJ'}(m,k)_\epsilon$, $I(m,k) := \lim_{\epsilon\to 0} I(m,k)_\epsilon$, and we make use of the relation $I^\sigma_1(m,k)_\epsilon - I^\sigma_2(m,k)_\epsilon = k^\sigma_2 I(m,k)_\epsilon$. It should be noted that the forms of mass-isotropic vector and axial vector CWIs (14) are completely determined by equations (8) and (13).

3 Regular values and identities of VV, AA, etc. amplitudes

3.1. The quantities in equations (9) are associated with the s-degree homogeneous p-polynomials in numerators of integrands. The proper divergence indices of them are equal to $\nu(s) := 2\omega + s = n - 4 + s$, $s = 0,1,2,3$, where $\omega := n/2 - 2$. Therefore, the main amplitudes, $I^{\alpha\beta}_{JJ'}(m,k)_\epsilon$, have the maximal divergence index, $\nu := \nu(2) = n - 2$, and the minimal one, $\nu - 2 \equiv \nu(0) = n - 4$, respectively. Associated amplitudes, $D^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $D^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $P^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, have corresponding divergence indices $\nu + 1 \equiv \nu(3) = n - 1$ and $\nu - 1 \equiv \nu(1) = n - 3$ respectively, but $M^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $M^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$ have merely the proper divergence index $\nu - 1 \equiv \nu(1) = n - 3$.

It is important to stress that $I(m,k)_\epsilon$ is the constituent of $I^{\alpha\beta}_{JJ'}(m,k)_\epsilon$ for all $J, J' \in \{V, A\}$. Similarly, $I^\sigma_1(m,k)_\epsilon$ and $I^\sigma_2(m,k)_\epsilon$ are constituents of $D^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $D^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $P^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $M^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $M^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$ for all $J, J' \in \{V, A\}$ as well. We have here examples of the general state of affairs, when precisely the same function can be involved in different expressions and relations simultaneously: in our case, in the vector and axial vector CWIs.

3.2. This reasonably raises the following question. Can we construct such regular (finite) values of quantities (1)–(3) which satisfy the relations (4) and the vector and axial vector CWIs (5) simultaneously as well? The answer is yes, we can. Such an universal, high-efficient, and invariant renormalization procedure which is applicable on equal grounds both to renormalizable and nonrenormalizable theories has been constructed by the author [10–13] and was named as the self-consistent renormalization (SCR). The SCR is based on ideas of the Bogoliubov–Parasiuk R-operation [14–16] and is supplemented with recurrence, compatibility, and differential relations fixing a renormalization arbitrariness of the R-operation in some universal way based on mathematical properties of Feynman amplitudes only.

3.3. Regular values $(R^\nu_0 I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 D)^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 D)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 P)^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 P)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 M)^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 M)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, obtained in the framework of the SCR retain completely the structure of equations (6)–(8) in which the quantities $I(m,k)_\epsilon$, $I^{\sigma\tau}_{ll'}(m,k)_\epsilon$, $I^\sigma_l(m,k)_\epsilon$, $I^\sigma_{l;l'}(m,k)_\epsilon$, $P^\sigma_{l;l'}(m,k)_\epsilon$ are replaced respectively by their regular values that are given by means of the following α-parametric integral representations:

$$
\begin{bmatrix}
(R^\nu_0 I)(m,k)_\epsilon\\
(R^\nu_0 I)^{\sigma\tau}_{ll'}(m,k)_\epsilon\\
(R^{\nu+1}_0 I)^\sigma_l(m,k)_\epsilon\\
(R^{\nu+1}_0 I)^\sigma_{l;l'}(m,k)_\epsilon\\
(R^{\nu+1}_0 P)^\sigma_{l;l'}(m,k)_\epsilon
\end{bmatrix}
= C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}}
\begin{bmatrix}
(R^\nu_0 \mathcal{I})(\alpha,m,k)_\epsilon\\
(R^\nu_0 \mathcal{I})^{\sigma\tau}_{ll'}(\alpha,m,k)_\epsilon\\
(R^{\nu+1}_0 \mathcal{I})^\sigma_l(\alpha,m,k)_\epsilon\\
(R^{\nu+1}_0 \mathcal{I})^\sigma_{l;l'}(\alpha,m,k)_\epsilon\\
(R^{\nu+1}_0 \mathcal{P})^\sigma_{l;l'}(\alpha,m,k)_\epsilon
\end{bmatrix},
\quad C_g := (2\pi)^n \delta(k) b(g),
\tag{15}
$$

