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Quantum Long-Range Interactions in General Relativity

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We have found one-loop effects in general relativity which can be interpreted as quantum corrections to the Schwarzschild metric. They induce quantum long-range corrections to the Newton law and to gravitational relativistic effects: spin-orbit, spin-spin, and Lense–Thirring interactions.

1 Introduction

It has been recognized long ago that quantum effects can generate long-range corrections in general relativity. Those corrections due to the graviton polarization operator with photons and massless neutrinos in the loop were calculated in [1–4]. The corresponding quantum correction to the Newton potential between two bodies with masses m_1 and m_2 is

$$U_{\gamma\nu} = -\frac{4 + N_\nu}{15\pi}\,\frac{k^2\hbar m_1 m_2}{c^3 r^3}, \tag{1}$$

where N_ν is the number of massless two-component neutrinos, k is the Newton gravitational constant (in the text below we put $\hbar = 1$, $c = 1$).

The reason why the problem allows for a closed solution is as follows. The Fourier-transform of $1/r^3$ is

$$\int d\boldsymbol{r}\,\frac{\exp(-i\boldsymbol{q}\boldsymbol{r})}{r^3} = -2\pi\ln\boldsymbol{q}^2. \tag{2}$$

This singularity in the momentum transfer $\boldsymbol{q}$ means that the correction discussed can be generated only by diagrams with two massless particles in the t-channel. The number of such diagrams of second order in k is finite, and their logarithmic part in q^2 can be calculated unambiguously.

The analogous diagrams with gravitons and ghosts in the loop (see Fig. 1; wavy and dashed lines in it refer to gravitons and ghosts, respectively, double wavy lines refer to the background gravitational field) were considered in [1, 5–7].

Clearly, other diagrams with two gravitons in the t-channel contribute as well to the discussed correction $\sim 1/r^3$. Some of these contributions were addressed, along with diagrams 1a,b, in [8–13]. However in these papers the set of considered diagrams was incomplete, and the results for these diagrams were incorrect. For the first time the complete set of relevant diagrams was pointed out in [14], however, only one of them was calculated therein correctly.

The problem of quantum correction to the Newton law, which is certainly interesting from the theoretical point of view, was then addressed in our previous article [15]. In it all relevant diagrams, except one (see Fig. 3b below), were calculated correctly.

In a recent paper [16] our criticisms are acknowledged, though implicitly. The diagram 3b is calculated in [16] correctly, and the results for all other contributions agree with ours.

The content of our present work is as follows. We demonstrate in an elementary way that the discussed corrections are the same both for scalar and (after averaging over spins) for spinor particles. The fact was proven previously in [17] by direct calculation of loop diagrams.

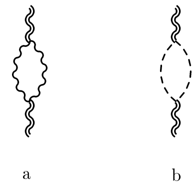

Figure 1. Graviton loops.

Then, using the background field technique [7], we construct effective amplitudes which describe quantum power corrections in general relativity. Since the derived corrections are universal (i.e. the same for scalar and spinor particles), in the limit when one of the interacting particles is heavy, these corrections can be interpreted as corrections to the Schwarzschild metric. (Our results for the latter differ from the results of [17].) In this way we not only simplify the calculation of the corrections to the Newton law, but obtain rather easily quantum corrections to gravitational relativistic effects.

We have also cross-checked and confirmed the result for the corrections to the Newton law by calculation in another technique (used previously in [14]) that with the gravitational variables $\psi^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$ in the harmonic gauge $\partial_\mu \psi^{\mu\nu} = 0$.

2 Effective amplitudes

Let us sketch first of all our proof of the fact that the corrections $\sim \ln|q^2|$ for spinor particles coincide, after averaging over spins, with those for scalar ones. To this end, it is sufficient in fact to consider tree diagrams, Fig. 2, which are building blocks of the logarithmic loops. Here and below wavy lines denote gravitons again, and solid lines refer to scalar or spinor particles. We average the spinor diagrams over the spins, and single out from the numerators of diagrams 2a,b, both for scalar and spinor particles, the structures that cancel the denominators therein. Thus obtained contact contributions combine with the initial contact diagram 2c into an effective seagull which is the same both for spins 0 and 1/2. As to the true s- and u-pole contributions in diagrams 2a,b, which are left after this procedure, they also coincide for scalar and spinor particles with the adopted accuracy.

