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Poisson Algebras on Elliptic Curves

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We describe Poisson algebras associated with classical analogs of some self-dual generalized eigenvalue problems. These algebras are related in a natural way to various elliptic curves.

1 Introduction

We work with the classical mechanical systems of one degree of freedom. Let $X(q,p)$ and $Y(q,p)$ are two independent dynamical variables of canonical variables q, p with the standard Poisson bracket $\{q,p\}=1$. Functions X and Y are called independent if they satisfy the condition

$$\frac{\partial(X,Y)}{\partial(q,p)} \equiv \frac{\partial X}{\partial q}\frac{\partial Y}{\partial p} - \frac{\partial X}{\partial p}\frac{\partial Y}{\partial q} = \{X,Y\} \neq 0, \tag{1}$$

in some domain of interest of the phase space (q,p). Here $\partial(X,Y)/\partial(q,p)$ denotes the Jacobian of a change of variables.

According to the definition proposed in [9], two independent variables X and Y form a *classical Leonard pair* (CLP) if there exist two different canonical transformations $(q,p)\to(x,y)$ and $(q,p)\to(\xi,\eta)$ such that the first transformation brings X and Y to the form

$$X = \varphi(x), \qquad Y = A_1(x)e^{y} + A_2(x)e^{-y} + A_3(x) \tag{2}$$

and in the second case we have the representation

$$X = B_1(\xi)e^{\eta} + B_2(\xi)e^{-\eta} + B_3(\xi), \qquad Y = \psi(\xi), \tag{3}$$

where (x,y) and (ξ,η) are canonical pairs (i.e., $\{x,y\}=\{\xi,\eta\}=1$) and $\varphi(x)$, $A_i(x)$, $\psi(\xi)$, $B_i(\xi)$ are some functions. Using canonical transformations $y\to\kappa y$, $x\to x/\kappa$ and taking the limit $\kappa\to 0$ one can obtain from (2) the limiting form $Y = a_1(x)y^2 + a_2(x)y + a_3(x)$. Therefore we shall assume that CLP admit such degenerate forms of Y in (2) (or of X in (3)) without further reservations.

It is convenient to introduce the variable

$$Z = \{X,Y\}. \tag{4}$$

We assume that there exists a region of values of X, Y where X and Y form independent variables, that is $Z\neq 0$. The latter means that in this domain we can invert the changes of variables and find $x = x(X,Y)$, $y = y(X,Y)$. As a result, we can consider Z as a function of X and Y, $Z = Z(X,Y)$. The condition that X and Y form a CLP allows us to establish the explicit form of this function $Z(X,Y)$. Namely, as shown in [9] there exist 9 arbitrary constants α_{ik}, $i,k=0,1,2$, such that

$$Z^2 = \sum_{i,k=0}^{2} \alpha_{ik} X^i Y^k \equiv -F(X,Y). \tag{5}$$

Vice versa, it can be shown that starting from the condition (5) for arbitrary α_{ik} one arrives at a CLP (including its degenerate form mentioned above). The condition $F=0$ determines the region of the phase space with complex values of q, p where such a consideration breaks down.

From (5) it follows that the dynamical variables X, Y and $Z=\{X,Y\}$ form a Poisson algebra with the defining relations (4) and

$$\{Z,X\} = \frac{1}{2}\frac{\partial F(X,Y)}{\partial Y}, \qquad \{Y,Z\} = \frac{1}{2}\frac{\partial F(X,Y)}{\partial X}, \tag{6}$$

which are known as the classical Askey–Wilson algebra relations [6]. This Poisson algebra generates relation (5) with the interpretation of the constant α_{00} as a value of the corresponding Casimir element [9]. In this way we get a particular example of the quadratic algebras, the most popular representative of which is given by the Sklyanin algebra [13]. In a more general setting, algebraic relations between dynamical variables involve polynomials of generators (see, e.g., particular polynomial quantum algebras in [14]).