where the integrands,

$$\begin{aligned}
&(R_0^\nu \mathcal{I})(\alpha,m,k)_\epsilon := (R_0^\nu \mathcal{F})_{00},\\
&(R_0^\nu \mathcal{I})^{\sigma\tau}_{ll'}(\alpha,m,k)_\epsilon := Y_l^\sigma Y_{l'}^\tau (R_0^\nu \mathcal{F})_{20} + (-2)^{-1} X_{ll'} g^{\sigma\tau} (R_0^\nu \mathcal{F})_{21},\\
&(R_0^{\nu+1} \mathcal{I})^{\sigma}_{l}(\alpha,m,k)_\epsilon := Y_l^\sigma (R_0^{\nu+1} \mathcal{F})_{10},\\
&(R_0^{\nu+1} \mathcal{I})^{\sigma}_{l;l'}(\alpha,m,k)_\epsilon := Y_l^2 Y_{l'}^\sigma (R_0^{\nu+1} \mathcal{F})_{30} + (-2)^{-1}\{n X_{ll} Y_{l'}^\sigma + 2 X_{ll'} Y_l^\sigma\} (R_0^{\nu+1} \mathcal{F})_{31}, \qquad (16)\\
&(R_0^{\nu+1} \mathcal{P})^{\sigma}_{l;l'}(\alpha,m,k)_\epsilon := m_l^2 (R_0^{\nu+1} \mathcal{I})^{\sigma}_{l'}(\alpha,m,k)_\epsilon - (R_0^{\nu+1} \mathcal{I})^{\sigma}_{l;l'}(\alpha,m,k)_\epsilon, \qquad (17)
\end{aligned}$$

are defined in terms of some basic functions, $(R_0^\nu \mathcal{F})_{00}$, $(R_0^\nu \mathcal{F})_{20}$, $(R_0^\nu \mathcal{F})_{21}$, and $(R_0^{\nu+1} \mathcal{F})_{10}$, $(R_0^{\nu+1} \mathcal{F})_{30}$, $(R_0^{\nu+1} \mathcal{F})_{31}$, associated with homogeneous polynomials of the degree $s-2j$, $s = 0,1,2,3$, $j = 0,[s/2]$, in external momenta k_1, k_2. Each monomial of the latter is a product of $s-2j$ linear Kirchhoff forms Y_l, $l = 1,2$, and j line-correlator functions $X_{ll'}$, $l,l' \in \{1,2\}$.

The explicit form of the basic functions $(R_0^\nu \mathcal{F})_{sj}$, $(R_0^{\nu+1} \mathcal{F})_{sj}$, and the determining numbers ν_{sj}, λ_{sj}, ν^1_{sj}, λ^1_{sj}, and ω appearing in them are as follows:

$$\begin{aligned}
&(R_0^\nu \mathcal{F})_{sj} := M_\epsilon^{\omega+j}\, \Gamma(\lambda_{sj})/\Gamma(2+\nu_{sj}) Z_\epsilon^{1+\nu_{sj}}\, {}_2F_1(1,\lambda_{sj};\, 2+\nu_{sj};\, Z_\epsilon),\\
&\nu_{sj} := [(\nu-s)/2] + j, \quad \lambda_{sj} := -\omega - j + 1 + \nu_{sj}, \quad \omega := n/2 - 2, \qquad (18)\\
&(R_0^{\nu+1} \mathcal{F})_{sj} := M_\epsilon^{\omega+j}\, \Gamma(\lambda^1_{sj})/\Gamma(2+\nu^1_{sj}) Z_\epsilon^{1+\nu^1_{sj}}\, {}_2F_1(1,\lambda^1_{sj};\, 2+\nu^1_{sj};\, Z_\epsilon),\\
&\nu^1_{sj} := [(\nu+1-s)/2] + j, \quad \lambda^1_{sj} := -\omega - j + 1 + \nu^1_{sj}, \quad \omega := n/2 - 2. \qquad (19)
\end{aligned}$$

