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On the Wave Functions of Ruijsenaars Model Related to q-Analogue of Symmetric Space $\mathrm{GL}(n)/\mathrm{SO}(n)$

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In [1], the wave functions of quantum trigonometric n-particle Ruijsenaars model are defined as matrix elements of operators of representations of Cartan subalgebra between vectors invariant with respect to q-deformation $U'_q(\mathrm{so}_n)$ of Lie algebra $\mathrm{so}(n)$. It was shown there, that the wave functions defined in such a way are simultaneous eigenfunctions of commuting set of Macdonald–Ruijsenaars difference operators. Using this information, the expressions for wave functions in terms of Macdonald polynomials were found. In this contribution, these expressions are obtained in a direct manner by using explicit expressions for invariant vectors in representation spaces in terms of Gel'fand–Tsetlin basis.

1 Introduction

In [2], M. Noumi introduced the notion of the quantum analogue $\mathcal{F}_q(\mathrm{GL}(n)/\mathrm{SO}(n))$ of algebra of functions on the homogeneous space $\mathrm{GL}(n)/\mathrm{SO}(n)$ and zonal spherical functions on it. It turned out that these spherical functions when restricted to Cartan subalgebra are Macdonald polynomials, which are simultaneous eigenfunctions of commuting set of Macdonald–Ruijsenaars difference operators. These difference operators appear as the radial components of Casimir elements of $U_q(\mathrm{gl}_n)$. In [2], the radial components of the "quadratic" Casimir elements were derived. In [1] the radial components of the other basis Casimir elements of $U_q(\mathrm{gl}_n)$ were found. From the other side, Macdonald–Ruijsenaars difference operators (up to a change of variables) coincide with commuting Hamiltonians of quantum trigonometric n-particle Ruijsenaars model. In [1], the wave functions of Ruijsenaars model are defined as matrix elements $\Psi(x_1, x_2, \ldots, x_n) = \langle v|q^{\sum_{k=1}^n x_k \epsilon_k}|v\rangle$, where q^{ϵ_k} belongs to Cartan subalgebra, $\langle v|$ and $|v\rangle$ are invariant vectors with respect to q-deformation $U'_q(\mathrm{so}_n)$ of Lie algebra $\mathrm{so}(n)$ introduced in [3]. It was shown that the wave functions defined in such a way are simultaneous eigenfunctions of commuting set of Macdonald–Ruijsenaars difference operators. Using this information, the expressions for wave functions in terms of Macdonald polynomials were found. In this contribution, these expressions are obtained in a direct manner by using explicit expressions for invariant vectors in representation spaces in terms of Gel'fand–Tsetlin basis. In order to express the obtained wave functions in terms of Macdonald polynomials we need the combinatorial formula [4] for such polynomials.

Quantum trigonometric Ruijsenaars model which we discuss in this paper is a q-analogue of quantum Sutherland model connected with symmetric space $\mathrm{GL}(n)/\mathrm{SO}(n)$. Thus this paper can be considered in the spirit of [5], where different integrable systems are connected with analysis on symmetric spaces.

In [6], the authors used representation theory of $\mathrm{GL}(n)$ to obtain the wave functions for open Toda model and Sutherland model. Their Gel'fand–Tsetlin formulas are different from that which are used in this contribution. An important consequence of this difference is that the wave functions obtained in [6] are wave functions in some separated variables.

2 The quantum algebras $U_q(\mathrm{gl}_n)$ and their finite-dimensional representations

By the definition [7], Drinfeld–Jimbo quantum algebra $U_q(\mathrm{gl}_n)$ is a unital associative algebra with the generators e_i, f_i, $i = 1, 2, \ldots, n-1$, and generators q^h, where $h = x_1\epsilon_1 + x_2\epsilon_2 + \cdots + x_n\epsilon_n$, $x_i \in \mathbb{R}$, are elements of a vector space with the basis ϵ_i, $i = 1, 2, \ldots, n$. Defining relations containing complex deformation parameter $q \neq 0, \pm 1$ are:

$$
\begin{aligned}
&q^0 = 1, \qquad q^{h_1}q^{h_2} = q^{h_1+h_2}, \qquad q^h e_i = q^{x_i - x_{i+1}} e_i q^h, \qquad q^h f_i = q^{x_{i+1}-x_i} f_i q^h,\\
&[e_i, f_j] = \delta_{ij}\frac{q^{\epsilon_i-\epsilon_{i+1}} - q^{\epsilon_{i+1}-\epsilon_i}}{q - q^{-1}},\\
&[e_i, e_j] = 0, \quad |i-j| > 1, \qquad e_i^2 e_j - (q+q^{-1})e_ie_je_i + e_je_i^2 = 0, \quad |i-j| = 1,\\
&[f_i, f_j] = 0, \quad |i-j| > 1, \qquad f_i^2 f_j - (q+q^{-1})f_if_jf_i + f_jf_i^2 = 0, \quad |i-j| = 1.
\end{aligned}
$$