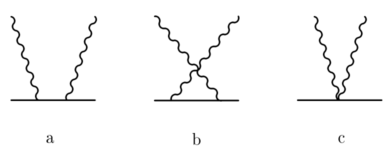

Figure 2. Tree diagrams.

We start the discussion of the loops with the vacuum polarization diagrams, Fig. 1. The effective Lagrangian corresponding to the sum of these loops, derived in [7], can be rewritten for our purpose as [8]

$$L_{rr} = -\frac{1}{1920\pi^2}\ln|q^2|\left(42R_{\mu\nu}R^{\mu\nu} + R^2\right); \tag{3}$$

here $R_{\mu\nu}$ is the Ricci tensor, $R = R^\mu_\mu$ is the scalar curvature. We are interested at the moment in the situation when at least one of the particles is considered in the static limit. In this case $|q^2| \to \boldsymbol{q}^2$, and in the coordinate representation we obtain

$$L_{rr} = \frac{1}{3840\pi^3 r^3}\left(42R_{\mu\nu}R^{\mu\nu} + R^2\right). \tag{4}$$

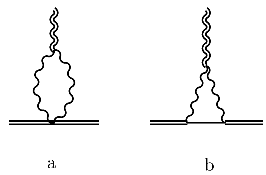

Figure 3. Vertex diagrams.

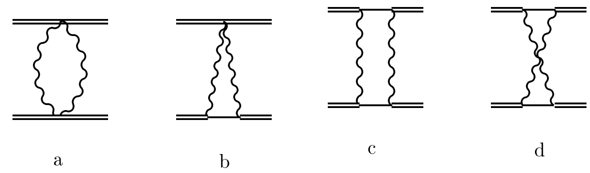

Figure 4.

The next set of diagrams, Fig. 3, refers to the vertex part. The corresponding effective operator is

$$L_{rt} = -\frac{k}{8\pi^2 r^3}\left(3R_{\mu\nu}T^{\mu\nu} - 2RT\right), \qquad T = T^{\mu}_{\mu}. \tag{5}$$

Here $T^{\mu\nu}$ is the energy-momentum tensor (EMT) averaged over spin.

At last diagrams of Fig. 4. Their sum is

$$L_{tt} = \frac{k^2}{\pi r^3}\, T^2. \tag{6}$$

In fact, the box diagrams in Fig. 4 not only contribute to amplitude (6). They also generate a more complicated amplitude which cannot be reduced to a product of energy-momentum tensors. We will come back to this irreducible amplitude later.

In virtue of the Einstein equations

$$R_{\mu\nu} = 8\pi k\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{7}$$

the three Lagrangians (4), (5), (6) can be conveniently combined into

$$L_{\text{tot}} = -\frac{k^2}{60\pi r^3}\left(138\, T_{\mu\nu}T^{\mu\nu} - 31T^2\right). \tag{8}$$

3 Quantum corrections to metrics

Now quantum corrections to some gravitational effects can be most easily derived as follows. Let us split the total EMT $T_{\mu\nu}$ into those of a static central body and of a light probe one, $T^{o}_{\mu\nu}$ and $t_{\mu\nu}$, respectively. Then, by variation the resulting expression in $t^{\mu\nu}$ we obtain a tensor which can be interpreted as a quantum correction $h^{(q)}_{\mu\nu}$ to the metric created by the central body:

$$h^{(q)}_{\mu\nu} = \frac{k^2}{15\pi r^3}\left(138T^{o}_{\mu\nu} - 31g_{\mu\nu}T^{o}\right). \tag{9}$$

It follows immediately from this expression that

$$h^{(q)}_{00} = \frac{107}{15}\,\frac{k^2 M}{\pi r^3}, \tag{10}$$

where M is the mass of the central body.

