

On Calculation of Finite-Gap Spectra for One-Dimensional Schrödinger Operator

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The general self-consistent approach for calculating of spectral parameters of the one-dimensional Schrödinger operator with the finite-gap Abelian potential is proposed. It is based on use of finite-band equations and solution of the Jacobi inverse problem in terms of hyperelliptic Weierstrass functions.

1 Introduction

General solutions of completely integrable systems are expressed in terms of the Riemann theta functions containing unknown parameters. Obtaining these solutions in an explicit form is reduced to the problem of calculation of the mentioned parameters. Usually, solving of this problem is realized in the framework of two following approaches. The first algebro-geometric approach is based on an use of the so-called Thomae formulae describing relations between the unknown parameters and so-called theta-constants (see [1–3]). In so doing, it is assumed that spectral branch points are given, and their calculation is a special problem. Role of independent parameters in this formulae is played by the period matrix τ generated by the hyperelliptic Riemann curve $\Gamma_g : y^2 = \prod_{i=1}^{2g+1} (\varepsilon - \varepsilon_i)$ (see [2]). The second approach is based on construction of algebraic equations for the unknown parameters by forward substitution of the general solutions in to an original integrable equation (see [3]). These two approaches assume that the explicit form of the generated Riemann curve is given and do not provide their calculation.

In the same time, the full and self-consistent solving of the considered problem assumes construction of a unique system of equations determining all unknown parameters. This can be realized in the form of so-called finite-band equations in the framework of the spectral problem for auxiliary linear operators (see [4–7]) corresponding to a given integrable equation.

In the case of the KdV equation under consideration this problem is connected with the spectral problem for the finite-gap Schrödinger operator. In the presented paper the effective method for full solving of the spectral problem for the one-dimensional Schrödinger operator is proposed on the basis of finite-band equations. As distinct from the above-mentioned second approach, the substitution of solutions (the Schrödinger potential) is realized with respect to the finite-band equations. In so doing, the problem is reduced to solving of a self-consistent system of algebraic equations determining all unknown parameters through theta-constants (see [1]). This provides calculation of both parameters of the potential and spectral parameters of the Schrödinger operator.

The presented paper consists of three sections. General analytic solving of the spectral problem for the finite-gap Schrödinger operator in terms of hyperelliptic Weierstrass functions is considered in Section 2. In Section 3 we obtain the system of algebraic equations for the unknown parameters including spectral parameters and consider particular cases of one- and two-gap spectra.

2 Finite-gap solutions

Eigenfunctions of linear differential operators with a finite-gap spectrum are characterized by their dependence on a spectral variable ε through polynomials in ε [5]. Therefore solving of the corresponding spectral problem is based on the comparison of asymptotic expansions in ε of general and finite-gap eigenfunctions. This is expressed in the form of so-called finite-band equations describing all unknown parameters of finite-gap solutions.

The spectral equation ($H\Psi = \varepsilon\Psi$) of the general and finite-gap one-dimensional Schrödinger operator (H) determines its eigenfunction in the form

$$\Psi_{\pm}(x) = C\sqrt{\chi(\varepsilon, x)} \exp\left(\pm i \int_{x_0}^x dx \chi(\varepsilon, x)\right)$$

with $\chi(\varepsilon, x)$ determined by the general spectral equation and $\chi(\varepsilon, x) = y(\varepsilon)/X(\varepsilon, x)$ for the finite-gap spectra. Here polynomial functions $y(\varepsilon)$ and $X(\varepsilon, x)$ are of $(2g + 1)$ -th and g -th orders in ε , respectively. Here the X -polynomial is characterized by a dependence on x of coefficients at powers of ε .

The function $X(\varepsilon, x)$ is the product of two eigenfunctions ($X(\varepsilon, x) = \Psi_+(\varepsilon, x)\Psi_-(\varepsilon, x)$) and satisfies to the equation, which in terms of X -polynomial roots $\mu_j(x)$ take the form (see [8])

$$\partial_x X(\varepsilon, x)|_{\varepsilon=\mu_j(x)} = 2iy(\mu_j(x)).$$

This equation is integrated by the Abelian map $\zeta_j = \sum_{k=1}^g \int_{a_k}^{\mu_k(x)} dv_j(\varepsilon)$, where $dv_j = \sum_{i=1}^g C_{ij}d\omega_i$