In the limit $q \to 1$, algebra $U_q(\mathrm{gl}_n)$ becomes the universal enveloping algebra for Lie algebra gl_n.

For every set of n integers $\boldsymbol{m}_n = (m_{1,n}, m_{2,n}, \ldots, m_{n,n})$ such that

$$
m_{1,n} \geq m_{2,n} \geq \cdots \geq m_{n,n},
$$

there corresponds a simple finite-dimensional left module $\mathcal{V}^{\mathrm{L}}_{\boldsymbol{m}_n}$ over the algebra $U_q(\mathrm{gl}_n)$. Its explicit construction will be given in Gel'fand–Tsetlin formalism. Basis elements of this module are labeled by the sets of integers $\boldsymbol{m}_j = (m_{1,j}, m_{2,j}, \ldots, m_{j,j})$, $j = 1, \ldots, n-1$, such that

$$
m_{i,j+1} \geq m_{i,j} \geq m_{i+1,j+1}, \qquad i = 1, 2, \ldots, j, \quad j = 1, 2, \ldots, n-1.
$$

It is useful to visualize them by the Gel'fand–Tsetlin tableaux

$$
M = \begin{pmatrix}
m_{1,n} & & m_{2,n} & & \cdots & & m_{n,n}\\
& m_{1,n-1} & & m_{2,n-1} & \cdots & m_{n-1,n-1} & \\
& & \cdots & \cdots & \cdots & & \\
& & & m_{1,1} & & &
\end{pmatrix}. \tag{1}
$$

To the tableau (1), there corresponds basis element denoted by $|M\rangle$.

The generators of $U_q(\mathrm{gl}_n)$ act on the Gel'fand–Tsetlin basis by the formulas [7]

$$
q^{\epsilon_j}|M\rangle = q^{a_j}|M\rangle, \qquad a_j = \sum_{i=1}^{j} m_{i,j} - \sum_{i=1}^{j-1} m_{i,j-1}, \qquad 1 \leq j \leq n, \tag{2}
$$

$$
e_j|M\rangle = \sum_{i=1}^{j} A_j^i(M)|M_j^{+i}\rangle, \qquad f_j|M\rangle = \sum_{i=1}^{j} B_j^i(M)|M_j^{-i}\rangle, \qquad 1 \leq j \leq n-1. \tag{3}
$$

Here $M_j^{\pm i}$ is the Gel'fand–Tsetlin tableau obtained from the tableau (1) by replacement of $m_{i,j}$ by $m_{i,j} \pm 1$, and

$$
A_j^i(M) = -\frac{\prod\limits_{s=1}^{j+1}[l_{s,j+1} - l_{i,j}]}{\prod\limits_{s\neq i}[l_{s,j} - l_{i,j}]}, \qquad B_j^i(M) = \frac{\prod\limits_{s=1}^{j-1}[l_{s,j-1} - l_{i,j}]}{\prod\limits_{s\neq i}[l_{s,j} - l_{i,j}]},
$$

where

$$
l_{s,j} := m_{s,j} - s, \tag{4}
$$

and the numbers in square brackets denote q-numbers defined by

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

In complete analogy, it is possible to construct a simple finite-dimensional right module $\mathcal{V}^{\mathrm{R}}_{\boldsymbol{m}_n}$ over the algebra $U_q(\mathrm{gl}_n)$ which is dual to the described above. The basis elements of this module are also parameterized by the same set of Gel'fand–Tsetlin tableaux. The basis element denoted by $\langle M|$ which is dual to $|M\rangle$ corresponds to the tableau (1).