Suppose that X is the Hamiltonian of some physical system. Then the first canonical transformation $(q,p)\to(x,y)$ is, in fact, an action-angle transformation: it maps X into a function depending on only one canonical variable x. Similarly, canonical transformation $(q,p)\to(\xi,\eta)$ is an action-angle variables transformation for a system with the Hamiltonian Y. Existence of a CLP can be considered as some duality property of two Hamiltonians with respect to prescribed dependence on the momenta y and η of the "conjugated" Hamiltonians (i.e., Y and X, respectively). From this point of view, the CLP property is equivalent to the notion of duality discussed in the theory of integrable systems, see, e.g., [12, 5].

After quantization, functions $e^{\pm p}$ become shift operators and the quantum analogue of the CLP property coincides with the standard Leonard duality [10] or the bispectrality condition [4] for two tridiagonal $N\times N$ matrices L, M. In this case, the matrix M is tridiagonal in the basis formed by eigenvectors ϕ_k of the matrix L, whereas M is tridiagonal in the basis formed by eigenvectors ψ_k of M:

$$L\phi_k = \lambda_k\phi_k, \qquad M\phi_k = \alpha_{k+1}\phi_{k+1} + \beta_k\phi_k + \gamma_k\phi_{k-1}, \tag{7}$$

and

$$M\psi_k = \mu_k\psi_k, \qquad L\psi_k = \xi_{k+1}\psi_{k+1} + \eta_k\psi_k + \zeta_k\psi_{k-1}, \tag{8}$$

where $k=1,2,\ldots,N$. It is assumed that eigenvalues λ_k, μ_k are nondegenerate, so that the vectors ϕ_k and ψ_k form two independent complete bases. In such a form the problem of classifying all ("quantum") Leonard pairs L, M was investigated by Terwilliger [18]. It is equivalent to the original Leonard problem [10] in the following sense. Let us decompose the vectors ψ_k in the basis of vectors ϕ_k

$$\psi_k = \sum_{s=1}^{N} P_{ks}\phi_s, \tag{9}$$

with some expansion coefficients P_{ks}. It follows from (7), (8) that the coefficients P_{ks} satisfy simultaneously two three-term recurrence relations

$$\zeta_{s+1}P_{k,s+1} + \eta_s P_{ks} + \xi_{s-1}P_{k,s-1} = \mu_k P_{ks} \tag{10}$$

and

$$\gamma_{k+1}P_{k+1,s} + \beta_k P_{ks} + \alpha_{k-1}P_{k-1,s} = \lambda_s P_{ks}, \tag{11}$$

which mean that P_{ks} can be expressed in terms of some orthogonal polynomials of the argument λ_s or μ_k. It appears that these polynomials are self-dual: permutation of the discrete variables k and s is equivalent to some permutation of parameters entering the recurrence coefficients η_s, ξ_s (for details, see [10, 18, 19]).

Leonard theorem [10] states that the q-Racah polynomials, discovered by Askey and Wilson [1], are the most general self-dual orthogonal polynomials. As shown in [19], the quantum analog of the algebra (6) with the generators L, M and $N = [L, M] \equiv LM - ML$ describes these polynomials via the representation theory (see also [18] for similar algebraic treatments). Relations of this algebra to the standard $sl_q(2)$ quantum algebra have been established in [7, 8].

2 Duality for a generalized eigenvalue problem

In [15–17], a new family of discrete biorthogonal rational functions $R_n(z)$ and $T_n(z)$, $n = 0, \ldots, N-1$, has been found. These functions satisfy the property

$$\sum_{s=0}^{N-1} w_s R_n(z_s) T_m(z_s) = h_n \delta_{nm}, \tag{12}$$

with some weight function w_s and normalization constants h_n. The sequence z_s is called the "grid" and it is expressed in terms of the Jacobi theta functions. Both $R_n(z_s)$ and $T_n(z_s)$ satisfy three term recurrence relations in the variable n and second order difference equations in s.

