Dynamics of Sharp Interfaces in One-, Two-Phase Flows in Porous Media: Asymmetry in the Boussinesq and Charny Equations

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For the Boussinesq and Charny nonlinear diffusion equations we employ the known similarity solutions and derive new ones using the Adomian decomposition method and separation of variables. For a periodic water-drive regime with a heavier fluid sweeping a lighter one from a porous formation we arrive at an explicit analytical expression for the interface and describe the phenomena of "superpropagation" and "counterslumping" that date back to tidal "superelevation" in coastal unconfined aquifers described by J.R. Philip. Similarly, for a two-phase flow with a straight sharp interface separating two fluids of contrasting viscosity the interface in a periodic drive regime propagates deeper than in a constant rate sweep. Applications to groundwater hydrology and petroleum engineering are discussed.

1 Conceptual model

Two-phase flows in porous media are often modeled as two ($i = 1, 2$) continua separated by a sharp interface between two zones in which phase saturations are constant [6]. The interface moves and one phase displaces another either completely or to a residual content, which can be always scaled out. If the fluids and skeleton are incompressible, formation is homogeneous, movement is Darcian, then on each side of the interface the total head h_i satisfies the Laplace equation

$$\Delta h_i(x, y, z, t) = 0, \qquad (1)$$

where x, y are the horizontal coordinates, z is the vertical coordinate, $\vec{V_i} = -K_i \nabla h_i$ is the velocity vector, $K_i = k\rho_i g/\mu_i$ are phase conductivities, k is intrinsic permeability, ρ_i and μ_i are fluid densities and viscosities, respectively. If capillary jump across the interface is ignored, then pressure there is continuous as well as the normal component of $\vec{V}$. If the displacement is not full, then k in each zone is not constant (as is postulated in the sharp interface model) but a function of the degree of saturation and (1) does not hold.

Classical groundwater hydrology [6] ignores the second (air) phase and (1) is satisfied for the water phase only. The corresponding free boundary problem is nonlinear owing to the kinematic and dynamic boundary conditions along the water table (phreatic surface) that can result in somewhat counterintuitive shapes, e.g. a wet zone hanging over a dry one [14]. In practice, most unconfined aquifers extend laterally much more than vertically and the free surface slope is gentle. It makes possible a hydraulic approximation or averaging of all flow parameters across the aquifer that eliminates z from (1) and, if y is physically irrelevant (e.g. seepage from irrigation channels), the free boundary is implicitly incorporated into the governing Boussinesq equation

$$S\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[KH\frac{\partial H}{\partial x}\right] + c(H, x, t), \tag{2}$$

where $H(x,t)$ is the thickness of the saturated zone (water table elevation above a horizontal impermeable bedrock), c is the sink-source (evaporation-infiltration) term and S is specific yield.

2 Similarity solutions and decomposition

Barenblatt in 1952 [5] and Sokolov in 1956 [23] obtained a number of similarity solutions to the nonlinear diffusion equation (2) at $c = 0$. Of particular interest is the instantaneous source solution ABC (Fig. 1a, solid line), which has a finite support $2L(t)$, i.e. two symmetrical tips, which propagate left- and right-ward due to gravitational slumping of a fixed volume of water released as a spike in a dry aquifer.

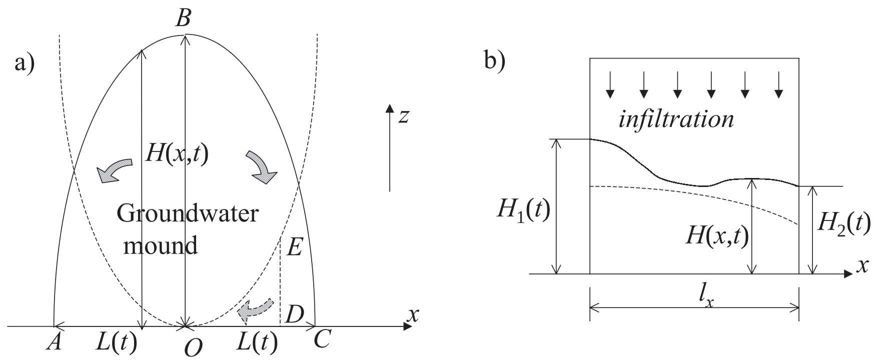

Figure 1.

We demonstrated [13] that the Barenblatt and Sokolov concave parabolas are identical and can be converted into a convex parabola satisfying (2) (Fig. 1a, dashed line). The two parabolas behave differently, in particular, the convex parabola has a frozen tip O, i.e. the two branches can be considered separately and we have localization or a finite size of the saturated zone even at blow-up boundary conditions. The question arises whether the convex parabola can appear in aquifers as a physically real initial condition to the transient equation (2). The answer is yes. We integrated the steady governing equation (LHP in (2) is zero) with a constant reservoir level (ED in Fig. 1a) and evaporation $c = \alpha h$ (α is a constant) and obtained just the convex parabola. Note that this linear evaporation function is an expedient model advocated by some groundwater hydrologists.

