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Computer Algebra Study of Spectral Invariants of Differential Operators on Curved Manifolds

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We consider asymptotic heat kernel expansion for elliptic differential operators acting on compact closed curved manifolds. The coefficients in this expansion are quantities of fundamental importance in quantum field theory, quantum gravity, spectral geometry and topology of manifolds. Obtaining explicit expressions for these quantities is very laborious task, especially in the problems of modern physics studying complicated operators (high order and *nonminimal*) in complicated geometric environment (in the presence of torsion and gauge fields in addition to the Riemannian curvature tensor). In fact, the task cannot be accomplished without computer algebra tools. In this paper we describe a covariant algorithm for computing the heat kernel coefficients. With the help of a ***C*** implementation of the algorithm some new results were obtained. The most significant of them concern nonminimal operators and manifolds with torsion.

1 Introduction

Determination of the internal structure of an object via the spectra of different radiations and waves around the object is one of the archetypal problems in physics. Particular instances of this problem arise practically in all fields of physics ranging from experimental physics of elementary particles to problems in seismology and medical tomography. More restricted but more precise mathematical version of this problem may be formulated as follows. A manifold (bundle) equipped with such structures as metric, curvature, torsion, gauge fields etc. and an elliptic (pseudo)differential operator acting on this manifold are given. What information about the manifold can one obtain by studying the spectral properties of the operator? M. Kac phrases the problem in an evocative title of his paper "Can one hear the shape of a drum?". Historically first studies were devoted mainly to the Laplace operator acting on a Riemannian manifold and for a while there was even impression that geometrical properties of the manifold can be completely restored by the spectrum. Only in 1964, J. Milnor [1] found the first counter-example, namely, a pair of 16-dimensional tori with different (flat) Riemannian metrics but with identical spectra of the Laplace operator acting on the tori. Later on many similar examples of *multiply connected* isospectral manifolds have been constructed, and recently, in 1999, D. Schüth [2] constructed continuous isospectral families of metrics on the product of spheres $S^4 \times S^3 \times S^3$, the first example of closed *simply connected* isospectral but non-isometric Riemannian manifolds.

Nevertheless, many important properties of manifold can be restored by the spectrum of elliptic operator acting on it. In particular, such global geometric invariants as dimension, volume, and total scalar curvature, are known to be spectrally determined. Moreover, various manifolds such as round spheres of dimension ≤ 6 and 2-dimensional flat tori are uniquely determined by the spectra of the Laplacian.

Whereas the classical spectral geometry deals mainly with Laplace operator acting on Riemannian manifold, the needs of modern physics force to consider similar problems for operators more general than Laplacian and acting on manifolds more general than Riemannian.

Approaches, based on studying *asymptotical* properties of spectrum, make the problem available for computational methods. The main and most constructive approach consists in investigation of the *heat kernel expansion.* This approach can be described briefly as follows. Starting with an elliptic operator A of the order $2r$, acting on a bundle whose base is a compact close n-dimensional manifold M, and introducing an additional "temporal" variable t one can construct the *heat* operator $A - \frac{\partial}{\partial t}$. Then one can compute the short-time asymptotic expansion of the diagonal elements of the kernel of this heat operator:

$$\langle x|e^{-tA}|x\rangle \sim \sum_{m\geq 0} E_m(x|A) t^{\frac{m-n}{2r}}, \qquad t \to +0. \tag{1}$$

The coefficients $E_m(x|A)$ in this expansion are spectral invariants of the operator A, and encode information about the asymptotic properties of the spectrum. The E_m's are of fundamental importance in spectral geometry and topology of manifolds, mathematical physics, quantum field theory and quantum gravity. These coefficients are called the *heat invariants* or *heat kernel coefficients.* They are also widely known under the (different combinations of) names Hadamard, DeWitt, Seeley, Gilkey, according to papers of these authors [3–6]. Hereinafter we shall use the term *DeWitt–Seeley–Gilkey (DWSG) coefficients*, since these authors pay special attention to the important for us differential geometric (covariant) aspect of the problem.