The $[(\nu-s)/2]$ and $[(\nu+1-s)/2]$ in equations (18)–(19) are integral parts of the numbers $(\nu-s)/2$ and $(\nu+1-s)/2$ respectively. There exist the following compatibility and recurrence relations:

$$\begin{aligned}
&(R_0^\nu \mathcal{F})_{sj} = \mathcal{F}_{sj} := M_\epsilon^{\omega+j} (1-Z_\epsilon)^{\omega+j}\, \Gamma(-\omega-j), \quad \text{if} \;\; \nu_{sj} \le -1, \qquad (20)\\
&(R_0^\nu \mathcal{F})_{sj} = (R_0^{\nu+1} \mathcal{F})_{s+1,j}, \qquad (21)\\
&M_\epsilon\, (R_0^\nu \mathcal{F})_{00} - A\, (R_0^\nu \mathcal{F})_{20} + (\omega+1)\, (R_0^\nu \mathcal{F})_{21} = 0,\\
&M_\epsilon\, (R_0^{\nu+1} \mathcal{F})_{10} - A\, (R_0^{\nu+1} \mathcal{F})_{30} + (\omega+1)\, (R_0^{\nu+1} \mathcal{F})_{31} = 0, \qquad (22)
\end{aligned}$$

between the basic functions $(R_0^\nu \mathcal{F})_{sj}$, $(R_0^{\nu+1} \mathcal{F})_{sj}$. In fact, due to compatibility relations (21) both recurrence relations (22) are different forms of the common one.

The α-parametric functions $Z_\epsilon \equiv Z(\alpha,m,k)_\epsilon$, $M_\epsilon \equiv M(\alpha,m)_\epsilon$, $A \equiv A(\alpha,k)$, $\Delta \equiv \Delta(\alpha)$, $Y_l \equiv Y_l(\alpha,k)$, and $X_{ll'} \equiv X_{ll'}(\alpha)$ entering into equations (16)–(19) have the form:

$$\begin{aligned}
&Z_\epsilon := A/M_\epsilon, \quad M_\epsilon := \alpha_1\mu_{1\epsilon} + \alpha_2\mu_{2\epsilon}, \quad A := \Delta\beta_1\beta_2\, k_2^2 = \alpha_1 Y_1^2 + \alpha_2 Y_2^2, \quad \Delta := \alpha_1 + \alpha_2,\\
&Y_1 := \beta_2\, k_2, \quad Y_2 := -\beta_1\, k_2, \quad \beta_l := \alpha_l/\Delta, \quad X_{ll'} := \Delta^{-1}, \; l,l' \in \{1,2\}, \quad Y_2 - Y_1 = -k_2 = k_1,\\
&Y_1 \cdot Y_2 = -A/\Delta, \quad Y_l^2 = -A/\Delta + \beta_{l'} k_2^2, \quad \alpha_l Y_l^2 = \beta_{l'} A, \quad l' \ne l, \quad \beta_1 + \beta_2 = 1. \qquad (23)
\end{aligned}$$

The integration measure $d\mu(\alpha)$, the integration region Σ^1, the metric dependent constant $b(g)$, and the overall δ-function $\delta(k)$ are defined as

$$\begin{aligned}
&d\mu(\alpha) := \delta(1-\alpha_1-\alpha_2) d\alpha_1\, d\alpha_2, \quad \Sigma^1 := \{\alpha_l |\, \alpha_l \ge 0, \; \forall\, l, \; \alpha_1 + \alpha_2 = 1\},\\
&b(g) := (\pi^{n/2} i^p)/(2\pi)^n, \quad \delta(k) := \delta(-k_1 - k_2), \quad p - \text{number of positive squares in } g. \quad (24)
\end{aligned}$$

3.4. From equations (6)–(8), (15)–(17) and (23) more specific formulae follows. So, the regular values of the main FAs (1), reduction identities (11), and their consequences take the form:

$$\begin{aligned}
(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon = C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \{&\mathrm{tr}[\gamma^\alpha \gamma \gamma^\beta \gamma']\, m_1 m_2 (R_0^\nu \mathcal{F})_{00}\\
&+ \mathrm{tr}[\gamma^\alpha \gamma \hat{k}_2 \gamma^\beta \gamma' \hat{k}_2](-\beta_1\beta_2)(R_0^\nu \mathcal{F})_{20} + \mathrm{tr}[\gamma^\alpha \gamma \gamma^\sigma \gamma^\beta \gamma' \gamma_\sigma](-1/2)\Delta^{-1}(R_0^\nu \mathcal{F})_{21}\}, \qquad (25)
\end{aligned}$$

