Relation of Majorization for $*$-Categories and $*$-Wild Categories

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In the present paper notions of majorization and $*$-wildness are defined for the class of $*$-categories.

1 Introduction

In the articles [1, 2] a relation of majorization in the class of $*$-algebras was defined and a definition of $*$-wildness of algebras was given: $*$-algebra is called $*$-*wild* if it majorizes $*$-algebra $C^*(\mathcal{F}_2)$. It is possible (and, seemingly, necessary) to extend these definitions to the class of $*$-categories [3], which is done in the present article.

2 $*$-categories and $*$-quivers

Let $\mathcal{K}$ be a category with involution $*$ over the field $\mathbb{C}$ of complex numbers, in which $a = a^*$ for each $a \in \operatorname{Ob}\mathcal{K}$. Thus, to each morphism $\alpha : a \to b$ a morphism $\alpha^* : b \to a$ is associated, such that:

1) $\alpha^{**} = \alpha$;
2) $(\alpha\beta)^* = \beta^*\alpha^*$;
3) $(z_1\alpha_1 + z_2\alpha_2)^* = \overline{z}_1\alpha_1^* + \overline{z}_2\alpha_2^* \quad (z_1, z_2 \in \mathbb{C})$.

We will assume the presence of zero object in $\mathcal{K}$. Hereinafter we will call such categories $*$-*categories*.

Along with category $\mathcal{K}$ we will consider as a system of generators an involutive quiver Q (see [3, 4]), for any point a of which $a^* = a$ and for any arrow $\alpha : a \to b$ of which an arrow α^* is associated such that $\alpha^{**} = \alpha$. Further such quivers are said to be $*$-*quivers*. Category $\mathcal{K}$ is obtained from the category of paths of the quiver Q by factorization.

The category $\mathcal{K}$ is said to be *finitely generated* if it can be defined in such a way by a finite $*$-quiver and a finite set of interrelations (linear combinations of quiver paths with common beginning and end are declared to be equal to zero). The set of interrelations we always consider to be closed under the involution.

Morphism $\varphi : a \to b$ of the category $\mathcal{K}$ will be called *isomorphism* if there exists a morphism $\varphi^{-1} : b \to a$ such that $\varphi^{-1}\varphi = \varepsilon_a$, $\varphi\varphi^{-1} = \varepsilon_b$; isomorphism φ is called *congruence* if $\varphi^{-1} = \varphi^*$ [3].

Representation of the category $\mathcal{K}$ is a functor concerted with involution (involutive functor) π over the field $\mathbb{C}$ from the category $\mathcal{K}$ to the category $\mathcal{H}$ of Hilbert spaces whose objects are separable Hilbert spaces, and morphisms are bounded linear operators from one space to another; the involution on objects is identical and on morphisms it is a transition to the adjoint operator.

Representations of the category $\mathcal{K}$ themselves form a $*$-category $\operatorname{Rep}\mathcal{K}$, whose objects are involutive functors π (representations) and morphisms are families of morphisms of the category $\mathcal{H}$ intertwining these functors (natural transformations of functors). Two representations are said to be *equivalent* if they are isomorphic in the category of representations, and *unitary equivalent* if they are congruent in the category of representations.

As it is well known, the category of representations $\operatorname{Rep}\mathcal{K}$ is quadratically closed (for any morphism α there exists a self-adjoint morphism $\beta = \beta^*$ such that $\beta^2 = \alpha^*\alpha$), therefore the equivalence of two representations implies their unitary equivalence (see [3]).

Let us define a category of matrices $\mathcal{M}(\mathcal{K})$ over the category $\mathcal{K}$. Its objects are ordered collections $(a_1, a_2, \ldots, a_n)$ of objects from $\mathcal{K}$ (not excluding the matching of a_i and a_j when $i \neq j$). Morphism from $(a_1, a_2, \ldots, a_n)$ to $(b_1, b_2, \ldots, b_m)$ is a matrix of dimension $m \times n$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \end{bmatrix},$$

where $\alpha_{ij} : a_j \to b_i$ is a morphism of the category $\mathcal{K}$. We will indicate over the j-th column an object a_j, on the left of i-th row we will indicate an object b_i such that α_{ij} "leads" from the object a_j to the object b_i. Final notation for the matrices-morphisms of the category $\mathcal{M}(\mathcal{K})$ has a form:

$$A = \begin{array}{c|cccc|} & a_1 & a_2 & \ldots & a_n \\ \hline b_1 & \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ b_2 & \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_m & \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \\ \hline \end{array}.$$

In this case it is possible to construct a composition and add the morphisms of the category $\mathcal{M}(\mathcal{K})$ by usual rules of multiplication and adding of matrices.

