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Spray Algebra

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Spray algebra is the algebra of equivariant multilinear tensor fields with respect to the spray of a linear connection. Many natural operations of tensor algebra on tangent bundle have interpretation in term of spray algebra. This give new point of view on well known differential objects and provides us with new means for investigation. In particular, such achievement allows to formulate an analog of so-called Bochner type condition for the case of general linear connection.

1 Introduction

The main purpose of recent report is to construct some tensor algebra on tangent bundle TM of a C^∞-manifold M^n provided with a torsion-free linear connection. This object named by author Spray Algebra is a graded subalgebra of tensor algebra $\mathfrak{T}(TM)$. Some of tensor operations on $\mathfrak{T}(TM)$ save grading and therefore have an interpretation in terms of the spray algebra operations. Many of them appeared earlier in connection with different problems of differential geometry and physics, some objects seem to be described for the first time.

Among operations defined for the leading terms of grading are inner differential, the Jacobi operator, the Killing equation, Ashtekar's pairing and other.

The construction of spray algebra is based on concept of multilinear tensor field on TM. In the case of vector field such ones were investigated by V.V. Kozlov in search of the polynomial integrals of Hamiltonian systems. However, the above concept can be extended over the whole algebra $\mathfrak{T}(TM)$. We describe this construction in details as we cannot give a direct reference.

The great deal of the report is devoted to the explicit description of the structures in particular cases.

The analysis of the exterior differential action on the kernel of Lie differentiation with respect to the spray of some linear connection leads to the classical problems of mobility. This allows to generalize the concept of Bochner type condition for the case of linear connection.

Notations:

$$\bar\imath = (i_1, \dots, i_k), \qquad |\bar\imath| = i_1 + \cdots + i_k, \qquad \|\bar\imath\| = k,$$
$$\bar 1 = (1, \dots, 1), \qquad T_{\bar\imath} dx^{\bar\imath} = T_{i_1 \cdots i_k} dx^{i_1} \cdots dx^{i_k}.$$

2 Multilinear tensor fields

Let us consider a C^∞-manifold M^n and its tangent bundle TM provided with standard maps $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{v})$. For tensor fields of type (k, s) on TM there exists a tensor analog of the (non-tensor) Taylor decomposition with respect to $\boldsymbol{v}$ along the 0-section of TM. We will define it recurrently for the field $\widetilde{T}$. The first term of decomposition is defined by the formula

$$\overset{(0)}{T}(\boldsymbol{x}, \boldsymbol{v}) = \widetilde{T}^{\bar\imath}_{\bar\jmath}(\boldsymbol{x}, 0) \frac{\partial}{\partial v^{\bar\imath}} \otimes dx^{\bar\jmath}.$$

For the tensor field $T = \widetilde{T} - \overset{(0)}{T}$ the following procedure is well defined

$$\overset{(p)}{T} = \lim_{\alpha\to\infty} \alpha^{p+s-k} h_\alpha^* \Big(T - \overset{(1)}{T} - \dots - \overset{(p-1)}{T} \Big),$$

where h_α^* is the transfer of the field with the fiber-wise homothety of TM. To describe such fields in coordinates let us introduce the **index** of the multi-indices triple $(\bar\imath, \bar\jmath, \bar m)$

$$\mathbf{Ind}(\bar\imath, \bar\jmath, \bar m) = |\bar 1 - [\bar\imath]| + |[\bar\jmath]| + \|\bar m\|,$$

where

$$[\bar\imath] = \left(\left[\frac{i_1 - 1}{n} \right], \dots, \left[\frac{i_k - 1}{n} \right] \right).$$

The formal Taylor decomposition of T with respect to $\boldsymbol{v}$ has the form

$$T = T^{\bar\imath}_{\bar\jmath|\bar m}(\boldsymbol{x}) v^{\bar m} \frac{\partial}{\partial z^{\bar\imath}} \otimes dz^{\bar\jmath}.$$

The general form of $\overset{(p)}{T}$ follows directly from the tensor coordinate relations

$$\overset{(p)}{T} = \sum_{\mathbf{Ind}(\bar\imath,\bar\jmath,\bar m)=p} T^{\bar\imath}_{\bar\jmath|\bar m}(\boldsymbol{x}) v^{\bar m} \frac{\partial}{\partial z^{\bar\imath}} \otimes dz^{\bar\jmath}.$$

Definition 1. The field $\overset{(p)}{T}$ is called multilinear tensor field of type (k, s, p). The space of such fields is denoted by $\overset{(k,s,p)}{\mathfrak{P}}$.