Let us introduce a set of N vectors $\Phi_s = (R_0(z_s), R_1(z_s), \ldots, R_{N-1}(z_s))^t$, $s = 0, \ldots, N-1$. Then there exist two tridiagonal matrices L_1, L_2 such that

$$L_1 \Phi_s = z_s L_2 \Phi_s, \tag{13}$$

where it is assumed that the matrices $L_{1,2}$ act on the vector Φ_s in the standard manner. Similarly, we introduce a set of N vectors $\Psi_n = (R_n(z_0), R_n(z_1), \ldots, R_n(z_N))^t$, $n = 0, \ldots, N-1$. Then there exist two tridiagonal matrices M_1, M_2 such that

$$M_1 \Psi_n = \lambda_n M_2 \Psi_n, \tag{14}$$

for some sequence of numbers λ_n (the dual "grid"). Since the functions $R_n(z_s)$ satisfy simultaneously two generalized eigenvalue problems (13) and (14), it is natural to consider the following problem.

Let X and Y be two invertible $N \times N$ matrices with different eigenvalues λ_k and μ_k, $k = 0, \ldots, N-1$. We denote as ϕ_k and ψ_k linearly independent eigenvectors of X and Y, respectively:

$$X\phi_k = \lambda_k \phi_k, \qquad Y\psi_k = \mu_k \psi_k. \tag{15}$$

Now we assume that in the basis of vectors ψ_k the matrix X takes the form

$$X\psi_k = X_2^{-1} X_1 \psi_k, \tag{16}$$

where X_1, X_2 are two tridiagonal matrices, that is

$$\begin{aligned} X_1\psi_k &= \alpha_{k+1}^{(1)}\psi_{k+1} + \beta_k^{(1)}\psi_k + \gamma_k^{(1)}\psi_{k-1}, \\ X_2\psi_k &= \alpha_{k+1}^{(2)}\psi_{k+1} + \beta_k^{(2)}\psi_k + \gamma_k^{(2)}\psi_{k-1}. \end{aligned} \tag{17}$$

In the same way, we assume that there exist two tridiagonal matrices Y_1, Y_2 such that

$$Y\phi_k = Y_2^{-1}Y_1\phi_k, \tag{18}$$

with the properties

$$\begin{aligned} Y_1\phi_k &= \xi_{k+1}^{(1)}\phi_{k+1} + \eta_k^{(1)}\phi_k + \zeta_k^{(1)}\phi_{k-1}, \\ Y_2\phi_k &= \xi_{k+1}^{(2)}\phi_{k+1} + \eta_k^{(2)}\phi_k + \zeta_k^{(2)}\phi_{k-1}. \end{aligned} \tag{19}$$

The main problem of interest consists in the classification of all matrices X and Y admitting such a representation. It implies an explicit description of algebraic structures behind this construction that generalize the Askey–Wilson algebra. We conjecture that complete solution of this problem yields discrete biorthogonal rational functions of [15–17]. Here we make a step towards solution of this problem and announce some results for its classical mechanics analog.

We take two independent dynamical variables $X(q,p)$ and $Y(q,p)$ depending on canonical variables q and p. Now we suppose that there exist two canonical transformations $(q,p) \to (x,y)$ and $(q,p) \to (\xi,\eta)$ such that the first transformation leads to

$$X = \varphi(x), \qquad Y = \frac{Y_1(x,y)}{Y_2(x,y)} \tag{20}$$

and in the second case we have

$$Y = \psi(\xi), \qquad X = \frac{X_1(\xi,\eta)}{X_2(\xi,\eta)}, \tag{21}$$

where X_r, Y_r, $r = 1,2$, are some classical "tridiagonal functions", that is

$$\begin{aligned} X_r(\xi,\eta) &= A_1^{(r)}(\xi)e^{\eta} + A_2^{(r)}(\xi)e^{-\eta} + A_0^{(r)}(\xi), \\ Y_r(x,y) &= B_1^{(r)}(x)e^{y} + B_2^{(r)}(x)e^{-y} + B_0^{(r)}(x). \end{aligned} \tag{22}$$

We shall call the pair (X,Y) satisfying such a property as generalized CLP.