The calculation of the space components $h^{(q)}_{mn}$ demands in fact some modification of formula (9). The point is that we work with the gauge condition $h^{\mu}_{\nu;\,\mu} - (1/2)h^{\mu}_{\mu;\,\nu} = 0$ for the graviton field. It is only natural to demand that the resulting effective field $h^{(q)}_{mn}$ should satisfy the same condition which simplifies now to $h^{(q)\mu}_{\quad\nu,\,\mu} - (1/2)h^{(q)\mu}_{\quad\mu,\,\nu} = 0$. Thus obtained space metric is

$$h^{(q)}_{mn} = \frac{k^2 M}{\pi r^3}\left\{\frac{31}{15}\delta_{mn} - \frac{76}{15}\left[\frac{r_m r_n}{r^2} + \ln\left(\frac{r}{r_0}\right)\left(\delta_{mn} - 3\frac{r_m r_n}{r^2}\right)\right]\right\}. \tag{11}$$

Technically, the expression in square brackets in (10) originates from the terms containing structures of the type $\partial_\mu T^{\mu\nu}$. Generally speaking, they arise when calculating Lagrangians (5), (6), and (8), but are omitted therein since they vanish on-mass-shell. Thus these terms are absent in (9). But they can be restored by rewriting the net result (8) with the Einstein equations (7) as

$$L_{tot} = -\frac{1}{3840\pi^3 r^3}\left(138 R_{\mu\nu}R^{\mu\nu} - 31R^2\right), \tag{12}$$

and then attaching energy-momentum tensors to the double wavy lines. The presence of $\ln(r/r_0)$, where r_0 is some normalization point, is quite natural here if one recalls $\ln \boldsymbol{q}^2$ in the momentum representation. Fortunately, this term in square brackets does not influence physical effects.

Our results (10), (11) differ from the corresponding ones of [17]. The main reason is that the contributions of diagrams 4 to metric are omitted in [17]. This omission does not look logical to us: on-mass-shell one cannot tell these diagrams from others (see (8), (12)). Besides, the Fourier-transformation of $(q_m q_n/\boldsymbol{q}^2 \ln \boldsymbol{q}^2)$ is performed in [17] incorrectly, with a wrong result ($r_m r_n/r^2$ only) for the term in the square brackets in (11).

4 Quantum corrections to gravitational effects

We start with the correction to the Newton law. In line with (10), we should take into account here the above mentioned contribution of the box diagrams 4, which cannot be reduced to metric. In the static limit for both particles this irreducible contribution is [15, 16, 19]

$$-\frac{23}{3}\,\frac{k^2 Mm}{\pi r^3}. \tag{13}$$

The net correction to the Newton law is

$$U^q(r) = -\frac{41}{10}\,\frac{k^2 Mm}{\pi r^3}. \tag{14}$$

Let us go over now to the quantum correction to the interaction of the orbital angular momentum $\boldsymbol{l}$ of a light particle with its own spin $\boldsymbol{s}$, i.e. to the common gravitational spin-orbit interaction. It is most easily obtained with the general expression for the frequency $\boldsymbol{\omega}$ of the spin precession in a gravitational field derived in [18]. For a nonrelativistic particle in a weak static centrally-symmetric field this expression simplifies to

$$\omega_i = \frac{1}{2}\,\epsilon_{imn}(\gamma_{mnk}v_k + \gamma_{0n0}v_m). \tag{15}$$

Here

$$\gamma_{mnk} = \frac{1}{2}\,(\partial_m h_{nk} - \partial_n h_{mk}), \qquad \gamma_{0n0} = -\frac{1}{2}\,\partial_n h_{00}$$

are the Ricci rotation coefficients, $\boldsymbol{v}$ is the particle velocity (the present sign convention for $\boldsymbol{\omega}$ is opposite to that of [18]). A simple calculation results in

$$U^q_{ls}(r) = -\frac{169}{20}\,\frac{k^2}{\pi r^5}\,\frac{M}{m}\,(\boldsymbol{ls}). \tag{16}$$

Now let us derive the quantum correction to the classical velocity-dependent gravitational interaction. We start with the amplitude (8) written in the momentum representation:

$$L_{\rm tot} = \frac{k^2}{30}\,\ln|q^2|\left(138 T_{\mu\nu}T^{\mu\nu} - 31T^2\right). \tag{17}$$

