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Nonlinear Amplitude Maxwell–Dirac Equations. Optical Leptons

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We apply the method of slowly-varying amplitudes of the electrical and magnetic fields to integro-differential system of nonlinear Maxwell equations. The equations are reduced to system of differential Nonlinear Maxwell amplitude Equations (NME). The electric and magnetic fields are presented as sums of circular and linear components. Thus, NME is written as a set of Nonlinear Dirac Equations (NDE). Exact solutions of NDE with classical orbital momentum $\ell = 1$ and opposite directions of the spin (opposite charge) $j = \pm 1/2$ are obtained. Using the Poynting vector for solutions with spin $j = 1/2$ we find that the energy flow through arbitrary closed surface around our vortex solutions is zero and the localized energy of our solutions circulate in x, y plane. Other important result is that the vortex solutions with spin $j = 1/2$ without external fields are immovable. The initial investigations on stability of these solutions show that vortices with spin $j = 1/2$ are stable while the vortices with opposite spin (charge) $j = -1/2$ are not. The possible generalization of NME to higher number of optical components and higher number of ℓ and j is discussed.

1 Introduction

The interest in the nonlinear generalizations of the quantum field equations [1–3] and in the possibility of obtaining exact stationary solitary solutions of the field equations [4, 5] increases rapidly in the last years. As a rule, different kinds of nonlinearity have been introduced in an *ad-hoc* fashion in the Klein–Gordon equation and also for all four spinor components of the Dirac equations. For the usual case of a cubic nonlinearity, exact $3D + 1$ localized solutions are not found. Our present work, reported in this paper, shows that *the optical analogy* of the Nonlinear Dirac Equations of field (NDE) leads to a nonlinear part *only in the first* coupled equation. This result allows to solve the NDE by separation of variables and to obtain solutions representing optical vortices with classical momenta one and spin one-half.

2 Maxwell's equations with non-stationary linear and nonlinear polarization

Consider the Maxwell's equations in the next case: A source-free medium with non-stationary linear and nonlinear electric polarization and non-stationary magnetic polarization. For this case, the Maxwell's equations can be written:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \qquad \nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}, \tag{1}$$

$$\nabla \cdot \vec{D} = 0, \qquad \nabla \cdot \vec{B} = \nabla \cdot \vec{H} = 0. \tag{2}$$

The linear and nonlinear electric polarization and the the linear magnetic polarization are presented as:

$$\vec{D} = \vec{P}^{\rm lin} + 4\pi \vec{P}_{\rm nl}, \qquad \vec{B} = \vec{H} + 4\pi \vec{M}_{\rm lin}, \tag{3}$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic intensity fields, $\vec{D}$ and $\vec{B}$ are the electric and magnetic induction fields, $\vec{P}^{\text{lin}}$, $\vec{P}_{\text{nl}}$ are the linear and nonlinear polarizations of the medium respectively and $\vec{M}_{\text{lin}}$ is the linear magnetic polarization. The magnetic polarization (magnetization) $\vec{M}_{\text{lin}}$ is written as the product of the linear magnetic susceptibility $\eta^{(1)}$ and the magnetic field $\vec{H}$. The nonstationary linear electric polarization can be written as:

$$\begin{aligned}\vec{P}^{\text{lin}} &= \int_{-\infty}^{t} \left(\delta(t-\tau) + 4\pi\chi^{(1)}(t-\tau)\right)\vec{E}(\tau,x,y,z)d\tau \\ &= \int_{-\infty}^{t} \varepsilon_0(t-\tau)\,\vec{E}(\tau,x,y,z)\,d\tau, \end{aligned} \tag{4}$$

where $\chi^{(1)}$ and ε_0 are the linear electric susceptibility and the dielectric constant respectively. Similar expression describes the dependence of $\vec{B}$ on $\vec{H}$ in the case of nonstationary linear magnetic polarization [6]:

$$\begin{aligned}\vec{B} &= \int_{-\infty}^{t} \left(\delta(t-\tau) + 4\pi\eta^{(1)}(t-\tau)\right)\vec{H}(\tau,x,y,z)d\tau \\ &= \int_{-\infty}^{t} \mu_0(t-\tau)\,\vec{H}(\tau,x,y,z)\,d\tau, \end{aligned} \tag{5}$$