$$(R_0^{\nu+1}P)^{\sigma}_{l\epsilon;l'}(m,k)_\epsilon = (-1)^l k_2^\sigma C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \big\{\mu_{l\epsilon}\beta_l (R_0^{\nu+1}\mathcal{F})_{10}$$
$$- \beta_l Y_l^2 (R_0^{\nu+1}\mathcal{F})_{30} + [(n/2)\beta_l - \beta_{l'}]\Delta^{-1}(R_0^{\nu+1}\mathcal{F})_{31}\big\} = 0, \quad l,l' \in \{1,2\},\ l' \neq l, \qquad (26)$$
$$(R_0^{\nu+1}P_{2\epsilon-1\epsilon})^\sigma(m,k)_\epsilon := (R_0^{\nu+1}P)^\sigma_{2\epsilon;1}(m,k)_\epsilon - (R_0^{\nu+1}P)^\sigma_{1\epsilon;2}(m,k)_\epsilon = 0$$
$$= k_2^\sigma C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \frac{1}{\Delta} \big\{M_\epsilon (R_0^{\nu+1}\mathcal{F})_{10} - A(R_0^{\nu+1}\mathcal{F})_{30} + (\omega+1)(R_0^{\nu+1}\mathcal{F})_{31}\big\} = 0, \qquad (27)$$
$$(R_0^{\nu+1}I)^\sigma_1(m,k)_\epsilon - (R_0^{\nu+1}I)^\sigma_2(m,k)_\epsilon = k_2^\sigma C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} (R_0^{\nu+1}\mathcal{F})_{10} = k_2^\sigma (R_0^\nu I)(m,k)_\epsilon. \qquad (28)$$

3.5. After isolating tensor structures in regular values of all amplitudes involving in CWIs (5), for example, $k_{1\alpha}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\beta\gamma'\gamma(-\hat{k}_2)](R_0^\nu I)_{JJ';1}(m,k)_\epsilon$, $k_{2\beta}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\alpha\gamma\gamma'\hat{k}_2](R_0^\nu I)_{JJ';2}(m,k)_\epsilon$, $(R_0^{\nu+1}D)^{\cdot\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\beta\gamma'\gamma(-\hat{k}_2)](R_0^{\nu+1}D)_{JJ';1}(m,k)_\epsilon$, $(R_0^{\nu+1}D)^{\alpha\cdot}_{JJ'} = \mathrm{tr}[\gamma^\alpha\gamma\gamma'\hat{k}_2](R_0^{\nu+1}D)_{JJ';2}(m,k)_\epsilon$, etc., the scalar factors of them can be represented as follows:

$$\begin{bmatrix} (R_0^\nu I)_{JJ';i}(m,k)_\epsilon \\ (R_0^{\nu+1}D)_{JJ';i}(m,k)_\epsilon \\ (R_0^{\nu+1}P)_{JJ';i}(m,k)_\epsilon \\ (R_0^{\nu+1}M)_{JJ';i}(m,k)_\epsilon \end{bmatrix} = C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \begin{bmatrix} (R_0^\nu \mathcal{I})_{JJ';i}(\alpha,m,k)_\epsilon \\ (R_0^{\nu+1}\mathcal{D})_{JJ';i}(\alpha,m,k)_\epsilon \\ (R_0^{\nu+1}\mathcal{P})_{JJ';i}(\alpha,m,k)_\epsilon \\ (R_0^{\nu+1}\mathcal{M})_{JJ';i}(\alpha,m,k)_\epsilon \end{bmatrix}, \quad i=1,2, \qquad (29)$$
$$(R_0^\nu \mathcal{I})_{JJ';i}(\alpha,m,k)_\epsilon := m_1 m_2 (-1)^{\pi_i}(R_0^\nu\mathcal{F})_{00} - A/\Delta (R_0^\nu\mathcal{F})_{20} + (\omega+1)/\Delta (R_0^\nu\mathcal{F})_{21}$$
$$\cong \big(m_1 m_2(-1)^{\pi_i} - M_\epsilon/\Delta\big)(R_0^\nu\mathcal{F})_{00},$$
$$(R_0^{\nu+1}\mathcal{D})_{JJ';i}(\alpha,m,k)_\epsilon := m_1 m_2(-1)^{\pi_i}(R_0^{\nu+1}\mathcal{F})_{10} - A/\Delta(R_0^{\nu+1}\mathcal{F})_{30} + (\omega+1)/\Delta(R_0^{\nu+1}\mathcal{F})_{31}$$
$$\cong \big(m_1 m_2(-1)^{\pi_i} - M_\epsilon/\Delta\big)(R_0^{\nu+1}\mathcal{F})_{10},$$
$$(R_0^{\nu+1}\mathcal{P})_{JJ';i}(\alpha,m,k)_\epsilon := \Delta^{-1}\big\{M(R_0^{\nu+1}\mathcal{F})_{10} - A(R_0^{\nu+1}\mathcal{F})_{30} + (\omega+1)(R_0^{\nu+1}\mathcal{F})_{31}\big\}$$
$$\cong iE/\Delta(R_0^{\nu+1}\mathcal{F})_{10}, \quad M := \alpha_1 m_1^2 + \alpha_2 m_2^2, \quad E := \alpha_1\epsilon_1 + \alpha_2\epsilon_2,$$
$$(R_0^{\nu+1}\mathcal{M})_{JJ';i}(\alpha,m,k)_\epsilon := \big(m_1 m_2(-1)^{\pi_i} - M/\Delta\big)(R_0^{\nu+1}\mathcal{F})_{10}, \quad i=1,2. \qquad (30)$$