The generators of $U_q(\mathrm{gl}_n)$ act on these basis elements by the formulas

$$\langle M|q^{\epsilon_j} = q^{a_j}\langle M|, \qquad a_j = \sum_{i=1}^{j} m_{i,j} - \sum_{i=1}^{j-1} m_{i,j-1}, \qquad 1 \le j \le n, \tag{5}$$

$$\langle M|e_j = \sum_{i=1}^{j} A^i_j(M^{-i}_j)\langle M^{-i}_j|, \qquad \langle M|f_j = \sum_{i=1}^{j} B^i_j(M^{+i}_j)\langle M^{+i}_j|, \qquad 1 \le j \le n-1. \tag{6}$$

3 Some invariant elements in $U_q(\mathrm{gl}_n)$-modules

In this section, we give explicit formulas for elements in left and right modules over $U_q(\mathrm{gl}_n)$ which are annihilating by

$$\theta_k = q^{\epsilon_k} f_k - q q^{\epsilon_{k+1}} e_k, \qquad k = 1, 2, \ldots, n-1.$$

In the limit $q \to 1$, the elements θ_k become generators of the enveloping algebra for the Lie algebra so_n embedded into gl_n. In fact, the elements $q^{-\epsilon_k}\theta_k$ generate [2] the non-standard deformation $U'_q(\mathrm{so}_n)$ of the enveloping algebra $U(\mathrm{so}_n)$ introduced by Gavrilik and Klimyk [3].

The following theorem is due to Noumi [2].

Theorem 1. *The element $|v\rangle \in \mathcal{V}^{\mathrm{L}}_{\boldsymbol{m}_n}$ (resp. $\langle v| \in \mathcal{V}^{\mathrm{R}}_{\boldsymbol{m}_n}$) such that $\theta_k|v\rangle = 0$, (resp. $\langle v|\theta_k = 0$), $k = 1, \ldots, n-1$, exists if and only if $(m_{i,n} - m_{i+1,n})$ are even for $i = 1, \ldots, n-1$. If such element $|v\rangle$ (resp. $\langle v|$) exists, it is unique up to a multiplier.*

The elements $|v\rangle$ and $\langle v|$ which are annihilating by θ_k are called invariant elements with respect to action of θ_k. Now, for the case when conditions of theorem for $\boldsymbol{m}_n$ are satisfied, we present explicit formulas for such elements $|v\rangle$ and $\langle v|$. Let $\mathcal{S}$ be the set of all Gel'fand–Tsetlin tableaux corresponding to $\boldsymbol{m}_n$ and satisfying the additional conditions:

$$(m_{i,j+1} - m_{i,j}) \quad \text{is even for all} \quad i = 1, 2, \ldots, j, \quad j = 1, 2, \ldots, n-1.$$

Then

$$|v\rangle = \sum_{M \in \mathcal{S}} \alpha(M)|M\rangle, \qquad \langle v| = \sum_{M \in \mathcal{S}} \beta(M)\langle M|, \tag{7}$$

where

$$\alpha(M) = \prod_{k=2}^{n} \alpha_k, \qquad \beta(M) = \prod_{k=2}^{n} \beta_k, \tag{8}$$

$$\alpha_k = q^{\gamma_k} \prod_{1 \le i < j \le k} \frac{[l_{i,k-1} - l_{j-1,k-1}]!![l_{i,k-1} - l_{j,k} - 2]!!}{[l_{i,k} - l_{j,k} - 2]!![l_{i,k} - l_{j-1,k-1}]!!}, \tag{9}$$

$$\beta_k = q^{-\gamma_k} \prod_{1 \le i < j \le k} \frac{[l_{i,k} - l_{j,k} - 1]!![l_{i,k} - l_{j-1,k-1} - 1]!!}{[l_{i,k-1} - l_{j-1,k-1} - 1]!![l_{i,k-1} - l_{j,k} - 1]!!}, \tag{10}$$

$$\gamma_k = \frac{1}{2} \sum_{1 \le i < j \le k} (l_{i,k} - l_{i,k-1})(l_{j-1,k-1} - l_{j,k} - 2). \tag{11}$$

We used the notations (4) and

$$[s]!! := [s][s-2] \cdots [2] \text{ (or } [1]), \qquad [0]!! = [-1]!! = 1.$$

These expressions for elements $|v\rangle$ and $\langle v|$ were obtained by straightforward calculation using action formulas (2)–(3) and (5)–(6).