is normalized holomorphic differential on the Riemann curve Γ_g : $y^2 = \prod_{i=1}^{2g+1} (\varepsilon - \varepsilon_i)$ and $d\omega_j = \varepsilon^{j-1}d\varepsilon/y(\varepsilon)$. Corresponding solutions have the form $\zeta_j = 2iC_{gj}x$. Thus calculation of symmetrized products of the set $\{\mu_j(x)\}$ is reduced to inverse of the Abelian map (see for instance [1, 2]). Symmetrized products of $\{\mu_j(x)\}$, considered as the point set on the Riemann curve Γ_g , can be calculated with help of the Riemann theta functions

$$\theta[\alpha](z|\tau) = \sum_{n \in \mathbb{Z}^g} \exp[\pi i(n + \alpha_1)\tau(n + \alpha_1) + 2\pi i z \cdot (n + \alpha_2)], \tag{1}$$

where

$$z = \int_{x_0}^x d\omega - \mathbf{C}u - \mathbf{K}, \quad u_j = \sum_{k=1}^g \int_{a_k}^{x_k} d\omega_j(x)$$

which vanish in points $\{\mu_j(x)\}$.

Here α is the $2 \times g$ half-integer matrix with rows represented by vectors α_1 and α_2 ; τ is the period matrix on the Riemann curve Γ_g , \mathbf{K} is a so-called Riemann constant vector.

In the framework of the Kleinian construction (see [9]) the so-called hyperelliptic σ -function

$$\sigma(u) = C \exp(\mathbf{u}\mathbf{k}\mathbf{u} + 2\pi i\mathbf{q}'(-\mathbf{C}u + \mathbf{r}\mathbf{q}'/2 - \mathbf{q}'))\theta\left(\int_{x_0}^x \mathbf{C}d\omega - \mathbf{C}u - \mathbf{K}|\tau\right), \tag{2}$$

plays the role of the generating functions for hyperelliptic Abelian functions. Absence of $\theta(z)$ implies the function (1) at $\alpha = 0$.

Here \mathbf{r}_j is determined by the expression $dr_j = r_j dx$ ($r_j = \sum_{k=j}^{2g+1-j} (k + 1 - j)\lambda_{k+1+j}x^k/2y$), which describes canonical Abelian differential of second kind on the Riemann curve Γ_g ; \mathbf{q}' and \mathbf{q} are half-integer characteristics of the Riemann constant vector \mathbf{K} .

Hyperelliptic zeta and Weierstrass functions are defined as logarithmic derivatives of the function (2) by the formulae

$$\zeta_{j_1}(\mathbf{u}) = \frac{\partial}{\partial u_{j_1}} \ln \sigma(\mathbf{u}), \quad \wp_{j_1 j_2 \dots j_n}(\mathbf{u}) = -\frac{\partial^{j_1+j_2+\dots+j_n}}{\partial u_{j_1} \partial u_{j_2} \dots \partial u_{j_n}} \ln \sigma(\mathbf{u}), \quad n \geq 2. \quad (3)$$

The Weierstrass functions in (3) implicitly contain unknown parameters $\{C_{ij}\}$ and the period matrix $\boldsymbol{\tau}$ as an independent parameter. Taking into account the dependence of \mathbf{u} on points $\{x_i\}$ on the Riemann curve Γ_g it is seen that second derivative of the σ -function (2) represents a polynomial in x_i with coefficient functions $\wp_{j_1 j_2}(\mathbf{u})$. Evidently the equality for a such polynomial enable to determine symmetrized products of $\mu_j(x)$ through hyperelliptic functions $\wp_{j_1 j_2}(\mathbf{u})$, which, with account of above-mentioned relations between vectors $\boldsymbol{\zeta}$ and \mathbf{u} , will contain unknown parameters C_{ij} .

This equality can be obtained on the basis of the known Baker relation (see [9])

$$\int_{\nu}^x \sum_{i=1}^g \int_{\nu_i}^{x_i} d\Omega(x, x_i) = \ln \left(\sigma \left(\int_{x_0}^x d\omega - \sum_{i=1}^g \int_{x_0}^{x_i} d\omega \right) \sigma^{-1} \left(\int_{x_0}^x d\omega - \sum_{i=1}^g \int_{x_0}^{\nu_i} d\omega \right) \right) \Big|_{x=\nu}^x, \quad (4)$$

where the fundamental 2-differential of the second kind $d\Omega(x, x_i)$ is defined by the expression

$$d\Omega(x, x_i) = \frac{\partial}{\partial x} \left(\frac{y(x) + y(x_i)}{2y(x)(x - x_i)} \right) dx dx_i + d\omega(x) d\mathbf{r}(x_i).$$