3 Main results

We consider the general situation when none of the coefficients $A_i^{(r)}$, $B_i^{(r)}$ in (22) vanishes identically. Note, however, that some of the potentials can be set constant by changes of variables.

Theorem 1. *Suppose that independent variables X and Y admit representations* (20) *and* (21) *via two canonical transformations. Then they necessarily satisfy the following quadratic equation determining a particular elliptic curve:*

$$Z^2 + F_1(X,Y)Z + F_2(X,Y) = 0, \tag{23}$$

where $Z \equiv \{X,Y\}$ is the Poisson bracket of X and Y,

$$F_2(X,Y) = \frac{1}{4}F_1(X,Y)\left(F_1(X,Y) - q^2(X,Y)\right), \tag{24}$$

and

$$F_1(X,Y) = \sum_{i,k=0}^{2} \alpha_{ik}X^iY^k, \qquad q(X,Y) = \sum_{i,k=0}^{1} \beta_{ik}X^iY^k \tag{25}$$

for some 13 constants α_{ik} and β_{ik}. Condition (23) *is also sufficient.*

For the "grid" function $\varphi(x)$ we obtain equation

$$(\varphi'(x))^2 = \pi_4(\varphi(x)), \tag{26}$$

where $\pi_4(y)$ is a polynomial of fourth degree depending on parameters α_{ik} and β_{ik}. In the general situation, the polynomial $\pi_4(y)$ is not degenerate. It is known that solution of equation (26) is given by general elliptic function of the second order, which can be represented in the form [3]:

$$\varphi(x) = \gamma\frac{\theta_1(\beta x + u_1)\theta_1(\beta x + u_2)}{\theta_1(\beta x + v_1)\theta_1(\beta x + v_2)}, \qquad u_1 + u_2 = v_1 + v_2, \tag{27}$$

where $\theta_1(u)$ is the standard Jacobi theta function. Recall that the order of the elliptic function is defined as the number of zeros or poles (counting their multiplicity) inside the fundamental parallelogram of periods [3]. Elliptic functions of the second order are the simplest non-trivial double-periodic meromorphic functions. Well-known examples are given by the Weierstrass function $\wp(x)$ and Jacobi functions $\mathrm{sn}\,(x)$, $\mathrm{cn}\,(x)$, $\mathrm{dn}\,(x)$.

The potentials $A_i^{(r)}$, $B_i^{(r)}$ are also given by elliptic functions, but their expressions are rather complicated and we omit them here. A proof of the above theorem will be given elsewhere.

There is an important subcase when equation (23) is reduced to the complete square:

$$Z = \sum_{i,k=0}^{2} \alpha_{ik} X^i Y^k \tag{28}$$

for some coefficients α_{ik}. This situation corresponds to symmetric, or two-diagonal representation for the variables X, Y. More exactly, we have

Theorem 2. *Dynamical variables X and Y form a generalized symmetric CLP, that is they admit representations* (20) *and* (21) *with $A_0^{(r)} = B_0^{(r)} = 0$, $r = 1, 2$, if and only if their Poisson bracket $Z = \{X, Y\}$ takes the form* (28) *with arbitrary 9 coefficients α_{ik}.*

We have shown that condition (23) is necessary and sufficient for the dynamical variables X and Y to satisfy the classical analog of the generalized eigenvalue problem for two tridiagonal matrices. The corresponding Poisson algebra takes the form

$$\{X, Y\} = Z, \qquad \{Z, X\} = -Z\frac{\partial Z}{\partial Y}, \qquad \{Y, Z\} = -Z\frac{\partial Z}{\partial X}. \tag{29}$$