Computation of the DWSG coefficients in the case of curved space-time includes complicated manipulations with tensor expressions, integrations, leading in many cases to expressions in terms of hypergeometric and gamma functions, and simplification of these expressions. To cope with these difficulties we elaborated (and implemented in $\boldsymbol{C}$) an algorithm [7].

With the help of our program we succeed in obtaining some new results. Most substantial and difficult among them is computation of complete expression [8] of E_4 for so-called *nonminimal* operator important in the quantum theory of gauge fields and quantum gravity. Note that just this coefficient is most important for the physical 4-dimensional space-time in view of the *Atiyah–Bott formula* [9,10] expressing the index of elliptic operator A on n-dimensional manifold in terms of the coefficient E_n. Moreover, we obtained some new results for higher order operators and for manifolds with torsion. In particular, the spectral invariant E_2 for nonminimal operator on manifold with torsion has been computed in [11].

The problems with torsion are especially difficult from the computational standpoint and we focused special attention on them. The torsion is defined as antisymmetric part of affine (or linear) connection $T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu}$, where $\Gamma^\lambda{}_{\mu\nu}$ are the coefficients of connection. The Einstein's General Relativity is based on a special connection called *Levi-Civita* connection, i.e., *symmetric* and compatible with metric affine connection. This torsionless connection can be expressed completely in terms of metric. The General Relativity describes well the interaction of the matter with the gravity as far as macroscopic bulk matter is considered. However, on the microscopic level, it seems reasonable to take into account the influence of spin on the geometry of space-time. To describe the interaction of spinning particles with the gravitation, a gravitation theory should include the non-vanishing torsion. In 1922 Elie Cartan first pointed out that there is no *a priori* reason to assume an affine connection to be symmetric in the context of General Relativity. He proposed also a theory of gravitation with torsion which is known now as the Einstein–Cartan theory. The torsion arises naturally in the different (based on Poincaré and affine groups) gauge theories of gravity developed in the recent years. Moreover, all kinds of modern superstring theories also predict the existence of torsion.

2 Operators on manifolds and applications of heat kernel

Let us consider a compact closed n-dimensional manifold M equipped with Riemann metric g and linear (or affine) connection defined in the bundle of linear (affine) frames over M. We assume that the connection (locally expressed in terms of the *Christoffel symbols* $\Gamma^\alpha{}_{\mu\nu}$) is compatible with the metric g, but not necessarily symmetric. We shall denote the curvature and torsion of this connection by the symbols $R^{\alpha}{}_{\beta\mu\nu}$ and $T^{\alpha}{}_{\mu\nu}$, respectively.

In more general setup, we consider also a vector bundle over M with the gauge connection in this bundle. The curvature of the gauge connection we denote by the symbol $W_{\mu\nu}$.

Using all these connections we can construct the covariant differentiation D_μ acting in the smooth sections of the bundle E. Using the operators D_μ one can construct covariant differential operators of different kinds. For example, the operator $\Box = g^{\mu\nu}D_\mu D_\nu$ is a covariant generalization of the Laplace operator.

Typical examples of so-called *minimal* operators studied in physics and mathematics are $-\Box + X$, and $\Box^2 + V^{\mu\nu}D_\mu D_\nu + N^\mu D_\mu + X$. Here $V^{\mu\nu}$, N^μ, X are matrix valued tensor fields (sections of bundle of endomorphisms $\mathrm{End}\,E$ of the bundle E). We write explicitly only "Lorentz" (i.e., corresponding to the coordinates in the manifold M) indices. The "gauge" indices are assumed implicitly. The term *minimal* means the operator symbol is Lorentzian scalar and its leading term is power of the Laplace operator.