We now consider the hydraulic model for a nearly horizontal homogeneous unconfined aquifer of length l_x, bounded by two time-dependent boundary conditions, $H_1(t)$ and $H_2(t)$, respectively, representing the seasonal fluctuations in river stage (Fig. 1b). With location of the origin, $x = 0$, at the left boundary, and the datum at the bottom of the aquifer, (2) can be rewritten as

$$\frac{\partial H}{\partial t} - \frac{\partial}{\partial x}\left[KH\frac{\partial H}{\partial x}\right] = \frac{I}{S} \tag{3}$$

with boundary and initial conditions

$$H(0,t) = H_1(t), \qquad H(l_x,t) = H_2(t), \qquad H(x,0) = H_0(x), \tag{4}$$

where I is the mean daily recharge from rainfall; and $H_0(x)$ is the initial head across the aquifer (Fig. 1b, dashed line). We solve (3), (4) by using one of the Adomian's decomposition schemes [2–4]. Let us write (3) as

$$\frac{\partial H}{\partial t} = \frac{I}{S} + \frac{KH}{S}\frac{\partial^2 H}{\partial x^2} + \frac{K}{S}\left(\frac{\partial H}{\partial x}\right)^2. \tag{5}$$

Defining the operator $L_t = \partial/\partial t$, and pre-multiplying (5) by the inverse operator L_t^{-1},

$$H = H_0 + \frac{It}{S} + L_t^{-1}N(H), \qquad N(H) = \frac{K}{S}\left[H\frac{\partial^2 H}{\partial x^2} + \left(\frac{\partial H}{\partial x}\right)^2\right]. \tag{6}$$

Now define the series $H = \sum_{n=0}^{\infty} H_n^*$, where the first term satisfies the recharge and the initial condition, that is

$$H_0^* = H_0(x) + \frac{It}{S}. \tag{7}$$

Subsequent terms in the series are given as

$$H_1^* = L_t^{-1}A_0, \qquad H_2^* = L_t^{-1}A_1, \qquad \ldots, \qquad H_{n+1}^* = L_t^{-1}A_n. \tag{8}$$

Taking into account (6), (7) the Adomian polynomials, A_n, in (8) for the nonlinear term, N, are defined as

$$A_0 = N(H_0^*), \qquad A_1 = H_1^*\frac{dN(H_0^*)}{dH_0^*}, \qquad A_2 = H_2^*\frac{dN(H_0^*)}{dH_0^*} + \frac{H_1^{*2}}{2!}\frac{d^2N(H_0^*)}{dH_0^{*2}},$$
$$A_3 = H_3^*\frac{dN(H_0^*)}{dH_0^*} + H_1^*H_2^*\frac{d^2N(H_0^*)}{dH_0^{*2}} + \frac{H_1^{*3}}{3!}\frac{d^3N(H_0^*)}{dH_0^{*3}}, \qquad \ldots. \tag{9}$$

According to (9) the polynomials A_n are generated for each non-linearity so that A_0 depends only on H_0^*, A_1 depends only on H_0^* and H_1^*, A_2 depends only on H_0^*, H_1^*, H_2^*, etc. All of the H_n^* components are calculable. It is now established that the series $\sum_{n=0}^{\infty} A_n$ for $N(H)$ is equal to a generalized Taylor series for $N(H_0)$, that $\sum_{n=0}^{\infty} H_n^*$ is a generalized Taylor series about the function H_0^*, and that the series terms approach zero as $1/(mn)!$, if m is the order of the highest linear differential operator. Since the series converges and does so very rapidly, the n-term partial sum $\Phi_n = \sum_{i=0}^{n-1} H_i^*$ usually serves as an accurate enough and practical solution. Thus, the second term in the series is

$$H_1 = \frac{K}{S}L_t^{-1}\left[H_0^*\frac{\partial^2 H_0^*}{\partial x^2} + \left(\frac{\partial H_0^*}{\partial x}\right)^2\right] = \frac{K}{S}L_t^{-1}\left[\left(H_0(x) + \frac{It}{S}\right)\frac{\partial^2 H_0}{\partial x^2} + \left(\frac{\partial H_0(x)}{\partial x}\right)^2\right]. \tag{10}$$

Higher terms in the series are derived similarly to (10). The third term would require information on the third-order spatial derivative of the initial condition. Clearly $H_0(x)$ must be sufficiently smooth for the calculation of its first and second-order spatial derivative. In practical applications a smooth surface should be fitted through the heads measured at individual wells. The spatial derivatives are then calculated analytically or numerically. In most practical modeling applications [21, 22], the first few terms in the decomposition series constitute an accurate solution. However it has been shown that decomposition series converge to the exact closed-form solution. For example, it has been shown in [18] that decomposition series converge to Sokolov's [23] solution. For more comparisons see [16, 19, 20]. Applications to the hydrodynamic model in porous media, that is the Laplace's equation subject to a dynamic nonlinear free surface are shown in [18]. Applications to the advective-dispersive equation subject to nonlinear reactions are shown in [17].