4 Macdonald polynomials

Macdonald polynomials constitute a linear basis in the space of symmetric polynomials of $X = \{X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_n^{\pm 1}\}$ with coefficients being rational functions in two formal variables q and t. Each Macdonald polynomial $P_{\boldsymbol{\lambda}}(X; q, t)$ is labeled by the set of n integers $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ such that

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n.$$

We introduce a partial ordering among such sets. For two such sets $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, we put

$$\boldsymbol{\lambda} \succeq \boldsymbol{\mu} \Leftrightarrow \begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_n = \mu_1 + \mu_2 + \cdots + \mu_n & \text{and} \\ \lambda_1 + \lambda_2 + \cdots + \lambda_r \ge \mu_1 + \mu_2 + \cdots + \mu_r, & r = 1, 2, \dots, n-1. \end{cases}$$

The set $\boldsymbol{\lambda}$ defines a monomial $X^{\boldsymbol{\lambda}} = X_1^{\lambda_1} \cdots X_n^{\lambda_n}$. The monomial symmetric function $m_{\boldsymbol{\lambda}}(X)$ is the sum of all *distinct* monomials obtainable from $X^{\boldsymbol{\lambda}}$ by permutations of X's. In particular, if $\boldsymbol{\lambda}$ such that $\lambda_i = 1$ for $i \le r$ and $\lambda_j = 0$ for $j > r$, we have $m_{\boldsymbol{\lambda}}(X) = e_r(X)$, the r-th elementary symmetric polynomial.

To present the definition of Macdonald polynomials we need a commuting family of q-difference operators

$$M_r = t^{-r(n-r)/2} \sum_{\substack{I \subset \{1,2,\dots,n\} \\ |I| = r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tX_i - X_j}{X_i - X_j} \prod_{i \in I} \tau_i, \qquad r = 1, \dots, n,$$

where τ_i represents the q-shift operator with respect to the variable X_i. Namely, τ_i is the automorphism of algebra of polynomials of $X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_n^{\pm 1}$ uniquely defined by $\tau_i(X_j) = q^{\delta_{ij}} X_j$. Macdonald proved [4] the following

Theorem 2. *There exists a unique linear basis $\{P_{\boldsymbol{\lambda}}(X)\}$ in the space of symmetric polynomials of $X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_n^{\pm 1}$ satisfying the following two conditions:*

- *For each $\boldsymbol{\lambda}$, $P_{\boldsymbol{\lambda}}$ can be presented as*

  $$P_{\boldsymbol{\lambda}}(X) = m_{\boldsymbol{\lambda}}(X) + \sum_{\boldsymbol{\mu} \prec \boldsymbol{\lambda}} u_{\boldsymbol{\lambda}\boldsymbol{\mu}} m_{\boldsymbol{\mu}}(X),$$

  where $u_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ are some rational functions of q and t.
- *For each $\boldsymbol{\lambda}$, $P_{\boldsymbol{\lambda}}(X)$ is a joint eigenfunction of M_r, $r = 1, 2, \dots, n$:*

  $$M_r P_{\boldsymbol{\lambda}}(X) = e_r(q^{\lambda_1} t^{\rho_1}, \dots, q^{\lambda_n} t^{\rho_n}) P_{\boldsymbol{\lambda}}(X),$$

  where $\rho_i = (n - 2i + 1)/2$.

We give now some explicit formulas for Macdonald polynomials. For the set $\boldsymbol{\lambda}$ labeling $P_{\boldsymbol{\lambda}}$, it corresponds Gel'fand–Tsetlin tableaux (like (1))

$$\Lambda = \left(\begin{array}{ccccccc} \lambda_{1,n} & & \lambda_{2,n} & & \cdots & & \lambda_{n,n} \\ & \lambda_{1,n-1} & & \lambda_{2,n-1} & \cdots & \lambda_{n-1,n-1} & \\ & & \cdots & \cdots & \cdots & & \\ & & & \lambda_{1,1} & & & \end{array} \right),$$

where $\lambda_{i,n} = \lambda_i$, $i = 1, \dots, n$, and $\lambda_{i,j}$, $i = 1, \dots, j$, $j = 1, \dots, n-1$, are integers satisfying

$$\lambda_{i,j+1} \geq \lambda_{i,j} \geq \lambda_{i+1,j+1}, \qquad i = 1, 2, \dots, j, \quad j = 1, 2, \dots, n-1.$$