where $\eta^{(1)}$ and μ_0 are the linear magnetic susceptibility and magnetic permeability respectively. In the following, we will study such media with nonstationary cubic nonlinear polarization, where the nonlinear polarization in the case of one carrying frequency can be expressed as:

$$\begin{aligned}\vec{P}_{\text{nl}}^{(3)} &= \frac{3}{4}\int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} \chi^{(3)}(t-\tau_1, t-\tau_2, t-\tau_3) \\ &\quad\times \vec{E}(\tau_1,x,y,z)\vec{E}^*(\tau_2,x,y,z)\vec{E}(\tau_3,x,y,z)d\tau_1 d\tau_2 d\tau_3. \end{aligned} \tag{6}$$

3 Amplitude equations

We derived in a recent paper [7] the slowly varying amplitude approximation from nonlinear integro-differential Maxwell's equations (1)–(3) in the standard way, as it was done in [10, 11]. The electric and magnetic field amplitudes are determined by the relations:

$$\vec{E}(x,y,z,t) = \vec{A}(x,y,z,t)\exp\left(i\left(\omega_0 t - g(x,y,z)\right)\right), \tag{7}$$

$$\vec{H}(x,y,z,t) = \vec{C}(x,y,z,t)\exp\left(-i\left(\omega_0 t - q(x,y,z)\right)\right), \tag{8}$$

where $\vec{A}$, $\vec{C}$, ω_0, g and q are the amplitudes of the electric and magnetic fields, the optical frequency and the real spatial phase functions respectively. After using Fourier representation of the response functions and of the amplitude functions $\vec{A}$ (and $\vec{C}$), and also the fact that $\nabla\cdot\vec{D} \approx \nabla\cdot\vec{E} \approx \nabla\cdot\vec{A} \approx 0$ [8], we obtain the following system of Nonlinear Maxwell vector amplitude Equations (NME), written in dimensionless variables:

$$\nabla\times\vec{A} = i\alpha_2\vec{C} - \delta\frac{\partial\vec{C}}{\partial t}, \tag{9}$$

$$\nabla\times\vec{C} = i\alpha_1\vec{A} + \frac{\partial\vec{A}}{\partial t} + i\gamma_1\left(\vec{A}\cdot\vec{A}^*\right)\vec{A}, \tag{10}$$

$$\nabla\cdot\vec{A} = 0, \qquad \nabla\cdot\vec{C} = 0. \tag{11}$$

where $\alpha_{1,2}$, δ and γ_1 are constants. The gradient of the spatial phase functions g and q satisfied the additional relations:

$$\nabla g \times \vec{A} = 0, \qquad \nabla q \times \vec{C} = 0. \tag{12}$$

The phase functions which satisfied (12) are determined in Section 6.

4 Dirac representation of NME

To solve the NME (9)–(11), we apply the method of separation of variables. The slowly varying amplitude vector of the electric field $\vec{A}$ and the magnetic field $\vec{C}$ are represented as:

$$\vec{A}(x,y,z,t) = \vec{F}(x,y,z)\exp(i\Delta\alpha t), \qquad \vec{C}(x,y,z,t) = \vec{G}(x,y,z)\exp(i\Delta\alpha t). \tag{13}$$

Substituting these forms into the NME (9)–(11) we obtain:

$$\nabla \times \vec{F} = -i\nu_2\vec{G}, \tag{14}$$

$$\nabla \times \vec{G} = i\nu_1\vec{F} + i\gamma_1\big(\vec{F}\cdot\vec{F}^*\big)\vec{F}, \tag{15}$$

$$\nabla \cdot \vec{F} = 0, \qquad \nabla \cdot \vec{G} = 0, \tag{16}$$

where $\nu_1 = \alpha_1 + \Delta\alpha$, $\nu_2 = \delta\Delta\alpha - \alpha_2 > 0$ are constants. When the electric and magnetic fields are represented as a sum of a linear polarization component and a circular polarized one it is possible to reduce equations (14)–(16) to a system of four nonlinear equations. Substituting [9]:

$$\psi_1 = iF_z, \qquad \psi_2 = iF_x - F_y, \qquad \psi_3 = -G_z, \qquad \psi_4 = -G_x - iG_y \tag{17}$$

into the nonlinear system (14)–(16), we obtain a stationary nonlinear Dirac system of equations (NDE):