Operators of more complicated type with non-Laplacian leading terms are called *nonminimal*. The simplest example of nonminimal operator is the Navier–Lamé operator of classical elasticity, $\mu\Delta\vec{V} + (\lambda+\mu)\nabla(\nabla\vec{V})$ (the Lamé constants, λ and μ, characterize the material). H. Weyl was apparently the first who investigated the asymptotics of spectrum of operators of the like type. We shall consider nonminimal operator of the form

$$-g^{\mu\nu}\Box + aD^\mu D^\nu + X^{\mu\nu}, \tag{2}$$

where $X^{\mu\nu}$ is a tensor field, a is a scalar parameter.

In recent years nonminimal operators of a similar sort have been encountered by physicists studying the quantization of gauge fields. For example, the quantization of Yang–Mills field in an arbitrary covariant background gauge leads to the operator $A^{ab}_{\mu\nu} = -\delta_{\mu\nu}\Box^{ab} - \left(\frac{1}{\alpha} - 1\right) D^{ac}_\mu D^{cb}_\nu - 2f^{acb}G^c_{\mu\nu}$, where D_μ is a covariant derivative containing the external field potential A_μ, $G_{\mu\nu}$ is a corresponding field strength and f^{abc} are the structure constants of a corresponding Lie algebra. An analogous operator arises also in quantum gravity.

We present below some examples of applications of the DWSG coefficients in different fields of mathematics and physics:

- **Spectral geometry and topology.** The index of elliptic operator A on a n-dimensional manifold M can be expressed via the Atiyah–Bott formula:

  $$\mathrm{ind}\,(A) \equiv \dim \mathrm{Ker}\, A - \dim \mathrm{Coker}\, A = \mathrm{Tr}\big[e^{-tA^+A} - e^{-tAA^+}\big].$$

  For the trace of the operator exponent the following asymptotic formula holds $\mathrm{Tr}\, e^{-tA} = \sum\limits_i e^{-t\lambda_i} \sim \sum\limits_{m\geq 0} B_m(A)t^{\frac{m-n}{2r}}$, $t \to 0_+$, where $B_m(A)$ are invariants of the manifold expressed in terms of the DWSG coefficients: $B_m(A) = \int_M d^n x\sqrt{g}\,\mathrm{tr}\, E_m(x|A)$. Here, Tr means the trace of the operator in the functional space, tr means the trace with respect to the Lorentz indices, g is the determinant of metric. In particular cases the invariants $B_m(A)$ can be interpreted as: B_0 is the volume of the manifold; if $n = 2$ and $A = -\Box$, then $E_2 = (4\pi)^{-1}R/6$ and $B_2 \sim \int d^2x\sqrt{g}R$ is the *Gauss–Bonnet invariant* of two-dimensional manifold; if $m = n$, then B_m is the index of A.
- **Korteweg–de Vries hierarchy.** The DWSG coefficients, computed for the Sturm–Liouville (1-dimensional Schrödinger) operator $A = -\frac{d^2}{dx^2} - U$, form the hierarchy of the higher Korteweg–de Vries equations. The mth KdV equation takes the form $\frac{\partial U}{\partial\tau} = \frac{\partial}{\partial x}G_m(U)$, where $G_m = \frac{m!}{2(m/2)!}E_m$. This approach extends somewhat the standard construction of the KdV hierarchy. For example, one can consider the matrix valued operator and construct the matrix KdV equation: $\frac{\partial U}{\partial\tau} = \frac{\partial}{\partial x}G_4(U) = \partial^3_x U + 3(\partial_x U)U + 3U(\partial_x U)$.