We suppose that Λ runs over all the possible Gel'fand–Tsetlin tableaux corresponding to the fixed $\boldsymbol{\lambda}$. Next theorem is a reformulation in terms of Gel'fand–Tsetlin tableaux of combinatorial formula [4] for Macdonald polynomials. We need the notation:

$$(a; q)_s := (1-a)(1-aq)\cdots(1-aq^{s-1}).$$

Theorem 3. *The expression for Macdonald polynomial $P_{\boldsymbol{\lambda}}$ has form*

$$P_{\boldsymbol{\lambda}}(X_1, X_2, \dots, X_n; q, t) = \sum_{\Lambda} \varphi(\Lambda; q, t) X^{\Lambda}, \tag{12}$$

where

$$X^{\Lambda} = \prod_{k=1}^{n} X_k^{a_k}, \qquad a_k = \sum_{i=1}^{k} \lambda_{i,k} - \sum_{i=1}^{k-1} \lambda_{i,k-1},$$

$$\varphi(\Lambda; q, t) = \prod_{k=2}^{n} \varphi_k(\boldsymbol{\lambda}_{k-1}, \boldsymbol{\lambda}_k; q, t), \qquad \boldsymbol{\lambda}_k = (\lambda_{1,k}, \lambda_{2,k}, \dots, \lambda_{k,k}),$$

$$\begin{aligned} \varphi_k(\boldsymbol{\lambda}_{k-1}, \boldsymbol{\lambda}_k; q, t) = & \prod_{1 \leq i < j \leq k} \frac{(t^{j-i}; q)_{\lambda_{i,k} - \lambda_{j-1,k-1}}}{(t^{j-i-1}q; q)_{\lambda_{i,k} - \lambda_{j-1,k-1}}} \cdot \frac{(t^{j-i-1}q; q)_{\lambda_{i,k} - \lambda_{j,k}}}{(t^{j-i}; q)_{\lambda_{i,k} - \lambda_{j,k}}} \\ & \times \frac{(t^{j-i}; q)_{\lambda_{i,k-1} - \lambda_{j,k}}}{(t^{j-i-1}q; q)_{\lambda_{i,k-1} - \lambda_{j,k}}} \cdot \frac{(t^{j-i-1}q; q)_{\lambda_{i,k-1} - \lambda_{j-1,k-1}}}{(t^{j-i}; q)_{\lambda_{i,k-1} - \lambda_{j-1,k-1}}} \\ = & \prod_{1 \leq i < j \leq k} \frac{(qt^{j-i-1} q^{\lambda_{i,k} - \lambda_{j-1,k-1}}; q)_{\lambda_{j-1,k-1} - \lambda_{j,k}}}{(t^{j-i} q^{\lambda_{i,k} - \lambda_{j-1,k-1}}; q)_{\lambda_{j-1,k-1} - \lambda_{j,k}}} \\ & \times \frac{(t^{j-i} q^{\lambda_{i,k-1} - \lambda_{j-1,k-1}}; q)_{\lambda_{j-1,k-1} - \lambda_{j,k}}}{(qt^{j-i-1} q^{\lambda_{i,k-1} - \lambda_{j-1,k-1}}; q)_{\lambda_{j-1,k-1} - \lambda_{j,k}}}. \end{aligned} \tag{13}$$

Proof. From [4], it follows that

$$\varphi(\Lambda; q, t) = \prod_{k=2}^{n} \frac{b_{\boldsymbol{\lambda}_k}(C_k)}{b_{\boldsymbol{\lambda}_k}} \frac{b_{\boldsymbol{\lambda}_{k-1}}}{b_{\boldsymbol{\lambda}_{k-1}}(C_k)} = \frac{1}{b_{\boldsymbol{\lambda}_n}} \prod_{k=1}^{n} \frac{b_{\boldsymbol{\lambda}_k}(C_k)}{b_{\boldsymbol{\lambda}_{k-1}}(C_k)}.$$

Combinatorial definition of $b_{\boldsymbol{\lambda}_k}$ and $b_{\boldsymbol{\lambda}_k}(C_j)$ is given in [4]. This combinatorics rewritten in terms of Gel'fand–Tsetlin tableaux gives

$$\frac{b_{\boldsymbol{\lambda}_k}(C_k)}{b_{\boldsymbol{\lambda}_k}} = \prod_{1 \leq i < j \leq k} \frac{(t^{j-i}; q)_{\lambda_{i,k} - \lambda_{j-1,k-1}}}{(t^{j-i-1}q; q)_{\lambda_{i,k} - \lambda_{j-1,k-1}}} \cdot \frac{(t^{j-i-1}q; q)_{\lambda_{i,k} - \lambda_{j,k}}}{(t^{j-i}; q)_{\lambda_{i,k} - \lambda_{j,k}}},$$