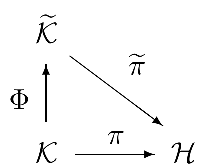
The involution on $\mathcal{M}(\mathcal{K})$ is defined naturally:

$$A^* = \begin{array}{c|cccc|} & b_1 & b_2 & \ldots & b_m \\ \hline a_1 & \alpha^*_{11} & \alpha^*_{21} & \ldots & \alpha^*_{m1} \\ a_2 & \alpha^*_{12} & \alpha^*_{22} & \ldots & \alpha^*_{m2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & \alpha^*_{1n} & \alpha^*_{2n} & \ldots & \alpha^*_{mn} \\ \hline \end{array}.$$

3 Relation of majorization for ∗-categories

Let us define a wrapping category.

Definition 1. Let $\mathcal{K}$, $\widetilde{\mathcal{K}}$ be ∗-categories. A pair $(\widetilde{\mathcal{K}}, \Phi : \mathcal{K} \to \widetilde{\mathcal{K}})$, where Φ is involutive functor from ∗-category $\mathcal{K}$ to ∗-category $\widetilde{\mathcal{K}}$, is called a *wrapping category* of the category $\mathcal{K}$ if for any ∗-representation $\pi : \mathcal{K} \to \mathcal{H}$ there exists a unique representation $\widetilde{\pi} : \widetilde{\mathcal{K}} \to \mathcal{H}$ such that the diagram



is commutative, and any morphism of the category $\operatorname{Rep}\mathcal{K}$ intertwining two representations π_1 and π_2 of the category $\mathcal{K}$ also intertwines representations $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ of the category $\widetilde{\mathcal{K}}$.

Let $\mathcal{M}(\mathcal{K})$ be a category of matrices over ∗-category $\mathcal{K}$, $(\widetilde{\mathcal{M}(\mathcal{K})}, \Phi : \mathcal{M}(\mathcal{K}) \to \widetilde{\mathcal{M}(\mathcal{K})})$ is its wrapping category.

Any ∗-representation $\pi : \mathcal{K} \to \mathcal{H}$ induces a representation $\mathcal{M}(\mathcal{K}) \to \mathcal{H}$ and, therefore, representation $\widetilde{\pi} : \widetilde{\mathcal{M}(\mathcal{K})} \to \mathcal{H}$.

If $\Psi : \mathcal{L} \to \widetilde{\mathcal{M}(\mathcal{K})}$ is involutive functor, then in a natural way a functor

$$F_\Psi : \operatorname{Rep}\mathcal{K} \to \operatorname{Rep}\mathcal{L}$$

can be constructed.

By definition $F_\Psi(\pi) = \widetilde{\pi} \circ \Psi$, and if $C = (C_a)_{a\in \operatorname{Ob}\mathcal{K}}$ is a morphism of representations from $\operatorname{Rep}\mathcal{K}$, $C_a : \pi(a) \to \pi_1(a)$, $(a_1, a_2, \dots, a_k) \in \operatorname{Ob}\mathcal{M}(\mathcal{K})$, then

$$F_\Psi(C) = \operatorname{diag}(C_{a_1}, C_{a_2}, \dots, C_{a_k}) \\ \in \operatorname{Hom}_{\mathcal{M}(\mathcal{K})}((\pi(a_1), \pi(a_2), \dots, \pi(a_k)); (\pi_1(a_1), \pi_1(a_2), \dots, \pi_1(a_k))).$$

Definition of the faithful and complete functor see in [5].

Definition 2. $*$-category $\mathcal{L}$ *directly majorizes* a $*$-category $\mathcal{K}$ ($\mathcal{L} \rhd \mathcal{K}$) if at the category of matrices $\mathcal{M}(\mathcal{K})$ exist a wrapping category $(\widetilde{\mathcal{M}(\mathcal{K})},\ \Phi)$ and $*$-functor $\Psi : \mathcal{L} \to \widetilde{\mathcal{M}(\mathcal{K})}$ such that the functor $F_\Psi : \operatorname{Rep}\mathcal{K} \to \operatorname{Rep}\mathcal{L}$ (faithful by definition) will be complete.

$*$-category *majorizes* a $*$-category $\mathcal{K}$ ($\mathcal{L} \succ \mathcal{K}$) if there exist $*$-categories $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_n$ such that $\mathcal{L} \equiv \mathcal{K}_0 \rhd \mathcal{K}_1 \rhd \mathcal{K}_2 \rhd \cdots \rhd \mathcal{K} \equiv \mathcal{K}_{n+1}$.