The second lines of equations (30) are due to equation (27). The congruence relation $\mathcal{A}(\alpha,m,k) \cong \mathcal{B}(\alpha,m,k)$ denotes the equality $\int_{\Sigma^1} d\mu(\alpha)\Delta^{-n/2}\mathcal{A}(\alpha,m,k) = \int_{\Sigma^1} d\mu(\alpha)\Delta^{-n/2}\mathcal{B}(\alpha,m,k)$.

3.6. Taking into account compatibility relations (21), the identities are verified:

$$(R_0^\nu I)_{JJ';i}(m,k)_\epsilon = (R_0^{\nu+1}D)_{JJ';i}(m,k)_\epsilon = (R_0^{\nu+1}P)_{JJ';i}(m,k)_\epsilon + (R_0^{\nu+1}M)_{JJ';i}(m,k)_\epsilon, \quad (31)$$

$i = 1,2$, which produce the general mass-anisotropic vector and axial vector CWIs,

$$k_{1\alpha}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon = (R_0^{\nu+1}D)^{\cdot\beta}_{JJ'}(m,k)_\epsilon = (R_0^{\nu+1}P)^{\cdot\beta}_{JJ'}(m,k)_\epsilon + (R_0^{\nu+1}M)^{\cdot\beta}_{JJ'}(m,k)_\epsilon,$$
$$k_{2\beta}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon = (R_0^{\nu+1}D)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon = (R_0^{\nu+1}P)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon + (R_0^{\nu+1}M)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon, \quad (32)$$

but now for regular values; it is seen that equations (32) retain the form of equations (5).

In the mass-isotropic case, $m_1 = m_2 = m \geq 0$, $\epsilon_1 = \epsilon_2 = \epsilon \geq 0$, from equations (28)–(32) imply the following mass-isotropic CWIs for ϵ-limiting values of amplitudes:

$$k_{1\alpha}(R_0^\nu I)^{\alpha\beta}_{VJ'}(m,k) = 0, \quad k_{1\alpha}(R_0^\nu I)^{\alpha\beta}_{AJ'}(m,k) = \mathrm{tr}[\gamma^\beta\gamma'\,\widetilde{\gamma}^5(-\hat{k}_2)](-2m^2)(R_0^\nu I)(m,k),$$
$$k_{2\beta}(R_0^\nu I)^{\alpha\beta}_{JV}(m,k) = 0, \quad k_{2\beta}(R_0^\nu I)^{\alpha\beta}_{JA}(m,k) = \mathrm{tr}[\gamma^\alpha\gamma\,\widetilde{\gamma}^5\hat{k}_2](-2m^2)(R_0^\nu I)(m,k),\ m>0,$$
$$k_{1\alpha}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)|_{m=0} = \mathrm{tr}\,[\gamma^\beta\gamma'\gamma(-\hat{k}_2)] \lim_{\epsilon\to 0}[i\epsilon\,(R_0^\nu I)(m,k)_\epsilon|_{m=0}],$$
$$k_{2\beta}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)|_{m=0} = \mathrm{tr}\,[\gamma^\alpha\gamma\gamma'\hat{k}_2] \lim_{\epsilon\to 0}[i\epsilon\,(R_0^\nu I)(m,k)_\epsilon|_{m=0}], \quad J,J'\in\{V,A\}, \qquad (33)$$

which retain the form of equations (14). Recall that the proper divergence index of $I(m,k)_\epsilon$ is $\nu - 2$.