$$\frac{b_{\boldsymbol{\lambda}_{k-1}}}{b_{\boldsymbol{\lambda}_{k-1}}(C_k)} = \prod_{1 \leq i < j \leq k} \frac{(t^{j-i}; q)_{\lambda_{i,k-1} - \lambda_{j,k}}}{(t^{j-i-1}q; q)_{\lambda_{i,k-1} - \lambda_{j,k}}} \cdot \frac{(t^{j-i-1}q; q)_{\lambda_{i,k-1} - \lambda_{j-1,k-1}}}{(t^{j-i}; q)_{\lambda_{i,k-1} - \lambda_{j-1,k-1}}}. \qquad \blacksquare$$

5 Wave functions of Ruijsenaars model

In [1], the wave function for Schrödinger equation for quantum n-particle Ruijsenaars model was defined as

$$\Psi_{\boldsymbol{m}_n}(x_1, x_2, \ldots, x_n) := \langle v|q^h|v\rangle = \langle v|q^{\sum_{k=1}^n x_k\epsilon_k}|v\rangle,$$

where the invariant elements $\langle v|$ and $|v\rangle$ are defined in Section 3 (conditions when they exist are also given there). Here we give a direct proof that $\Psi_{\boldsymbol{m}_n}(x_1, x_2, \ldots, x_n)$ can be expressed in terms of Macdonald polynomials.

From (2) and (7)–(8), it follows

$$\Psi_{\boldsymbol{m}_n}(x_1, x_2, \ldots, x_n) = \langle v|q^{\sum_{k=1}^n x_k\epsilon_k}|v\rangle = \sum_{M\in\mathcal{S}}\prod_{k=2}^n \alpha_k\beta_k \cdot \prod_{k=1}^n q^{x_k(\sum_{i=1}^k m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1})},$$

where α_k and β_k are given by (9)–(11). We make change of variables $\lambda_{i,j} = (m_{i,j} - m_{n,n})/2$ for all possible i and j. Since $M \in \mathcal{S}$, all the $\lambda_{i,j}$ are non-negative integers. For $\alpha_k\beta_k$ we have

$$\begin{aligned}
\alpha_k\beta_k &= \prod_{1\le i<j\le k} \frac{[l_{i,k} - l_{j,k} - 1]!![l_{i,k} - l_{j-1,k-1} - 1]!![l_{i,k-1} - l_{j,k} - 2]!![l_{i,k-1} - l_{j-1,k-1}]!!}{[l_{i,k} - l_{j-1,k-1}]!![l_{i,k} - l_{j,k} - 2]!![l_{i,k-1} - l_{j-1,k-1} - 1]!![l_{i,k-1} - l_{j,k} - 1]!!} \\
&= \prod_{1\le i<j\le k} \frac{[2\lambda_{i,k} - 2\lambda_{j,k} - i + j - 1]!!}{[2\lambda_{i,k} - 2\lambda_{j-1,k-1} - i + j - 1]!!} \cdot \frac{[2\lambda_{i,k} - 2\lambda_{j-1,k-1} - i + j - 2]!!}{[2\lambda_{i,k} - 2\lambda_{j,k} - i + j - 2]!!} \\
&\times \frac{[2\lambda_{i,k-1} - 2\lambda_{j,k} - i + j - 2]!!}{[2\lambda_{i,k-1} - 2\lambda_{j-1,k-1} - i + j - 2]!!} \cdot \frac{[2\lambda_{i,k-1} - 2\lambda_{j-1,k-1} - i + j - 1]!!}{[2\lambda_{i,k-1} - 2\lambda_{j,k} - i + j - 1]!!} \\
&= \prod_{1\le i<j\le k} \frac{[2\lambda_{i,k} - 2\lambda_{j,k} - i + j - 1][2\lambda_{i,k} - 2\lambda_{j,k} - i + j - 3]\cdots[2\lambda_{i,k} - 2\lambda_{j-1,k-1} - i + j + 1]}{[2\lambda_{i,k} - 2\lambda_{j,k} - i + j - 2][2\lambda_{i,k} - 2\lambda_{j,k} - i + j - 4]\cdots[2\lambda_{i,k} - 2\lambda_{j-1,k-1} - i + j]} \\
&\times \frac{[2\lambda_{i,k-1} - 2\lambda_{j,k} - i + j - 2][2\lambda_{i,k-1} - 2\lambda_{j,k} - i + j - 4]\cdots[2\lambda_{i,k-1} - 2\lambda_{j-1,k-1} - i + j]}{[2\lambda_{i,k-1} - 2\lambda_{j,k} - i + j - 1][2\lambda_{i,k-1} - 2\lambda_{j,k} - i + j - 3]\cdots[2\lambda_{i,k-1} - 2\lambda_{j-1,k-1} - i + j + 1]} \\
&= \prod_{1\le i<j\le k} \frac{(q^4 q^{2(j-i-1)} q^{4(\lambda_{i,k} - \lambda_{j-1,k-1})}; q^4)_{\lambda_{j-1,k-1} - \lambda_{j,k}}}{(q^{2(j-i)} q^{4(\lambda_{i,k} - \lambda_{j-1,k-1})}; q^4)_{\lambda_{j-1,k-1} - \lambda_{j,k}}} \\
&\times \frac{(q^{2(j-i)} q^{4(\lambda_{i,k-1} - \lambda_{j-1,k-1})}; q^4)_{\lambda_{j-1,k-1} - \lambda_{j,k}}}{(q^4 q^{2(j-i-1)} q^{4(\lambda_{i,k-1} - \lambda_{j-1,k-1})}; q^4)_{\lambda_{j-1,k-1} - \lambda_{j,k}}} = \varphi_k(\boldsymbol{\lambda}_{k-1}, \boldsymbol{\lambda}_k; q^4, q^2),
\end{aligned}$$