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\Psi_4 + \frac{\partial}{\partial z}\Psi_3 = -i\left(\nu_1 + \gamma_1\sum_{i=1}^{2}|\Psi_i|^2\right)\Psi_1, \tag{18}$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\Psi_3 - \frac{\partial}{\partial z}\Psi_4 = -i\left(\nu_1 + \gamma_1\sum_{i=1}^{2}|\Psi_i|^2\right)\Psi_2, \tag{19}$$

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\Psi_2 + \frac{\partial}{\partial z}\Psi_1 = -i\nu_2\Psi_3, \tag{20}$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\Psi_1 - \frac{\partial}{\partial z}\Psi_2 = -i\nu_2\Psi_4. \tag{21}$$

The system (18)–(21) is the optical analog of the nonlinear Dirac equations (NDE). Note that the *optical* NDE are significantly different from the NDE in the field theory. The nonlinear part appears *only* in the first two coupled equations of the system.

5 Hamiltonian representation of the NDE. First integrals for vortex solutions with spin $j = \pm 1/2$

It is not difficult to show that for the NDE system of equations (18)–(21) the Hamiltonian has the form:

$$H = \big(\vec{\sigma}\cdot\vec{P}\big) + \sum_{i=1}^{2}|\Psi_i|^2, \tag{22}$$

By virtue of it, equations (18)–(21) can be rewritten:

$$H\Psi = \varepsilon\Psi, \tag{23}$$

where $\varepsilon = (-i\nu_1, -i\nu_1, -i\nu_2, -i\nu_2)$ is the energy operator. Here we investigate the case where the nonlinear part of the equation is represented as a number of spinors with a scalar sum that depends only on the radial component

$$\sum_{i=1}^{2} |\Psi_i(r,\theta,\varphi)|^2 = F(r). \tag{24}$$

We also introduce here the well known orbital momentum operator $\vec{L}$, own orbital (spin) momentum $\vec{S}$, and the full momentum $\vec{J}$. It is straightforward to show that the Hamiltonian (22) of equation (23) commutes with the operators $\vec{J}^2$ and J_z (the z-projections must be x or y). Using these symmetries and the condition that the nonlinearity is of Kerr type, we can solve the NDE equations (23) by a separation of variables technique. We look for solutions in the form:

$$\Psi_1 = a(r)\,\Omega_{jlM}, \tag{25}$$
$$\Psi_2 = a(r)\,\Omega_{jlM}, \tag{26}$$
$$\Psi_3 = ib(r)\,\Omega_{jl'M}, \tag{27}$$
$$\Psi_4 = ib(r)\,\Omega_{jl'M}, \tag{28}$$

where Ω_{jlm} is the spherical spinor, $l + l' = 1$, and $a(r)$ and $b(r)$ are arbitrary radial functions. Using the symmetries of (23) and the fact that the nonlinear parts depend on r we separate variables and obtain the following system of equations for the radial part:

$$\frac{\partial a(r)}{\partial r} + \frac{1+\chi}{r}a(r) = -\nu_2 b(r), \tag{29}$$
$$\frac{\partial b(r)}{\partial r} + \frac{1-\chi}{r}b(r) = \nu_1 a(r) + \gamma|a(r)|^2 a(r), \tag{30}$$

where

$$\chi = l(l+1) - j(j+1) - 1/4. \tag{31}$$

Excluding $b(r)$ from the system (29)–(30), we obtain the equation for $a(r)$:

$$\nu_1\nu_2 a(r) + \frac{\partial^2 a}{\partial r^2} + \frac{2}{r}\frac{\partial a}{\partial r} - \frac{(1+\chi)\chi}{r^2}a + \nu_2\gamma|a|^2 a = 0. \tag{32}$$