- **Spectral ζ-function and functional determinants in quantum field theory** (one-loop divergences of the effective action, the Green functions and the axial and trace anomalies etc.). Many problems in the theoretical physics are reduced to the computation of the functional integrals of the form $Z = \int D\Phi\, e^{-S(\Phi)}$ (or $Z = \int D\Phi\, e^{-iS(\Phi)}$). In accordance with the Laplace method (or saddle-point method) the main contribution into the integral gives the vicinity of the stationary point $\overline{\Phi}$ of the action $S(\Phi)$: $\frac{\delta S}{\delta \Phi}\big|_{\Phi=\overline{\Phi}} = 0$. The integral can be written approximately as $Z = \int D\phi e^{-S(\overline{\Phi}+\phi)} = e^{-S(\overline{\Phi})} \int D\phi e^{-\phi S''(\overline{\Phi})\phi + O(\phi^3)} \sim e^{-S(\overline{\Phi})}\det^{-1/2} S''(\overline{\Phi})$. In this formula $S''(\overline{\Phi}) \equiv A$ and $\det A = \prod_i \lambda_i$, that is, we have to compute the determinant of a matrix of infinite size. To cope with this difficulty we can use the regularization with the help of spectral (also called *generalized*) ζ-function which is an analog of the Riemann ζ-function with eigenvalues of the operator A in place of integer numbers in the classical Riemann function $\zeta(s) = \sum_i \frac{1}{\lambda_i^s} \equiv \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathrm{Tr}\, e^{-tA}$. As is seen from the definition of the spectral ζ-function $\ln \det A = -\zeta'(s)\big|_{s=0}$. On the other hand, $\mathrm{Tr}\, e^{-tA}$ can be expressed via the DWSG coefficients.

Besides there are many other applications like *Huygens' principle for hyperbolic operators*, *high temperature expansion of the partition function* in statistical physics etc.

3 Algorithm for computing the DWSG coefficients

Following the approach based on the covariant generalization of the pseudodifferential calculus [12], let us consider a positive elliptic operator which spectrum lies inside a contour C. The Dunford formula $e^{-tA} = \int_C \frac{id\lambda}{2\pi} e^{-t\lambda}(A-\lambda)^{-1}$, allows to express the heat operator $\exp(-tA)$ in terms of the resolvent $(A-\lambda)^{-1}$ of the operator A. In the pseudodifferential calculus the following asymptotic representation for the matrix elements of the resolvent is used

$$G(x,x',\lambda) \equiv \langle x | \frac{1}{A-\lambda} | x' \rangle = \int \frac{d^n k}{(2\pi)^n \sqrt{g(x')}}\, e^{il(x,x',k)} \sigma(x,x',k;\lambda), \tag{3}$$

where $\sigma(x,x',k;\lambda)$ is an amplitude, $l(x,x',k)$ is a (real) phase function which is a biscalar with respect to general coordinate transformations, k is a wave vector.

The resolvent of operator A satisfies the equation $(A-\lambda)G = \mathbb{1}$ (unit operator) which leads to an equation for the amplitude:

$$(A(x, D_a + iD_a l) - \lambda)\sigma(x,x',k;\lambda) = I(x,x'), \tag{4}$$

where $I(x,x')$ is a transport function having both bundle and Lorentz indices.

The covariant generalization of the pseudodifferential calculus is reduced to the proper definitions of the phase and transport functions occurring in formulas (3) and (4). Suitable definitions, suggested by H. Widom, can be described as follows. In the flat space the phase function is a linear function of the coordinates x, $l(x,x',k) = k_a(x-x')^a$, and the transport function is (in the functional space) the unit matrix $I(x,x') = \mathbb{1}$, i.e., constant. One can provide these properties by postulating vanishing of all derivatives for the transport function I and derivatives of the order > 1 for the phase function l. The covariant analog of these properties of the phase and transport functions is that the correspondent *symmetrized* covariant derivatives should be equal to zero at the point $x = x'$, i.e.:

$$[\{D_{a_1} \dots D_{a_m}\} l] = 0, \quad m > 1; \qquad [\{D_{a_1} \dots D_{a_m}\} I] = 0, \quad m \geq 1, \tag{5}$$

where $\{\dots\}$ means symmetrizing in all Lorentz indices, and $[\dots]$ means transition to coincidence limit $(x = x')$.