As distinct from the previous corrections, here we go beyond the static approximation, and expand $\ln|q^2| = \ln(\boldsymbol{q}^2 - \omega^2)$ to first order in ω^2. Then (in spirit of [20] where this trick was applied to the calculation of the classical velocity-dependent c^{-2} correction to the Newton law) we rewrite ωT_{00} as $q_n T_{0n}$, neglect terms of 4th and higher orders in c^{-2}, and come back to the coordinate representation. The resulting velocity-dependent quantum correction is

$$U^q_{vv}(\boldsymbol{r}) = -\frac{k^2 m_1 m_2}{60\pi r^3}\,[445(\boldsymbol{v}_1\boldsymbol{v}_2) + 321(\boldsymbol{n}\boldsymbol{v}_1)(\boldsymbol{n}\boldsymbol{v}_2)], \qquad \boldsymbol{n} = \frac{\boldsymbol{r}}{r}. \tag{18}$$

With formula (18) we derive (in spirit of [21], § 106, Problem 4) the quantum correction to the interaction of the orbital momentum $\boldsymbol{l}$ of a light particle with the internal angular momentum (spin) $\boldsymbol{s}$ of a compound central body, i.e. to the Lense–Thirring effect. The result is

$$U^q_{\mathrm{LT}}(r) = -\frac{69}{5}\,\frac{k^2}{\pi r^5}\,(\boldsymbol{l}\boldsymbol{s}). \tag{19}$$

In the same way one obtains with (18) the correction to the spin-spin interaction of two compound bodies:

$$U^q_{ss}(\boldsymbol{r}) = \frac{69}{10}\,\frac{k^2}{\pi r^5}\,[3(\boldsymbol{s}_1\boldsymbol{s}_2) - 5(\boldsymbol{n}\boldsymbol{s}_1)(\boldsymbol{n}\boldsymbol{s}_2)]. \tag{20}$$

The quantum correction to the classical velocity-dependent gravitational interaction is generated also by the irreducible parts of the box diagrams (a sort of relativistic analogue of (13)). To derive it, we expand the corresponding expressions for those diagrams to first order in the small parameter $(p_1p_2 - m_1m_2)/m_1m_2$ (but not to zeroth order, as it was the case with (13)). Thus obtained net result can be written in the momentum representation as

$$-k^2 m_1 m_2 \ln\left(\boldsymbol{q}^2 - \omega^2\right)\frac{2}{3}\left(23 + \frac{524}{5}\,\frac{p_1p_2 - m_1m_2}{m_1m_2}\right). \tag{21}$$

Now we expand again $\ln(\boldsymbol{q}^2 - \omega^2)$ to first order in ω^2, and use the obvious identity

$$\omega p_0 \approx \omega m \approx m\boldsymbol{v}\boldsymbol{q}.$$

At last, going over into the coordinate representation (and omitting the zeroth-order contribution (13)), we arrive at the following result for the irreducible contribution to the quantum velocity-dependent correction:

$$U^{q,\,irr}_{vv}(\boldsymbol{r}) = \frac{k^2 m_1 m_2}{10\pi r^3}\,[311(\boldsymbol{v}_1\boldsymbol{v}_2) + 115(\boldsymbol{n}\boldsymbol{v}_1)(\boldsymbol{n}\boldsymbol{v}_2)]. \tag{22}$$

From (22) we obtain now the irreducible contributions to the quantum corrections to the Lense–Thirring and spin-spin interactions, respectively:

$$U^{q,\,irr}_{LT}(r) = \frac{262}{5}\,\frac{k^2}{\pi r^5}\,(\boldsymbol{l}\boldsymbol{s}), \tag{23}$$

$$U^{q,\,irr}_{ss}(\boldsymbol{r}) = -\frac{131}{5}\,\frac{k^2}{\pi r^5}\,[3(\boldsymbol{s}_1\boldsymbol{s}_2) - 5(\boldsymbol{n}\boldsymbol{s}_1)(\boldsymbol{n}\boldsymbol{s}_2)]. \tag{24}$$

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