For instance, the Liouville field $v^i \partial/\partial v^i$ is of type $(1, 0, 1)$, the spray

$$S = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{mn}(\boldsymbol{x}) v^m v^n \frac{\partial}{\partial v^i}$$

is of type $(1, 0, 2)$. Notice that the index is additive with respect to tensor product. It implies that the space

$$\mathfrak{P}(TM) = \oplus_{k,s=0}^{\infty} \oplus_{p=0}^{\infty} \overset{(k,s,p)}{\mathfrak{P}}$$

is graded tensor algebra. Moreover, the straightforward calculation shows that for the vector fields $T \in \overset{(1,0,p)}{\mathfrak{P}}$ and $S \in \overset{(1,0,q)}{\mathfrak{P}}$ holds

$$\mathbf{Ind}([T, S]) = \mathbf{Ind}\, T + \mathbf{Ind}\, S - 1.$$

It means that the space $\oplus_p \overset{(1,0,q)}{\mathfrak{P}}$ is the graded Lie algebra.

3 Spray algebra

Let tensor field T be equivariant with respect to the spray S, i.e.

$$\mathfrak{L}_S T = 0,$$

where $\mathfrak{L}_S$ is the Lie differentiation. Straightforward calculation gives $\overset{(0)}{T} = 0$. For $\mathfrak{L}_S$ and h^*_α commute the following series of equations hold for any $p \geqslant 0$

$$\mathfrak{L}_S \overset{(p)}{T} = 0. \tag{1}$$

The space of solutions of equation (1) will be denoted by $\overset{(k,s,p)}{\mathfrak{Y}}$.

Definition 2. The graded algebra

$$\mathfrak{Y} = \oplus_{k,s=0}^{\infty} \oplus_{p=0}^{\infty} \overset{(k,s,p)}{\mathfrak{Y}}$$

is called *spray algebra* of symmetry linear connection with spray S.

The equation (1) in the standard map on TM is the system of PDE's on tensor field T components. In many interesting cases this system allows the explicit description in terms of tensor fields on manifold M itself and operations over them! This reduction sheds light on many invariant constructions of tensor calculus which turned out to be related. The algebraic structure of spray algebra provides us with new instruments applicable to different problems of differential geometry.

Let us give explicit description of some spaces and operations appeared.

The space $\overset{(0,0,p)}{\mathfrak{Y}}$. For the function f on TM the equation $\mathfrak{L}_S u = f$ is called the kinetic equation. Its solutions are defined up to the equivariant functions u: $\mathfrak{L}_S u = 0$. For the function $\overset{(p)}{u} = u_{|\bar{m}}(\boldsymbol{x}) v^{\bar{m}}$ which is the multilinear field in our notations the object $\overset{(p)}{u} = u_{|\bar{m}}(\boldsymbol{x})$ defines the tensor field of type $(0,p)$, $p = \|\bar{m}\|$, on M. The equation (1) can be written in terms of field u

$$\nabla_{(i} \overset{(p)}{u}_{|\bar{m})} = 0,$$

where $(,)$ means symmetrization with respect to covariant indices i, $\bar{m}$. Therefore the space $\overset{(0,0,p)}{\mathfrak{Y}}$ is the space of p-Killing fields, i.e. the first integrals of geodesic flow.