Equations (5) together with the "initial conditions" $[l]=0$, $[D_a l]=k_a$ and $[I]=\mathbb{1}$ allow one to compute the coincidence limits (these are just only values we need for the *diagonal* elements of the heat kernel) for nonsymmetrized covariant derivatives $[D_{a_1}\dots D_{a_m} l]$ and $[D_{a_1}\dots D_{a_m} I]$. The coincidence limits of nonsymmetrized derivatives are obtained directly from (5) by reducing all terms to a unified index ordering with the help of the Ricci identity for the commutator of covariant derivatives. The resulting expressions are polynomials in the curvature tensors R^a_{bcd} and W_{ab}, torsion T^a_{bc}, and covariant derivatives of all these tensors. The functions $l(x,x',k)$ and $I(x,x')$, introduced with the help of relations (5), play an important role in so-called *intrinsic symbolic calculus* developed by Widom. In fact, just these universal functions manifest the geometric properties of a base manifold and a bundle.

Expanding the amplitude σ in the degrees of homogeneity of k: $\sigma=\sum\limits_{m=1}^{\infty}\sigma_m(x,x',k;\lambda)$, we obtain the recursion equations for σ_m from equation (4). Solving the recursion equations we obtain σ_m. The heat invariants are expressed in terms of integrals of the coincidence limits $[\sigma_m]$ by the formula

$$E_m(x|A)=\int\frac{d^nk}{(2\pi)^n\sqrt{g}}\int_C\frac{id\lambda}{2\pi}e^{-\lambda}[\sigma_m](x,k,\lambda)\equiv J([\sigma_m]). \tag{6}$$

The integrals in (6) can be expressed in terms of gamma and Gauss hypergeometric functions for a wide class of operators A. The typical integral of terms in (6) takes the form

$$\begin{aligned}&J\left(\frac{k^{2p}k_{a_1}\dots k_{a_{2s}}}{(k^{2r}-\lambda)^l[(1-a)k^{2r}-\lambda]^m}\right)\\&\quad=g_{\{a_1\dots a_{2s}\}}\frac{\Gamma\left(\left(p+s+n/2\right)/r\right)}{(4\pi)^{n/2}2^sr\Gamma(n/2+s)\Gamma(l+m)}F\left(m,(p+s+n/2)/r;l+m;a\right),\end{aligned} \tag{7}$$

where $g_{\{\mu_1\dots\mu_{2s}\}}$ is a symmetrized sum of products of metric tensors.

Since m and l are integer numbers, the hypergeometric function in (7) can be expressed in terms of elementary functions with the help of one of the *Gauss relations* which is a second order recurrence with respect to the parameter a:

$$a(1-z)F(a+1,b;c;z)=(2a-c-az+bz)F(a,b;c;z)+(c-a)F(a-1,b;c;z). \tag{8}$$

To finish transition to the elementary functions we should solve this recurrence with the following initial conditions: $F(0,b;m;z)=1$, $F(1,b;m;z)=(m-1)!\frac{(-z)^{1-m}}{(1-b)_{m-1}}\Big[(1-z)^{m-b-1}-\sum\limits_{k=0}^{m-2}\frac{(b-m+1)_k}{k!}z^k\Big]$, $m=1,2,\dots$, $m-b\neq1,2,\dots$.

Here, $(a)_k=a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol (shifted factorial).

Besides, some additional work on structuring large output expressions is needed to make them as short and readable as possible. For this purpose we should first of all to find a suitable basis of independent elements in accordance with some criteria. The problem of constructing such bases is very difficult and up to now have no satisfactory algorithmic solution. As one can see, any coefficient E_m in expansion (1) is a linear combination of tensor monomials of a given mass dimensionality. These monomials are constructed from tensors by multiplication and index contraction. Factors in the monomials could be, in general case, non-commutative and posses symmetries leading to non-obvious linear dependencies among the monomials. Simplifying the tensor expressions one should take into account the symmetry properties of the tensors $R^a{}_{bcd}$, $T^a{}_{bc}$, W_{ab}, the Ricci identity:

$$[D_a,D_b]\varphi^{b_1\dots b_l}_{a_1\dots a_k}=\sum_{i=1}^{l}R^{b_i}{}_{jab}\varphi^{b_1\dots b_{i-1}jb_{i+1}\dots b_l}_{a_1\dots a_k}-\sum_{i=1}^{k}R^j{}_{a_iab}\varphi^{b_1\dots b_l}_{a_1\dots a_{i-1}ja_{i+1}\dots a_k}$$

$$+ \ T^{j}{}_{ab} D_j \varphi^{b_1 \ldots b_l}_{a_1 \ldots a_k} + W_{ab} \varphi^{b_1 \ldots b_l}_{a_1 \ldots a_k}, \tag{9}$$

the Bianchi identities for both affine $R^a{}_{bcd}$ and gauge W_{ab} curvatures:

$$D_a R^b{}_{cde} + D_d R^b{}_{cea} + D_e R^b{}_{cad} + T^i{}_{ad} R^b{}_{cei} + T^i{}_{de} R^b{}_{cai} + T^i{}_{ea} R^b{}_{cdi} = 0, \tag{10}$$

$$D_a W_{bc} + D_b W_{ca} + D_c W_{ab} + W_{ai} T^i{}_{bc} + W_{bi} T^i{}_{ca} + W_{ci} T^i{}_{ab} = 0, \tag{11}$$

the cyclic identity:

$$\begin{aligned} R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} + D_b T^a{}_{cd} + D_c T^a{}_{db} + D_d T^a{}_{bc} + T^a{}_{bi} T^i{}_{cd} \\ + T^a{}_{ci} T^i{}_{db} + T^a{}_{di} T^i{}_{bc} = 0, \end{aligned} \tag{12}$$

and also different consequences of these identities obtained by covariant differentiations and index contractions. In addition, tensor expressions are invariant with respect to renaming of dummy indices (i.e., with respect to the group S_n of all permutations of these indices, where n is the number of pairs of dummy indices in a given tensor monomial). All these identities and symmetries make the problem of choice of canonical basis rather nontrivial. Certainly, the problem have no unique solution and different bases are useful for different purposes. The problems with torsion are especially difficult, as the Bianchi and cyclic identities are non-linear in the presence of torsion. It may happen, that, in general, the problem of canonizing tensor expressions has no algorithmic solution at all, the usual situation for non-commutative and non-associative structures. Taking all the above into account, we elaborated a heuristic approach based on choosing a certain ordering of tensors and tensor indices and reducing tensor monomials to minimal ones with respect to this ordering. Of course, such approach does not ensure the full independence of the tensor invariants in the final expressions (especially in the case of high degree tensor polynomials arising in the DWSG coefficients of high order). Nevertheless, the approach works quite satisfactory and allows, in most cases, to decrease the expressions essentially.

In the case of non-minimal and high-order operators, the scalar coefficients at tensor invariants depend on the dimension of the manifold n and parameters of operator (like a in (2)). As a rule, there are many linear dependencies among these scalar coefficients. The dependences should be eliminated by the linear algebra methods to make the resulting formulas as compact as possible.

The algorithm has been implemented in the ***C*** language. The ***C*** code of total length about 11000 lines contains about 250 functions for different manipulations with tensors and scalars. These functions are gathered into two programs ***CoincidenceLimits*** and ***DWSGCoefficient***.

The ***CoincidenceLimits*** program computes the coincidence limits of non-symmetrized derivatives of the functions $l(x, x', k)$ and $I(x, x')$ and writes them to the disk. Once computed and stored they are used in many calculations for different operators A.