However, the variable Z considered as a function of X, Y is given by a root of the quadratic equation (23). The algebra obtained after the substitution $Z = (-F_1 \pm q\sqrt{F_1})/2$ into (29) has much less attractive form with respect to the Askey–Wilson case (6). Therefore we need to find a simpler approach to this problem. For instance, we can try to express X, Y, Z in terms of some other dynamical variables U_i, $i = 1, 2, \ldots$, so that the algebra (29) is reproduced by relatively simple Poisson algebraic relations between generators U_i.

This idea can be explained as follows. From the very beginning we have demanded that in the picture (20) the variable Y is presented as the ratio $Y = U_3/U_4$ where both U_3 and U_4 are "tridiagonal" in y. Similarly, in the dual picture (21) one should be able to represent $X = U_1/U_2$, where U_1, U_2 have the same tridiagonality property with respect to η. It is therefore natural to set

$$X = \frac{U_1(q,p)}{U_2(q,p)}, \qquad Y = \frac{U_3(q,p)}{U_4(q,p)}$$

and seek 4 dynamical variables U_1, U_2, U_3, U_4 such that for some canonical transformation $(q, p) \to (x, y)$ all U_i are reduced to the tridiagonal form and, additionally, $U_1 = \varphi(x)U_2$ for some function $\varphi(x)$. Similarly, there should exist a dual canonical transformation $(q,p) \to (\xi,\eta)$ which reduces again all U_i to tridiagonal form and, additionally, guarantees that $U_3 = \psi(\xi)U_4$ for some function $\psi(\xi)$. All these requirements do not bring anything new to the picture we have considered so far. The crucial additional requirements look as follows:

i) pairwise Poisson brackets of $\{U_i, U_j\}$ should be quadratic polynomials in all U_1, U_2, U_3, U_4;

ii) the linear transformations $\tilde{U}_1 = m_{11}U_1 + m_{12}U_2$, $\tilde{U}_2 = m_{21}U_1 + m_{22}U_2$ and $\tilde{U}_3 = \ell_{11}U_3 + \ell_{12}U_4$, $\tilde{U}_4 = \ell_{21}U_3 + \ell_{22}U_4$ with two arbitrary nonsingular matrices ℓ_{ij}, m_{ij} do not change the form of the Poisson algebra for U_i, that is it should be covariant with respect to such linear fractional transformations.

In [11], some generalizations of the Sklyanin algebra were discussed. They are generated by two polynomials $Q_1(U)$ and $Q_2(U)$ depending on four dynamical variables U_i, $i = 1,\ldots,4$, whose Poisson brackets are defined in the following nice way

$$\{U_i, U_j\} = (-1)^{i+j} \det\left(\frac{\partial Q_k}{\partial U_l}\right), \qquad l \neq i, j, \quad i > j. \tag{30}$$

The functions Q_1, Q_2 serve as Casimir elements of these algebraic relations, that is $\{U_i, Q_k\} = 0$, $k = 1, 2$.

In order to get quadratic Poisson algebra it is necessary to fix $Q_1(U)$ and $Q_2(U)$ as quadratic polynomials. For example, the standard Sklyanin algebra is obtained from

$$Q_1(U) = U_1^2 + U_2^2 + U_3^2,$$
$$Q_2(U) = U_4^2 + J_1U_1^2 + J_2U_2^2 + J_3U_3^2.$$

We shall use this construction in order to model our classical Poisson algebraic relation (23).

We consider the case of two-diagonal representation, when

$$Z = \{X, Y\} = F(X, Y) = \sum_{i,k=0}^{2} \alpha_{ik} X^i Y^k \tag{31}$$

is an arbitrary polynomial of the second degree in each variable X and Y.