4 Logical trap as a cause of quantum anomalies

4.1. Up till now no explicit forms of the basic functions $(R_0^\nu \mathcal{F})_{sj}$ and $(R_0^{\nu+1}\mathcal{F})_{sj}$ have been used. In fact, we have exploited only the following items: i) $(R_0^\nu \mathcal{F})_{sj}$ and $(R_0^{\nu+1}\mathcal{F})_{sj}$ are regular functions in the vicinity of zero values of external momenta k_2 that is denoted by the subscript 0 on R; ii) *each* $(R_0^\nu \mathcal{F})_{sj}$ *or* $(R_0^{\nu+1}\mathcal{F})_{sj}$, which are defined in some reasonable way according to indices ν or $\nu+1$ that is denoted by superscripts ν or $\nu+1$ on R respectively, take *the same value in all relations*; iii) the compatibility relation (21); iv) the reduction identity (27) which produces the more convenient second form of quantities in equations (30) and predicts the recurrence relation (22) as well. It should be particularly emphasized as to pure mathematical nature of equation (27) associated with some symmetry property of integrals under consideration.

In the mass-isotropic case, $m_1 = m_2 = m > 0$, the mass term functions $(R_0^{\nu+1}M)_{JJ';i}(m,k)$ are zero for vector CWIs and are expressed in terms of the quantity $(-2m^2)(R_0^\nu I)(m,k)$ for axial vector CWIs. In so doing, the Schwinger term contribution (STC) functions $(R_0^{\nu+1}P)_{JJ';i}(m,k)$, closely associated with the reduction identity (27), are zero for both vector and axial vector CWIs. Nevertheless, in the commonly used renormalization techniques (e.g. in the minimal renormalization schemes) the mass term functions for axial vector CWIs are expressed as rule in terms of the quantity $(-2m^2)(R_0^{\nu-2}I)(m,k)$, calculated according to its proper divergence index $\nu-2$, and in addition it is required the zero value for STC functions that provides the conserved vector CWIs. In so doing, we fall into a *logical trap* since *two different regular values*, $(R_0^{\nu-2}I)(m,k)$ and $(R_0^\nu I)(m,k)$, are prescribed to *the same UV-divergent quantity* $I(m,k)$ involving into two different relations that is prohibited. As a result, we come to the well-known quantum anomalies. It remains to show that *just the function* $(R_0^\nu \mathcal{F})_{00}$ *rather than* $(R_0^{\nu-2}\mathcal{F})_{00} = (R_0^\nu \mathcal{F})_{20}$ *to secure realizability of the identity* $M_\epsilon (R_0^\nu \mathcal{F})_{00} - A(R_0^\nu \mathcal{F})_{20} + (\omega+1)(R_0^\nu \mathcal{F})_{21} = 0$.

4.2. If $n = 2r + \delta_n$, $\delta_n = 0,1$, then $\omega = r - 2 + \delta_n/2$, and from (18) it follows: $\nu_{00} = r-1$, $\lambda_{00} = 2 - \delta_n/2$; $\nu_{10} = r - 2 + \delta_n$, $\lambda_{10} = 1 + \delta_n/2$; $\nu_{20} = r-2$, $\lambda_{20} = 1 - \delta_n/2$; $\nu_{21} = r-1$, $\lambda_{21} = 1 - \delta_n/2$. As a result, the explicit form of basic functions $(R_0^\nu \mathcal{F})_{sj}$ take the form:

$$\begin{aligned}
(R_0^\nu \mathcal{F})_{00} &= M_\epsilon^\omega\, \Gamma(2-\delta_n/2)/\Gamma(r+1) Z_\epsilon^r\, {}_2F_1(1, 2-\delta_n/2;\, r+1;\, Z_\epsilon),\\
(R_0^\nu \mathcal{F})_{20} &= M_\epsilon^\omega\, \Gamma(1-\delta_n/2)/\Gamma(r) Z_\epsilon^{r-1}\, {}_2F_1(1, 1-\delta_n/2;\, r;\, Z_\epsilon),\\
(R_0^\nu \mathcal{F})_{21} &= M_\epsilon^{\omega+1}\, \Gamma(1-\delta_n/2)/\Gamma(r+1) Z_\epsilon^r\, {}_2F_1(1, 1-\delta_n/2;\, r+1;\, Z_\epsilon),\\
(R_0^\nu \mathcal{F})_{10} &= \delta_n (R_0^\nu \mathcal{F})_{00} + (1-\delta_n)(R_0^\nu \mathcal{F})_{20}.
\end{aligned} \tag{34}$$

Properties of the ${}_2F_1$, see Ref. [17], give rise to the important relations

$$(R_0^\nu \mathcal{F})_{00} - (R_0^{\nu-2}\mathcal{F})_{00} = (R_0^\nu \mathcal{F})_{00} - (R_0^\nu \mathcal{F})_{20} = -M_\epsilon^\omega\, \Gamma(\lambda_{00}-1)/\Gamma(1+\nu_{00})\, Z_\epsilon^{\nu_{00}}, \tag{35}$$

$$(R_0^\nu \mathcal{F})_{00} \overset{M_\epsilon \to 0}{\sim} \frac{(-1)\Gamma(\lambda_{00}-1)\, A^{\nu_{00}}}{\Gamma(1+\nu_{00})\, M_\epsilon^{\lambda_{00}-1}}, \quad \text{if } \nu_{00} \geq 0 \;\text{ and }\; \lambda_{00} - 1 > 0. \tag{36}$$

By using equations (34) and factorizing of $M_\epsilon^{\omega+1} Z_\epsilon^r \Gamma(1-\delta_n/2)/\Gamma(r+1)$ in equation (22) one obtains a particular case of the recurrence relation between contiguous hypergeometric functions ${}_2F_1$:

$$b\, {}_2F_1(a, b+1;\, c;\, Z_\epsilon) - (c-1)\, {}_2F_1(a, b;\, c-1;\, Z_\epsilon) + (c-b-1)\, {}_2F_1(a, b;\, c;\, Z_\epsilon) = 0, \tag{37}$$

[see [17], Sec. (2.8)(eq. 42)] for $a = 1$, $b = 1 - \delta_n/2$, $c = r+1$; $c - b - 1 = r - 1 + \delta_n/2 = \omega + 1$.

Equations (34)–(37) show clearly that just the function $(R_0^\nu \mathcal{F})_{00}$ rather than $(R_0^{\nu-2}\mathcal{F})_{00} = (R_0^\nu \mathcal{F})_{20}$ satisfy to the recurrence relation (22). In the mass-isotropic case, $m_1 = m_2 = m \geq 0$, from equations (35)–(36) imply the phenomenologically important limiting values:

$$\begin{aligned}
&\lim_{\epsilon\to 0} [i\epsilon (R_0^\nu I)(m,k)_\epsilon|_{m=0}] = C_g (k_2^2)^{r-1}\Gamma(r)/\Gamma(2r), \quad \lim_{\epsilon\to 0} [i\epsilon (R_0^\nu I)(m,k)_\epsilon|_{m\neq 0}] = 0,\\
&\lim_{m\to 0} [-2m^2 (R_0^\nu I)(m,k)|_{m\neq 0}] = C_g (k_2^2)^{r-1} 2\Gamma(r)/\Gamma(2r), \quad n = 2r,
\end{aligned} \tag{38}$$

of quantities appearing in CWIs (32)–(33) and involving the function $(R_0^\nu I)(m,k)_\epsilon$. In the general mass-anisotropic case, equations (38) must be replaced by

$$(R_0^\nu I)_{JJ';i}(m,k) \overset{m_1,m_2\to 0}{\sim} C_g(k_2^2)^{r-1}\Gamma(r)/\Gamma(2r)\big[1-(-1)^{\pi_i}x\,{}_2F_1(1,r;\,2r;\,1-x^2)\big], \qquad (39)$$

where $x := m_1/m_2$, see Ref. [9]. In fact, equations (38) are particular cases of (39) since for $a_{2r}(x) := x\,{}_2F_1(1,r;\,2r;\,1-x^2)$ there holds $a_{2r}(x) = a_{2r}(1/x)$ and $a_{2r}(0) = a_{2r}(\infty) = 0$, $a_{2r}(1) = 1$.