Several works have been devoted to the convergence problem of decomposition series. See for example the rigorous mathematical framework developed in [1, 7–11].

3 Superpropagation

A fascinating "superelevation" property of (2) has been elucidated in [15]. He considered a BVP with $H(0,t) = H_0 + a \sin \omega t$ and $H_\infty = H_0$ and found that a cyclostationary excited reservoir water level results in H whose time average value is higher than H_0. In other words, the aquifer is prone to accept water rather than to drain. This effect has been later validated experimentally in coastal aquifers subject to tidal variations of the sea level [24]. Clearly, the linear diffusion equation does not exhibit any "superelevation".

We inspected other nonlinear sharp-front equations to find out whether the "superelevation" property is generic. We considered the Charny [6] equation

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[qH + H(H-1)\frac{\partial H}{\partial x}\right] = 0 \tag{11}$$

which describes a sharp interface $H(x,t)$ between two phases in a formation of a constant thickness $b = 1$ with a transient drive of phase 1 displacing phase 2. The nonlinear diffusion equation (11) is based on the same vertical averaging as the Boussinesq equation and invokes additionally the assumption $\mu_1 = \mu_2$. However, $\rho_1 > \rho_2$ that is common in coastal aquifers where denser sea water encroaches into the fresh water zone.

We found [12] a straight line solution, which holds for an arbitrary dimensionless flow rate $q(t)$. In particular, we detected the phenomenon of "counterslumping" when the trailing front B in Fig. 2a slumps for a while against the main flow. For a cyclostationary excitation (11) the interface propagates throughout one period deeper than at a regime with $q = \text{const}$. We called this enhanced sweeping "superpropagation" in congruity with Philip's "superelevation". Note, that we solved PDE (11) in a closed form.

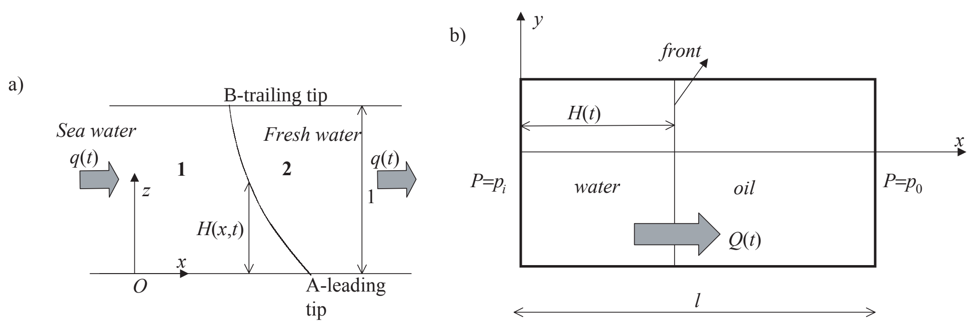

Figure 2.

We tested for the same non-symmetrical effect the Green–Ampt model describing a one-phase transient vertical infiltration. The governing equation is

$$m\frac{dH}{dt} = K\left(1 + \frac{d + h_c}{H}\right), \tag{12}$$

where $H(t)$ is the depth of the infiltration front counted from the soil surface, $d(t)$ is a given ponding water level at the surface and h_c is the constant invoking capillarity of the dry soil. By computer algebra routines we solved equation (12) with a cyclostationary d and arrived at the same conclusion that the wetting front propagates (on the time average) deeper than the front with $d = \text{const}$.

Eventually, we examined a two-phase displacement in x-direction when two phases contrast in viscosity only. The corresponding 1-D ODE according to [6] is

$$m\frac{dH}{dt} = k\frac{p_i - p_0}{\mu_1 H + \mu_2(l - H)}, \tag{13}$$

where $H(t)$ is the front location, l is the finite domain size, $p_i(t)$ and $p_o(t)$ are the injection and abstraction pressures (Fig. 2b). In (13) the densities of the two phases are identical that excludes z. Integrating (13) for a cyclostationary pressure drop $p_i - p_o$ across the layer we discovered the same effect of "superpropagation" as in all previous examples.

Acknowledgements

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