Calculating Poisson bracket of $X = U_1/U_2, Y = U_3/U_4$ we get

$$Z = \{X, Y\} = \frac{\{U_1, U_3\}}{U_2U_4} + \frac{U_1U_3\{U_2, U_4\}}{U_2^2U_4^2} - \frac{U_3\{U_1, U_4\}}{U_2U_4^2} - \frac{U_1\{U_2, U_3\}}{U_2^2U_4}. \tag{32}$$

It is seen that in order to get (31) it is necessary to demand that each Poisson bracket $\{U_i, U_j\}$ in (32) contains the same variables U_iU_j, their adjacent pair U_kU_l (where k or l are not equal to i or j) and two adjacent pairs U_iU_k and U_lU_j. These conditions can be satisfied if we choose Casimir elements in the form

$$Q_1(U) = \sum_{i=1}^{4} a_iU_i^2 + \xi_1U_1U_2 + \eta_1U_3U_4,$$
$$Q_2(U) = \sum_{i=1}^{4} b_iU_i^2 + \xi_2U_1U_2 + \eta_2U_3U_4 \tag{33}$$

with arbitrary parameters a_i, b_i, ξ_m, η_m. Direct calculations show that the equality (31) is indeed satisfied with 9 arbitrary parameters α_{ik} which are expressed through 12 parameters a_i, b_i, ξ_m, η_m. Note that the property ii) is fulfilled, e.g. the permutation $U_1 \leftrightarrow U_2$ leads only to the permutation of parameters a_1, a_2 and b_1, b_2 in Q_1, Q_2. We thus have the following statement.

Theorem 3. *Any two-diagonal generalized CLP with $X = U_1/U_2$ and $Y = U_3/U_4$ can be realized in terms of the quadratic Poisson algebra* (30) *with two Casimir elements given by* (33).

4 Further generalizations and perspectives

We consider now condition (23) without restriction (24), that is we assume that

$$F_1(X,Y) = \sum_{i,k=0}^{2} \alpha_{ik} X^i Y^k, \qquad F_2(X,Y) = \sum_{i,k=0}^{4} \gamma_{ik} X^i Y^k \tag{34}$$

are arbitrary polynomials of degrees at most two and four with respect to both variables X and Y. In this case, obviously, the basic properties (20) and (21) are not valid any more. Still, relation (23) in this case generates many interesting special systems.

For example, if we set $F_1(X,Y) = 0$ then we have the condition

$$Z^2 + F_2(X,Y) = 0, \tag{35}$$

which generates some Poisson algebra generalizing the Askey–Wilson algebra (the latter algebra is obtained when $\deg(F_2) = 2$). Some special cases of dynamical systems described by this algebra (e.g., the Euler and Lagrange tops) were presented in [9]. It was observed that in all these cases one of the dynamical variables X or Y is an elliptic function of time.

Here we present the general result.

Theorem 4. *Suppose that independent dynamical variables X and Y have the following property: if X is chosen as the Hamiltonian then $Y(t)$ is described by an elliptic function of the second order. Conversely, if Y is chosen as the Hamiltonian then $X(t)$ is also an elliptic function of the second order. Then the dynamical variables X, Y, and Z satisfy the key relation* (35) *with $F_2(X,Y)$ being an arbitrary polynomial of degree at most four with respect to each variable X and Y.*

On the one hand, this theorem can be considered as a generalization of the previous result concerning the classical Leonard pairs [20]. On the other hand, this result provides a characterization of the so-called "double-elliptic systems" [2, 11]. Examples of double-elliptic systems described by relation (35) will be considered separately.

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[1] Askey R. and Wilson J., Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Mem. Amer. Math. Soc.*, 1985, V.54, N 319.

[2] Braden H.W., Marshakov A., Mironov A. and Morozov A., On double-elliptic integrable systems 1. A duality argument for the case of $SU(2)$, *Nucl. Phys. B*, 2000, V.573, 553–572.