The partial derivation of (4) with respect to x and x_k taking into account the definition (3) leads to the relation

$$\sum_{i,j=1}^g \wp_{ij} \left(\int_{x_0}^x d\omega - \sum_{i=1}^g \int_{x_0}^{x_i} d\omega \right) x^i x_k^j = \frac{F(x, x_k) - 2yy_k}{4(x - x_k)^2}, \quad (5)$$

with known [9] F -function which has the form of polynomial in variables x and x_k . Using the known relation $F(x, x_k) \rightarrow x^{g+1} x_k^{g+1}$ at $x \rightarrow \infty$ from (5) in the asymptotic limit $x \rightarrow \infty$ we obtain the equation

$$x_k^g - \sum_{j=1}^g \wp_{g,g+1-j}(\mathbf{u}) x_k^{g-j} = 0. \quad (6)$$

In view of (6) the coefficient function $\wp_{g,g+1-j}(\mathbf{u})$ equals to the symmetrized product of a j degree of the above mentioned point set $\{x_k\}$ of the Riemann curve Γ_g . This is true also with respect to the point set $\{\mu_k(x)\}$.

Thus coefficient functions $b_j(x)$ are expressed through generalized Weierstrass \wp -functions by the relation

$$b_j(x) = -\wp_{g,g+1-j}(\mathbf{u}). \quad (7)$$

In terms of the above-mentioned normalized vectors $\boldsymbol{\zeta} = \mathbf{C}\mathbf{u}$ belonging to Abelian map on the basis of the normalized first-kind Abelian holomorphic differentials v , the relation (6) takes the form

$$b_j(x) = -\sum_{n,m=1}^g C_{gn} C_{g,g+1-j} \wp_{nm}(\boldsymbol{\alpha}x), \quad (\boldsymbol{\alpha})_j = C_{gj}. \quad (8)$$

Thus the spectral problem for the one-dimensional Schrödinger operator with a finite-gap spectrum is reduced to calculation of the normalized coefficient matrix C . Normalized coefficients in (8) are described by algebraic equations, which can be obtained by substitution of the expression (8) into known finite-band equations with transition to the theta function representation.

3 Calculation of unknown parameters

The system of finite-band equations represents relations between coefficient functions $b_j(x)$ and polynomials in derivatives of the Schrödinger potential $U(x)$ with respect to x . This system follows from comparison of asymptotic expansions in ε for general and finite-gap χ -functions entering in the above-mentioned expression for the eigenfunction $\Psi(\varepsilon, x)$ of the Schrödinger spectral equation (see [4]).

For the general solution, coefficients of the power expansion

$$\chi = \sqrt{\varepsilon} \left(1 + \sum_{n=0}^{\infty} ((-1)^n / 2^{2n+1}) \chi_{2n+1} \varepsilon^{-(n+1)} \right)$$

are polynomial in derivatives of $U(x)$. For the finite-gap solutions, coefficients of the power expansion $\chi(\varepsilon, x) = y(\varepsilon)/X(\varepsilon, x)$ are represented polynomial in spectral coefficients $\{a_j\}$ and coefficient functions $b_j(x)$. Comparison of power expansions for both cases leads to the so-called finite-band system of equations

$$\frac{1}{n!} \frac{\partial^n}{\partial z^n} \left(\frac{\left(\sum_{k=0}^{2g+1} a_k z^k \right)^{1/2}}{\sum_{k=0}^g b_k(x) z^k} \right)_{z=0} = \frac{(-1)^{n-1}}{2^{2n-1}} \chi_{2n-1}(x), \quad a_0, b_0 = 1,$$

$$\chi_{n+1} = \frac{\partial}{\partial x} \chi_n + \sum_{k=1}^{n-1} \chi_k \chi_{n-k}, \quad \chi_1 = -U(x), \tag{9}$$

in which $b_k(x) = 0$ at $k > g$. Feature of this equations is that the coefficient function $b_j(x)$ enters linearly to each equation (9) at $n = j$ and is algebraically expressed through all previous coefficient functions $b_i(x)$, $i = \overline{1, j-1}$. The same regularity takes place for spectral parameters a_j at $j > g$.

Therefore all coefficient functions b_j at $j \leq g$ are expressed as polynomials in derivatives of potential $U(x)$. The system of spectral parameters a_j at $j = \overline{g+1, 2g+1}$ is algebraically expressed through first g parameters a_1, \dots, a_g and polynomials in derivatives $U^{(i)}$.