The case of optical spinors with spin $j = \pm 1/2$ corresponds to localized solutions with $\chi = 1$ and spherical spinors with $l = 1$ and $j = \pm 1/2$. The spherical spinors are a spinor generalization of usual spherical functions and the spinors (25)–(28) can by written directly in this case:

$$\Psi_1 = a(r)\cos(\theta), \qquad \Psi_2 = a(r)\sin(\theta)\,e^{i\varphi}, \qquad \Psi_3 = -ib(r), \qquad \Psi_4 = 0, \tag{33}$$

for $j = 1/2$ and

$$\Psi_1 = a(r)\sin(\theta)\,e^{-i\varphi}, \qquad \Psi_2 = -a(r)\cos(\theta), \qquad \Psi_3 = 0, \qquad \Psi_4 = -ib(r), \tag{34}$$

for $j = -1/2$.

After substituting solutions (33)–(34) into equations (23) and excluding $b(r)$ from the radial system (29)–(30), we obtained the following equation for $a(r)$:

$$\nu_1\nu_2 a(r) + \frac{\partial^2 a}{\partial r^2} + \frac{2}{r}\frac{\partial a}{\partial r} - \frac{2}{r^2}a + \nu_2\gamma|a|^2 a = 0. \tag{35}$$

This system has exact vortex *de Broglie* soliton solutions [4] in the form:

$$a(r) = \frac{\sqrt{2}}{i}\frac{\exp\left(i\sqrt{\nu_1\nu_2}r\right)}{r}, \tag{36}$$

if $\nu_2\gamma_1 = 1$. The complete solutions for these two cases are written:

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{i}\frac{\exp(i\sqrt{\nu_1\nu_2}r)}{r}\cos(\theta)\\ \frac{\sqrt{2}}{i}\frac{\exp(i\sqrt{\nu_1\nu_2}r)}{r}\sin(\theta)\exp(i\varphi)\end{pmatrix}, \tag{37}$$

$$\begin{pmatrix}\Psi_3\\ \Psi_4\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{i\nu_2}\left(\frac{i\sqrt{\nu_1\nu_2}\exp\left(i\sqrt{\nu_1\nu_2}r\right)}{r} + \frac{\exp i(\sqrt{\nu_1\nu_2}r)}{r^2}\right)\\ 0\end{pmatrix}, \tag{38}$$

for $j = 1/2$ and

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{i}\frac{\exp(i\sqrt{\nu_1\nu_2}r)}{r}\sin(\theta)\exp-(i\varphi)\\ \frac{-\sqrt{2}}{i}\frac{\exp(i\sqrt{\nu_1\nu_2}r)}{r}\cos(\theta)\end{pmatrix}, \tag{39}$$

$$\begin{pmatrix}\Psi_3\\ \Psi_4\end{pmatrix} = \begin{pmatrix}0\\ \frac{-\sqrt{2}}{i\nu_2}\left(\frac{i\sqrt{\nu_1\nu_2}\exp\left(i\sqrt{\nu_1\nu_2}r\right)}{r} + \frac{\exp i(\sqrt{\nu_1\nu_2}r)}{r^2}\right)\end{pmatrix}, \tag{40}$$

for $j = -1/2$. The equation (32) admits exact "de Broglie" soliton solutions for arbitrary number of χ, but as we remember our solutions are limited by the conditions (24), the nonlinear part to depend only on the radial components. The condition (24) for a number $\chi \geq 1$ can be fulfilled also for a higher number of fields with different frequencies. This case includes also the parametric processes.

6 Spatial phase functions, Poynting vector and the flow of energy

In a previous paper [7] we investigate the localized energy of our vortex solutions. A spectral region and dispersion parameters of a paramagnetic media were found, where the solutions admit finite energy. The type of the phase functions which satisfied (12) is determined for vortex solutions (33) with spin $j = 1/2$. Using again the relations between the spinors of NDE and the amplitude functions (17) we have:

$$F_x = \frac{\psi_2 - \psi_2^*}{2i}, \qquad F_y = \frac{\psi_2 + \psi_2^*}{2}, \qquad F_z = \frac{\psi_1 - \psi_1^*}{2i}, \tag{41}$$

$$G_x = -\frac{\psi_4 + \psi_4^*}{2}, \qquad G_y = -\frac{\psi_4 - \psi_4^*}{2i}, \qquad G_z = -\frac{\psi_3 - \psi_3^*}{2i}. \tag{42}$$