[3] Erdélyi A., Magnus W., Oberhettinger F. and Tricomi F.G., Higher transcendental functions, Vol. II, New York, McGraw-Hill, 1953.

[4] Duistermaat J.J. and Grünbaum F.A., Differential equations in the spectral parameters, *Commun. Math. Phys.*, 1986, V.103, 177–240.

[5] Fock V., Gorsky A., Nekrasov N. and Rubtsov V., Duality in integrable systems and gauge theories, *J. High Energy Phys.*, 2000, N 7, paper 28, 40 p.

[6] Granovskii Ya.I., Lutzenko I.M. and Zhedanov A.S., Mutual integrability, quadratic algebras, and dynamical symmetry, *Ann. Phys. (N.Y.)*, 1992, V.217, 1–20.

[7] Granovskii Ya.I. and Zhedanov A.S., Hidden symmetry of the Racah and Clebsch–Gordan problems for the quantum algebra $sl_q(2)$, *J. Group Theory in Phys.*, 1993, V.1, 161–171.

[8] Granovskii Ya.I. and Zhedanov A.S., Linear covariance algebra for $\mathrm{SL}_q(2)$, *J. Phys. A: Math. Gen.*, 1993, V.26, L357–L359.

[9] Korovnichenko A. and Zhedanov A., Dual algebras with non-linear Poisson brackets, in Proc. NATO ARW "Integrable Structures of Exactly Solvable Two-Dimensional Models of Quantum Field Theory" (25–30 September, 2000, Kyiv, Ukraine), Dordrecht, Kluwer, 2001, 265–272.

[10] Leonard D.A., Orthogonal polynomials, duality and association schemes, *SIAM J. Math. Anal.*, 1982, V.13, 656–663.

[11] Odesskii A.V. and Rubtsov V.N., Polynomial Poisson algebras with regular structure of symplectic leaves, *Teor. Mat. Fiz.*, 2002, V.133, N 1, 3–23 (in Russian).

[12] Ruijsenaars S.N.M., Finite-dimensional soliton systems, in Integrable and Superintegrable Systems, Singapore, World Scientific, 1990, 165–206.

[13] Sklyanin E.K., On some algebraic structures connected with the Yang–Baxter equation, *Funktsional. Anal. i Prilozhen.*, 1982, V.16, N 4, 27–34 (in Russian).

[14] Spiridonov V.P., Universal superpositions of coherent states and self-similar potentials, *Phys. Rev. A*, 1995, V.52, 1909–1935; Erratum, *Phys. Rev. A*, 1995, V.53, 2903; quant-ph/9601030.

[15] Spiridonov V.P. and Zhedanov A.S., Spectral transformation chains and some new biorthogonal rational functions, *Commun. Math. Phys.*, 2000, V.210, 49–83.

[16] Spiridonov V.P. and Zhedanov A.S., Classical biorthogonal rational functions on elliptic grids, *C. R. Math. Rep. Acad. Sci. Canada*, 2000, V.22, N 2, 70–76.

[17] Spiridonov V.P. and Zhedanov A.S., Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids, in Proc. NATO ASI "Special Functions-2000" (29 May – 9 June, 2000, Tempe, USA), Dordrecht, Kluwer Academic Publishers, 2001, 365–388.

[18] Terwilliger P., Two linear transformations each tridiagonal with respect to an eigenbasis of the other, *Lin. Alg. Appl.*, 2001, V.330, 149–203.

[19] Zhedanov A.S., "Hidden symmetry" of the Askey–Wilson polynomials, *Teor. Mat. Fiz.*, 1991, V.89, 190–204 (in Russian); translation in *Theor. Math. Phys.*, 1991, V.89, 1146–1157.

[20] Zhedanov A. and Korovnichenko A., "Leonard pairs" in classical mechanics, *J. Phys. A: Math. Gen.*, 2002, V.35, 5767–5780.