Thus, use of an eliminating method concerning a_j and $b_j(x)$ reduces equations (9) at $n = \overline{2g+2, 3g+1}$ to the closed system (RS) of algebraic equations for g unknowns a_j , $j = \overline{1, g}$.

The potential $U(x)$ in (9) is connected with the coefficient function $b_1(x)$ by the relation

$$b_1(x) = \frac{1}{2} (U(x) + a_1),$$

which together with the equality (7) result in the explicit expression

$$U(x) = -2 \sum_{j,j'=1}^g \alpha_j \alpha_{j'} \wp_{jj'}(\alpha x | \tau) - a_1. \tag{10}$$

describing the Schrödinger potential in terms of generalized Weierstrass \wp -functions at the given module matrix τ of the Riemann curve Γ (in future for simplicity τ will be omitted). Here components α_j , $j = \overline{1, g}$ must be calculated from the above-mentioned reduced system.

Substitution of the expression (10) into equations (9) results in a system of nonlinear equations for unknowns a_j , $j = \overline{1, 2g+1}$ and α_j , $j = \overline{1, g}$ with coefficients expressed through derivatives of \wp -function with respect to x . In view of independence on x this system is represented by algebraic nonlinear equations with respect to a_j and α_j . In representation of above-mentioned

theta functions such system is reduced to algebraic nonlinear equations with coefficients which are expressed through theta-constants (i.e. theta functions with zero argument). Correspondingly, the above-mentioned RC system is transformed in to the algebraic system of equations concerning unknowns a_j and α_j at $j = \overline{1, g}$.

For obtaining above-mentioned equations it is convenient to introduce j -component vector $\mathbf{n}_j = (n_1, \dots, n_j)$ and notations $\alpha_{\mathbf{n}_j} = 2 \prod_{n_i=1}^j \alpha_{n_i}$ and $\wp_{\mathbf{n}_j} = \wp_{n_1 \dots n_j}$. Then the expression (10) can be rewritten more compactly as

$$U(x) = \alpha_{\mathbf{n}_2} \wp_{\mathbf{n}}(\alpha x) - a_1, \quad (11)$$

where summing by repetitive subscript components is implied.

Substitution of the expression (11) into the equations (9) results in the equation

$$\begin{aligned} b_2(x) = & \frac{1}{8} (3 (\alpha_{\mathbf{n}_{2g}} \wp_{\mathbf{n}_2} - a_1)^2 - (\alpha_{\mathbf{n}_{4g}} \wp_{\mathbf{n}_4})) \\ & - \frac{1}{4} a_1 (\alpha_{\mathbf{n}_{2g}} \wp_{\mathbf{n}_2} - a_1) + \frac{1}{2} a_2 - \frac{1}{8} a_1^3 \end{aligned} \quad (12)$$

and

$$\begin{aligned} b_3(x) = & \frac{1}{32} (\alpha_{\mathbf{n}_{4g}} \wp_{\mathbf{n}_4} + 10 (\alpha_{\mathbf{n}_{2g}} \wp_{\mathbf{n}_2} - a_1)^3 - 5 (C_{\mathbf{n}_{3g}} \wp_{\mathbf{n}_3})^2 \\ & - 10 (\alpha_{\mathbf{n}_{2g}} \wp_{\mathbf{n}_2} - a_1) \alpha_{\mathbf{n}_{4g}} \wp_{\mathbf{n}_4}) - \frac{1}{16} a_1 (3 (\alpha_{\mathbf{n}_{2g}} \wp_{\mathbf{n}_2} - a_1)^2 - \alpha_{\mathbf{n}_{4g}} \wp_{\mathbf{n}_4}) \\ & + \frac{1}{16} (\alpha_{\mathbf{n}_{2g}} \wp_{\mathbf{n}_2} - a_1) (a_1^2 - 4a_2) + \frac{1}{2} a_3 + \frac{1}{4} a_1 a_2 + \frac{1}{16} a_1^3. \end{aligned} \quad (13)$$

Finite-gap equations (12), (13) and of higher order are valid with any arguments of generalized Weierstrass \wp -functions. Therefore these equations can be represented in terms of theta-constants (theta-functions on zero argument) by the formal substitute

$$\wp_{\mathbf{n}_j} \rightarrow (\ln \theta)_{\mathbf{n}_j}, \quad (\ln \theta)_{\mathbf{n}_j} = \partial_{n_1, \dots, n_j} \ln \theta(z | \boldsymbol{\tau})_{z=0}. \quad (14)$$