4 New results

With the help of the programs ***CoincidenceLimits*** and ***DWSGCoefficient*** we have obtained several new results. Some of them were not known entirely, others were known only partly or at different simplifying assumptions. Among the new results we can mention the following:

- E_4 with gauge and Riemann curvatures and torsion for operator $-\Box + X$.
- E_2, E_4 (gauge, Riemann curvatures; torsion) and E_6 (without torsion) for operator $-\Box + B_i D^i + X$. For example, $E_2 = (4\pi)^{-\frac{n}{2}} \left\{ -X - \frac{1}{4} B^2 + \frac{1}{6} R + D_i \left(\frac{1}{2} B^i + \frac{1}{6} T^i \right) + \frac{1}{12} T^2 - \frac{1}{24} T_{ijk} T^{ijk} - \frac{1}{12} T_{ijk} T^{jik} \right\}$, where $T_i = T^j{}_{ji}$, $T^2 = T_i T^i$, $B^2 = B_i B^i$.
- E_2, E_4 (gauge, Riemann curvatures; torsion) for operator $\Box^2 + V^{\mu\nu} D_\mu D_\nu + N^\mu D_\mu + X$.

As to the nonminimal operator $-g^{\mu\nu}\Box + aD^\mu D^\nu + X^{\mu\nu}$, we succeeded in computing the coefficient E_2 in the presence of gauge and Riemannian curvatures and torsion and the coefficient E_4 without torsion. In view of importance of the last result, let us consider it in more detail.

Recall from Atiyah–Bott formula that the coefficient E_4 is connected intimately just with the dimension of physical space-time $n = 4$ due to the fact that the Atiyah–Singer index of an elliptic operator on a manifold of dimension n can be expressed in terms of E_n with the help of integration. The full expression for E_4 can not be present here (one can find it in [8]), since it consists in 73 tensor terms with 43 different scalar coefficients (with 28 linear dependencies among them). Here we reproduce only the trace (with respect to Lorentz indices) $\mathrm{tr}_L E_4$. The expression for the trace still contains some novelties in comparison with the result of Branson, Gilkey and Pierzchalski [13], since they computed $\mathrm{tr}_L E_4$ without gauge field and neglecting the terms with total derivative (essential if the topology of manifold is non-trivial).

$$\begin{aligned}\mathrm{tr}_L E_4 = (4\pi)^{-\frac{n}{2}}\Big\{ & - C_1 \Box X_i{}^i - C_2 D_i D_j X^{ij} + C_3\left(X_i{}^i X_j{}^j + X_{ij}X^{ij}\right) \\ & + C_4 X_{ij}X^{ji} + C_5 X_{ij}W^{ij} - C_6 W_{ij}X^{ij} + C_7 W_{ij}W^{ij} + C_8 R_{ijkl}R^{ijkl} \\ & - C_9 R_{ij}X^{ij} - C_{10}R_{ij}R^{ij} + C_{11}\Box R + C_{12}R^2 - C_{13} R X_i{}^i \Big\},\end{aligned} \tag{13}$$