The space $\overset{(1,0,p)}{\mathfrak{Y}}$. The elements of this space have the coordinate form

$$(A, B) = A^i_{|\bar{m}}(\boldsymbol{x}) v^{\bar{m}} \frac{\partial}{\partial x^i} + B^i_{|\bar{n}}(\boldsymbol{x}) v^{\bar{n}} \frac{\partial}{\partial v^i},$$

where $\|\bar{m}\| = p - 1, \|\bar{n}\| = p$. In terms of components of the objects A and B the equation (1) can be written as generalized Jacobi equation [1]

$$\nabla_{(i} \nabla_j A^l_{|\bar{m})} + R^l_{(isj} A^s_{|\bar{m})} = 0,$$

where R is curvature tensor. The object B has necessarily the form

$$B_{|\bar{n}} v^{\bar{n}} = \overset{h}{\nabla}_i A_{|\bar{m}} v^{\bar{m}} v^i,$$

where $\overset{h}{\nabla}$ is horizontal covariant differentiation corresponding to the Berwald connection [2] of semi-basic tensor field formed by A. Therefore the space $\overset{(1,0,p)}{\mathfrak{Y}}$ can be described in terms of generalized Jacobi fields on M. In the case of Hamiltonian systems on two-dimensional manifolds the existence of such fields was investigated by V.V. Kozlov [3], for greater dimensions – by I.A. Taymanov [4].

The space $\overset{(0,1,p)}{\mathfrak{Y}}$. In this case the affine-Killing fields appear as solutions of the equation

$$\nabla_{(i}\nabla_j A_{|l||\bar{m})} + A_{(s|\bar{m}}R^s_{ij)l} = 0.$$

In the metric case this equation is equivalent to the previous one for dual fields.

In above cases and some others the spaces $\overset{(k,s,p)}{\mathfrak{Y}}$ admit the description in terms of tensor algebra $\mathfrak{T}(M)$. Let us now give the interpretation of some operation on $\mathfrak{T}(TM)$ in $\mathfrak{T}(M)$.

Exterior differential. For $\mathfrak{L}_S$ and d commute the operators

$$d_p : \overset{(0,0,p)}{\mathfrak{Y}} \to \overset{(0,1,p)}{\mathfrak{Y}}$$

are well defined. It is easy to see that d_p are identical injections. For $p = 1$ this fact is equivalent to the Ricci identity and means that every Killing field is an affine-Killing one. The case $p = 2$ was explicitly formulated in [5]. Here we obtained the set of statements for p-Killing fields.

Contraction. Contraction of tensor fields from $\overset{(1,0,p)}{\mathfrak{Y}}$ and $\overset{(0,1,p)}{\mathfrak{Y}}$ get into $\overset{(0,0,p)}{\mathfrak{Y}}$. In language of spray algebra it means, for instance, that for $\vartheta \in \mathfrak{T}^0_1(M)$ and $X \in \mathfrak{T}^1_0(M)$ the field

$$(\nabla\vartheta)(X) - \vartheta(\nabla X) \tag{2}$$

belongs to $\mathfrak{T}^0_0(M)$. This operation is nontrivial pairing of Jacobi fields and affine-Killing ones with value in Killing fields space. The very particular case of the operation (2) was described by Ashtekar [6]. For two Killing fields ϑ and η of metric ω_{ij} he associated the Killing field

$$\eta_j\omega^{ij}(\nabla\vartheta_i) - \vartheta_j\omega^{ij}(\nabla\eta_i),$$

where ∇ is Levi-Civita connection of metric. This operation corresponds to the contraction $\widetilde{\omega}(df, dg)$, where $\widetilde{\omega}$ is equivariant bivector on $\mathfrak{T}(TM)$.

Tensor product. The usual product of equivariant functions on TM in language of spray algebra means the symmetrized tensor product of Killing fields. As a sequence we have the statement that symmetrized product of Killing and affine-Killing fields is the latter again.