4.3. So, it is shown that the genuine algorithmic cause of quantum anomalies is of prescribing two different regular values, $(R_0^{\nu-2}I)(m,k)$ and $(R_0^\nu I)(m,k)$, to the same UV-divergent quantity $I(m,k)$ involved into two different Ward identities that is logically prohibited. Quantum anomalies of triangle amplitudes are dictated by the same cause. The SCR [10–13], that makes use of the logically motivated oversubtracted operations, produces such regular values which satisfy the vector and axial vector canonical Ward identities simultaneously.

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[1] Novikov V.A., Okun L.B., Shifman M.A., Vainshtein A.I., Voloshin M.B. and Zakharov V.I., Charmonium and gluons, *Phys. Rep. C*, 1978, V.41, N 1, 1–133.

[2] Narison S., Techniques of dimensional regularization and the two-point functions of QCD and QED, *Phys. Rep. C*, 1982, V.84, N 4, 263–399.

[3] Johnson K., γ_5 invariance, *Phys. Lett.*, 1963, V.5, N 4, 253–254.

[4] Kummer W. and Schweda M., The interpretation of anomalies in Ward identities, *Nucl. Phys. B*, 1970, V.22, N 2, 412–422.

[5] Jackiw R., Topological investigations of quantized gauge theories, in Relativity, Groups, and Topology II, Editors B.S. De Witt and R. Stora, Amsterdam, North Holland, 1984, 221–332.

[6] Jackiw R., New directions for topological research in gauge theories, in "Symposium on Anomalies, Geometry, Topology", Editors W.A. Bardeen and A.R. White, Singapore, World Scientific Publishing Co., 1985, 153–166.

[7] Morozov A.Yu., Anomalies in gauge theories, *Uspehi Fiz. Nauk,* 1986 V.150, N 3, 337–416.

[8] Kucheryavy V.I., Dimensional regularization with natural n-dimensional analog of the γ^5-Dirac matrix, *Ukr. J. Phys.*, 1993, V.38, N 12, 1862–1871; Preprint ITP–92–66E, Kyiv, 1993, 37 p.

[9] Kucheryavy V.I., Symmetries and dynamical symmetry breaking of general n-dimensional self-consistently renormalized spinor diangles, in Proceedings of Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–15 July, 2001, Kyiv), Editors A.G. Nikitin, V.M. Boyko and R.O. Popovych, Kyiv, Insitute of Mathematics, 2002, V.43, Part 2, 629–640.

[10] Kucheryavy V.I., Feynman amplitude and the Meyer G-function. A unified representation for divergent and convergent graphs, *Theor. Math. Phys.*, 1974, V.20, N 1, 29–47.

[11] Kucheryavy V.I., On some algorithmic features of the subtraction procedure in quantum field theory, *Nucl. Phys. B*, 1977, V.127, N 1, 66–86.

[12] Kucheryavy V.I., Simple parametric integral representations for regular (finite) and singular parts of divergent Feynman amplitudes. I. Formulae of general character, *Theor. Math. Phys.*, 1982, V.51, N 3, 355–365.

[13] Kucheryavy V.I., The Bogoliubov–Parasiuk R-operation in quantum field theory, *Ukr. J. Phys.*, 1991, V.36, N 12, 1769–1785 (Ukrainian).

[14] Bogoliubov N.N. and Parasiuk O.S., On the subtraction formalism in multiplicating of causal singular functions, *Izv. AN SSSR, Ser. Mat.*, 1956, V.20, N 5, 585–610.

[15] Bogoliubov N.N. and Parasiuk O.S., Über die Multiplikation der Kausalfunk'tionen in der Quantentheorie der Felder, *Acta Math.*, 1957, V.97, N 3/4, 227–266.

[16] Parasiuk O.S., On the theory of the Bogoliubov R-operation, *Ukr. Math. J.*, 1960, V.12, N 3, 287–307.

[17] Bateman H. and Erdelyi A., Higher transcendental functions. Vol. 1. Hypergeometric function, Moscow, Nauka, 1965.