Substituting the solutions (33) with spin $j = 1/2$ in (41)–(42) for arbitrary real $a(r)$ and $b(r)$ we obtain:

$$F_x = -a(r)\frac{x}{r}, \qquad F_y = -a(r)\frac{y}{r}, \qquad F_z = -a(r)\frac{z}{r}, \tag{43}$$

$$G_x = 0, \qquad G_y = 0, \qquad G_z = b(r). \tag{44}$$

We rewrite again the conditions for the spatial phase functions:

$$\nabla g \times \vec{F} = 0, \qquad \nabla q \times \vec{G} = 0. \tag{45}$$

These relations for solutions of kind (43)–(44) are satisfied only when:

$$g = k_0 r \quad \text{or} \quad g = k_0 f(r) \qquad \text{and} \qquad q = k_0 z \quad \text{or} \quad q = k_0 f(z), \tag{46}$$

where k_0 is the carrying wave number. The spatial phase functions of kind $g = k_0 r$ and $q = k_0 z$ correspond to spectral limited pulses which satisfied additional relations $\triangle k \triangle r = \text{const}$. The spatial phase functions of kind $g = k_0 f(r)$ and $q = k_0 f(z)$ correspond to phase modulated pulses and for them the relations $\triangle k \triangle r = \text{const}$ is not satisfied. The Poynting vector can be expressed by the amplitude functions of the electrical and magnetic field:

$$\begin{aligned} \vec{S} &= \vec{E}(x,y,z,t) \times \vec{H}(x,y,z,t) \\ &= \exp\left(i\left(W(t) - q(z) \pm g(r)\right)\right) \vec{F}(x,y,z,t) \times \vec{G}(x,y,z,t), \end{aligned} \tag{47}$$

where q and g are scalar phase functions. Substituting the solutions with spin $j = 1/2$ in above expression we find that:

$$\vec{S} = \exp\left(i\left(W(t) - q(z) \pm g(r)\right)\right) \left(-a(r)b(r)\frac{y}{r};\ a(r)b(r)\frac{x}{r};\ 0\right). \tag{48}$$

We see that the Poynting vector $\vec{S}$ is one circulation vector for solutions with spin $j = 1/2$ and its divergency is zero:

$$\nabla \cdot \vec{S} = 0. \tag{49}$$

The relation (49) determines that the energy flow through arbitrary closed surface around our vortex solutions with spin $j = 1/2$ is zero. The relation (48) shows that flow of energy of our solutions circulate in x, y plane. We generalize the above results for solutions with spin $j = 1/2$: The vortex solutions with spin $j = 1/2$ without external fields are immovable and electromagnetic energy oscillates in x, y plane. The electrical field oscillates spherically, in 'r' direction, while the magnetic field oscillates in z direction. In the same way the Poynting vector was calculated for solutions with spin $j = -1/2$. For them we obtain that $\nabla \cdot \vec{S} \neq 0$ and we expect that they are not stable. Exact investigation of stability requires investigation also of the perturbation of the Poynting vector and will be discussed in a further paper.

7 Conclusion

We have shown that in cases of linear and circularly polarized components of the electric and magnetic fields, the NME reduces to the Nonlinear Dirac system of equations (NDE). The equations are represented in a spinor form. Using the method of separation of variables, exact vortex solutions have been obtained. The optical vortex solutions admit classical orbital momentum $l = 1$ and classical own momentum $j = \pm 1/2$. The energy integral of the vortex solutions is finite only in some special cases of paramagnetic media with suitable conditions on linear electric and magnetic dispersion. Using the Poynting vector for solutions with spin $j = 1/2$ we find that the energy flowing through arbitrary closed surface around our vortex solutions is zero and the localized energy of our solutions circulates in x, y plane. Other important result is, that the vortex solutions with spin $j = 1/2$ without external fields are immovable. The electrical field in the vortices oscillating spherically, while the magnetic field oscillating in z direction. The initial investigations on stability of these solutions show the following: While the vortices with spin $j = 1/2$ are stable, the vortices with opposite spin (charge) $j = -1/2$ are not. All of the above results will be discussed later in relation with nonlinear field theory.

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