where

$$\begin{aligned}
C_1 = {} & \frac{1}{96a^2\left(\frac{n}{2}-2\right)_4}\Big\{(1-a)^{1-\frac{n}{2}}\left(96 - 96a - 48an + 8a^2n + 6a^2n^2 + a^2n^3\right) \\
& - 96(1-a)^2 + 32a^2n + 2a^2n^2 - 5a^2n^3 + a^2n^4\Big\}, \\
C_2 = {} & \frac{1}{48a^2\left(\frac{n}{2}-2\right)_4}\Big\{(1-a)^{1-\frac{n}{2}}\left(-48n + 48a + 24an - 4a^2n + a^2n^3\right) \\
& + 48n - 48a - 72an + 24an^2 + 48a^2 + 4a^2n - 24a^2n^2 + 5a^2n^3\Big\}, \\
C_3 = {} & \frac{1}{16a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}(-4 + 2a + an) + 4 - 2a + an\Big\}, \\
C_4 = {} & \frac{1}{16a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}(4 + 4n - 6a - 3an) - 4 - 4n + 6a - 3an - 2an^2 + an^3\Big\}, \\
C_5 = {} & \frac{1}{192a^2\left(\frac{n}{2}-2\right)_5}\Big\{(1-a)^{-1-\frac{n}{2}}\big(192n + 48n^2 + 576a + 192an - 216an^2 - 24an^3 \\
& - 1152a^2 - 944a^2n + 136a^2n^2 + 62a^2n^3 + 2a^2n^4 + 576a^3 + 664a^3n + 86a^3n^2 \\
& - 31a^3n^3 - 5a^3n^4 - 104a^4n\big) - 62a^4n^2 - a^4n^3 + 2a^4n^4 - 192n - 48n^2 - 576a \\
& - 384an + 72an^2 + 576a^2 + 272a^2n - 208a^2n^2 + 10a^2n^3 + 4a^2n^4 - 24a^3n + 26a^3n^2 \\
& - 9a^3n^3 + a^3n^4\Big\}, \\
C_6 = {} & \frac{1}{192a^2\left(\frac{n}{2}-2\right)_5}\Big\{(1-a)^{-1-\frac{n}{2}}\big(- 192n - 48n^2 + 384a + 816an + 96an^2 + 12an^3 \\
& - 768a^2 - 1120a^2n - 148a^2n^2 - 8a^2n^3 - 2a^2n^4 + 384a^3 + 584a^3n + 150a^3n^2 + 7a^3n^3 \\
& - 88a^4n - 58a^4n^2 - 5a^4n^3 + a^4n^4\big) + 192n + 48n^2 - 384a - 624an + 48an^2 + 12an^3 \\
& + 384a^2 + 304a^2n - 164a^2n^2 + 8a^2n^3 + 2a^2n^4 - 24a^3n + 26a^3n^2 - 9a^3n^3 + a^3n^4\Big\}, \\
C_7 = {} & \frac{1}{48a\left(\frac{n}{2}-1\right)_2}\Big\{(1-a)^{1-\frac{n}{2}}\big(- 96 + 22an + an^2 + 2a^2n - a^2n^2\big) + 96 - 96a + 26an \\
& - 3an^2 + an^3\Big\}, \\
C_8 = {} & \frac{(1-a)^{2-\frac{n}{2}} - 16 + n}{180},
\end{aligned}$$

$$C_9 = \frac{1}{24a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}\big(12n-12a-8an-an^2+2a^2n+a^2n^2\big)-12n+12a$$
$$+8an-5an^2\Big\},$$
$$C_{10} = \frac{1}{1440a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}\big(-360n+360a+296an+60an^2+an^3-112a^2n$$
$$-60a^2n^2-2a^2n^3-4a^3n+a^3n^3\big)+360n-360a-296an+116an^2-an^3+an^4\Big\},$$
$$C_{11} = \frac{1}{480a^2\left(\frac{n}{2}-2\right)_4}\Big\{(1-a)^{1-\frac{n}{2}}\big(480-240n-240a-120an+16a^2n+26a^2n^2$$
$$+11a^2n^3+a^2n^4+4a^3n+4a^3n^2-a^3n^3-a^3n^4\big)-480+240n+720a-360an$$
$$+120an^2-240a^2+104a^2n-70a^2n^2+15a^2n^3-5a^2n^4+a^2n^5\Big\},$$
$$C_{12} = \frac{1}{576a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}\big(-144+72a+56an+12an^2+an^3-16a^2n-12a^2n^2$$
$$-2a^2n^3-4a^3n+a^3n^3\big)+144-72a+16an-16an^2-an^3+an^4\Big\},$$
$$C_{13} = \frac{1}{48a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}\big(-24+12a+8an+an^2-2a^2n-a^2n^2\big)+24-12a$$
$$-an^2+an^3\Big\}.$$

As one can see, the scalar coefficients in (13) depend in a rather non-trivial way not only on the gauge parameter a, but also on the space dimension n. This is a typical property of nonminimal and high order operators [12].

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