Remark 1. Let $F_{\Psi_i} : \operatorname{Rep}\mathcal{K}_i \to \operatorname{Rep}\mathcal{K}_{i-1}$ be a faithful and complete functor existing due to direct majorization $\mathcal{K}_i \rhd \mathcal{K}_{i-1}$. Then functor

$$F = F_{\Psi_1} \circ F_{\Psi_2} \circ \cdots \circ F_{\Psi_{n+1}} : \operatorname{Rep}\mathcal{K} \to \operatorname{Rep}\mathcal{L}$$

is faithful and complete.

We will say that a $*$-quiver Q_1 *majorizes* a $*$-quiver Q_2 ($Q_1 \succ Q_2$) if for categories corresponding to these quivers $\mathcal{K}(Q_1) \succ \mathcal{K}(Q_2)$.

Absolutely evident is

Proposition 1. *Relation of majorization is relation of quasiorder in the class of $*$-categories ($*$-quivers).*

4 On $*$-wild categories

Let us consider $*$-algebras:

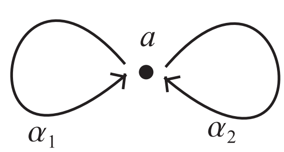
$$\mathcal{F}_2 = \mathbb{C}\langle u_1, u_2 \,|\, u_i^* = u_i^{-1},\ i = 1, 2\rangle, \\ \mathcal{A}_n = \mathbb{C}\langle \alpha_1, \alpha_2\ \dots, \alpha_n \,|\, \alpha_i^* = \alpha_i,\ i = \overline{1, n}\rangle.$$

In article [6], in fact, it was shown that $\mathcal{F}_2 \succ \mathcal{A}_2$, $\mathcal{A}_2 \succ \mathcal{F}_2$ and $\mathcal{A}_2 \succ \mathcal{A}_n$ when $n \geq 2$.

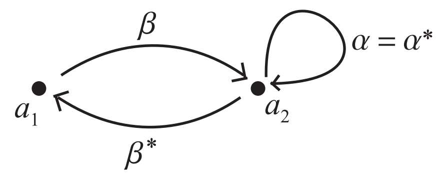
Definition 3. We will call $*$-category $\mathcal{K}$ ($*$-quiver $\mathcal{Q}$) $*$-*wild* if $\mathcal{K} \succ \mathcal{F}_2$ (category corresponding to the quiver $\mathcal{K}(Q) \succ \mathcal{F}_2$).

In [4, 7] it was proved that $*$-quiver Q (without interrelations) is $*$-wild iff Q (as a quiver without involution $*$) is wild.

Example 1. Let Q_1 be a $*$-quiver



with involution $\alpha_1^* = \alpha_1$, $\alpha_2^* = \alpha_2$ (without additional interrelations) and Q_2 be a $*$-quiver



with interrelation $\beta^*\beta = \varepsilon_{a_1}$.

Let us show that $Q_1 \prec Q_2$, i.e. quiver Q_2 is $*$-wild.

Let us construct a functor $\Psi : \mathcal{K}(Q_2) \to \mathcal{M}(\mathcal{K}(Q_1))$ putting $\Psi(a_1) = a$, $\Psi(a_2) = (a, a)$,

$$\Psi(\beta) = \begin{array}{c|c|} & a \\ \cline{2-2} a & \varepsilon_a \\ a & 0_a \\ \cline{2-2} \end{array}\,, \quad \Psi(\alpha) = \begin{array}{c|cc|} & a & a \\ \cline{2-3} a & \alpha_1 & \varepsilon_a \\ a & \varepsilon_a & \alpha_2 \\ \cline{2-3} \end{array}\,.$$

It is immediately checked on that functor F_Ψ is faithful and complete.

[1] Kruglyak S.A. and Samoilenko Yu.S., On the structural theorems for families of idempotents, *Ukr. Math. J.*, 1998, V.50, N 4, 523–533.

[2] Kruglyak S.A. and Samoilenko Yu.S., On the complexity of description of representations of $*$-algebras generated by idempotents, *Proc. Amer. Math. Soc.*, 2000, V.128, N 6, 1655–1664.

[3] Roiter A.V., Boxes with involution, in Representations and Quadratic Forms, Editor Yu.A. Mytropolskii, Kyiv, Institute of Mathematics, 1979, 124–126.

[4] Kruglyak S.A., Representations of free involutive quivers, in Representations and Quadratic Forms, Editor Yu.A. Mytropolskii, Kyiv, Institute of Mathematics, 1979, 149–151.

[5] Bass H., Algebraic K-theory, Moscow, Mir, 1973.

[6] Kruglyak S.A. and Samoilenko Yu.S., On the unitary equivalence of collections of self-adjoint operators, *Funct. Anal. and its Appl.*, 1980, V.14, N 1, 60–62.

[7] Kruglyak S.A., Representations of involutive quivers, Dep. VINITI N 7266–84, 1984, 26 pages.