Lie bracket. The explicit description of Lie bracket in spray algebra was obtained in [7]. For $A \in \overset{(1,0,p)}{\mathfrak{Y}}$ and $B \in \overset{(1,0,q)}{\mathfrak{Y}}$ the symmetrized expression holds

$$[A, B]_{\mathfrak{Y}} = \mathfrak{S}\left(\nabla_A B - \nabla_B A + (q-1)B(\nabla A) - (p-1)A(\nabla B)\right).$$

4 Bochner complex

Let us begin with the general arguments. For the arbitrary vector field X on a smooth manifold N we introduce the space of equivariant differential p-forms

$$\Omega^X_p = \{\vartheta \in \Omega_p(N) \,|\, \mathfrak{L}_X\vartheta = 0\}.$$

For $\mathfrak{L}_X$ and d commute the exterior differential $d\vartheta$ of equivariant p-form ϑ is also equivariant. Therefore the complex $\Omega^X_\bullet(N)$ with exterior differential is well defined. If $X = 0$ this complex coincides with initial de Rham complex. In the case $N = TM$ and $X = S$ we can apply all above constructions for the complex $\Omega^S_\bullet(TM)$. The decomposition of $\mathfrak{T}(TM)$ induces the decomposition of the complex and differential on the set of complexes

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & \dots & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow d_{-1} \\
\overset{(0,0,1)}{\Omega_0^S} & \oplus & \overset{(0,0,2)}{\Omega_0^S} & \oplus & \overset{(0,0,3)}{\Omega_0^S} & \dots & \Omega_0^S(TM) \\
\downarrow d_0^1 & & \downarrow d_0^2 & & \downarrow d_0^3 & & \downarrow d_0 \\
\overset{(0,1,1)}{\Omega_1^S} & \oplus & \overset{(0,1,2)}{\Omega_1^S} & \oplus & \overset{(0,1,3)}{\Omega_1^S} & \dots & \Omega_1^S(TM) \\
\downarrow d_1^1 & & \downarrow d_1^2 & & \downarrow d_1^3 & & \downarrow d_1 \\
\overset{(0,2,1)}{\Omega_2^S} & \oplus & \overset{(0,2,2)}{\Omega_2^S} & \oplus & \overset{(0,2,3)}{\Omega_2^S} & \dots & \Omega_2^S(TM) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}
$$

As was shown in Section 3, the operators d_0^p are identical injection of p-Killing fields into the space of affine-Killing fields. In the compact Riemannian case the Jacobi operator is reducible to the Weitzenböck form [8] and the Bochner technique is applicable. The Bochner condition is that guaranteeing every affine mapping be a Killing one, i.e. d_0^1 is a surjection. The strong Bochner condition means Riemannian immovability, i.e. d_{-1}^1 is surjection. All operators in above complexes consist of covariant differentiations and curvature tensor like in the Weitzenböck form. On the basis of these arguments we shall call the complex $\Omega_\bullet^S(TM)$ the Bochner complex and the surjection of the differential will be called the Bochner condition. These notions are defined for the case of non-compact manifolds equipped with a general linear connection.

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[1] Kalnitsky V.S., Algebra of generalized Jacobi fields, *Zap. Nauch. Sem. POMI*, 1995, V.231, 222–243 (translation in *J. Math. Sci. (New York)*, 1998, V.91, 3476–3491).

[2] Pestov L.P. and Sharafutdinov V.A., Integral geometry of tensor fields on manifolds of negative curvature, *Sibirsk. Mat. Zh.*, 1988, V.29, N 3, 114–130 (translation in *Siberian Math. J.*, 1988, V.29, N 3, 427–441).

[3] Kozlov V.V., On the polynomial integrals of the dynamical systems, *Mat. Zametki*, 1989, V.45, N 4, 46–52 (translation in *Math. Notes*, 1989, V.45, N 3–4, 296–300).

[4] Taymanov I.A., Topological properties of integrable geodesic flows, *Mat. Zametki*, 1988, V.44, N 2, 283–284 (in Russian).

[5] Singh K.D., Affine 2-Killing vector and tensor fields, *C. R. Acad. Bulgare Sci.*, 1983, V.36, N 11, 1375–1378.

[6] Ashtekar A. and Magnon-Ashtekar A., A technique for analyzing the structure of isometries, *J. Math. Phys.*, 1978, V.19, N 7, 1567–1572.

[7] Kalnitsky V.S., Automorphisms of a geodesic vector field, *Vestnik St. Petersburg Univ. Math.*, 1995, V.28, N 2, 19–20.

[8] Bochner S., Vector fields and Ricci curvature, *Bull. Am. Math. Soc.*, 1946, N 52, 776–797.