In the case of the one-gap spectrum, the generalized Weierstrass functions transform in to the elliptic Weierstrass function ($\wp_{\mathbf{n}_j} \rightarrow \wp_{11}$) and $\alpha_1 = 1$. In this case $b_j(x) = 0$ at $j > 1$. Therefore reducing in (12) and (13) similar terms and vanishing coefficients at degrees of \wp_{11} we immediately obtain the following solutions for spectral parameters a_1, a_2, a_3

$$\begin{aligned} a_1 = 0, \quad a_2 = & \wp(\omega_1) \wp(\omega_2) + \wp(\omega_1) \wp(\omega_3) + \wp(\omega_2) \wp(\omega_3), \\ a_3 = & \wp(\omega_1) \wp(\omega_2) \wp(\omega_3), \quad \omega_1 = \omega, \quad \omega_2 = \omega + \omega', \quad \omega_3 = \omega', \end{aligned}$$

where ω and ω' are real and imaginary self-periods of the elliptic Weierstrass function.

In the two-gap case ($g = 2$) normalized constants $\{C_j\}$ are not known and are determined by equations (13) and (14). Unknowns $a_j, \alpha_j, j = 1, 2$ are solutions of the above-mentioned RC system (corresponding to (9) at $n = \overline{6, 9}$). This system takes the form of algebraic nonlinear equations which can be written as

$$\begin{aligned} 0 = & B_{610}^{22} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_2^2 + B_{61j}^{21} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_2 a_1^j |_0^1 + B_{61j}^{20} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_1^j |_0^4, \\ 0 = & B_{71j}^{22} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_2^2 a_1^j |_0^1 + B_{71j}^{21} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_2 a_1^j |_0^3 + B_{71j}^{20} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_1^j |_0^5, \\ 0 = & B_{810}^{23} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_2^3 + B_{81j}^{22} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_2^2 a_1^j |_0^2 + B_{81j}^{21} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_2 a_1^j |_0^4 + B_{81j}^{20} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_1^j |_0^6, \\ 0 = & B_{91j}^{23} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_2^3 a_1^j |_0^1 + B_{91j}^{22} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_2^2 a_1^j |_0^3 \\ & + B_{91j}^{21} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_2 a_1^j |_0^5 + B_{81j}^{20} (\alpha_{\mathbf{n}_2} \wp_{\mathbf{n}_2}) a_1^j |_0^8, \end{aligned}$$

where summing by repetitive subindexes in boundaries which are denoted at vertical hyphens is implied. Also the formal transfer (14) is suggested. Here $B_{m_{1j}^{2i}}(z)$ implies coefficient function at $a_2^i a_1^j$ corresponding to the equation (9) at $n = m$. This coefficient function is a polynomial in derivatives of its argument with respect to x and therefore contains terms $\partial_x^k \wp_{\mathbf{n}_2} = \alpha_{\mathbf{n}_2+k} \wp_{\mathbf{n}_2+k}$ in accordance with (11). The latter system is represented by the closed system of the four algebraic nonlinear equations which determines two spectral parameters a_1, a_2 and components α_1, α_2 . The remaining unknowns $a_j, j = \overline{3, 5}$ are calculated by forward substitution of obtained a_1, a_2 and α_1, α_2 into the reduced equations (9) at $n = \overline{3, 5}$. In so doing, the matrix τ plays a role of a independent parameter.

Similar calculation can be applied in the case of the arbitrary number of spectral gaps in the spectrum of the one-dimensional Schrödinger operator. Then unknown spectral parameters a_1, \dots, a_g and the parameters $\alpha_1, \dots, \alpha_g$ will be determined by reduced equations a system of $2g$ reduced algebraic equations (9) at $n = \overline{2g+2, 4g+1}$. The rest spectral parameters $a_j, j = \overline{g+1, 2g+1}$ will be calculated by forward substitution of computed a_1, \dots, a_g and $\alpha_1, \dots, \alpha_g$ into the reduced equations (9) at $n = \overline{g+1, 2g+1}$.

The general reduced algebraic equations are represented by the necessary conditions that the Riemann τ -matrix will be the period matrix of holomorphic differentials on the Riemann curve Γ . Thus there are no restrictions on the number of gaps in spectra.

The suggested finite-gap method gives simple and common algorithm for solving the spectral problem for the finite-gap Schrödinger operator. This problem is reduced to calculation of the closed system of algebraic nonlinear equations. The method is represented by the hyperelliptic generalization of the earlier suggested approach [10] for solution of the spectral problem in the one-gap case for the one-dimensional elliptic Schrödinger operator.

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