where the definition (13) for $\varphi_k(\boldsymbol{\lambda}_{k-1}, \boldsymbol{\lambda}_k; q, t)$ is used. Thus

$$\begin{aligned}
\Psi_{\boldsymbol{m}_n}(x_1, x_2, \ldots, x_n) &= \sum_\Lambda \varphi(\Lambda; q, t) \prod_{k=1}^n q^{x_k(m_{n,n} + 2(\sum_{i=1}^k \lambda_{i,k} - \sum_{i=1}^{k-1}\lambda_{i,k-1}))} \\
&= q^{(x_1+\cdots+x_n)m_{n,n}} \sum_\Lambda \varphi(\Lambda; q, t) \prod_{k=1}^n q^{2x_k(\sum_{i=1}^k \lambda_{i,k} - \sum_{i=1}^{k-1}\lambda_{i,k-1})}.
\end{aligned}$$

Comparing with (12) we obtain

$$\Psi_{\boldsymbol{m}_n}(x_1, x_2, \ldots, x_n) = q^{(x_1+\cdots+x_n)m_{n,n}} P_{\boldsymbol{\lambda}}(q^{2x_1}, q^{2x_2}, \ldots, q^{2x_n}; q^4, q^2).$$

This wave function satisfies equations

$$\hat{H}_r \Psi_{\boldsymbol{m}_n}(x_1, x_2, \ldots, x_n) = E_\Lambda^{(r)} \Psi_{\boldsymbol{m}_n}(x_1, x_2, \ldots, x_n), \qquad r = 1, 2, \ldots, n, \tag{14}$$

where $\hat{H}_r$ is r-th Hamiltonian of Ruijsenaars model:

$$\hat{H}_r = q^{r(r-1)} \sum_{\substack{I\subset\{1,2,\ldots,n\}\\ |I|=r}} \left(\prod_{\substack{i\in I\\ j\notin I}} \frac{q^2 q^{2x_i} - q^{2x_j}}{q^{2x_i} - q^{2x_j}} \right) e^{2\sum_{i\in I}\partial_{x_i}},$$

$$E^{(r)}_{\boldsymbol{m}_n} = e_r(q^{2m_{1,n}+2n-2}, q^{2m_{2,n}+2n-4}, \ldots, q^{m_{n,n}}).$$

This statement is easily follows from Theorem 2. Note, the equation (14) at $r = 1$ is precisely Schrödinger equation for quantum trigonometric n-particle Ruijsenaars model.

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[1] Iorgov N., On the quantum Ruijsenaars model, some quantum homogeneous spaces and Hecke algebras, *Int. J. Mod. Phys.*, 2004, to appear.

[2] Noumi M., Macdonald's symmetric polynomials as zonal spherical functions on quantum homogeneous spaces, *Adv. Math.*, 1996, V.123, 16–77.

[3] Gavrilik A.M. and Klimyk A.U., q-Deformed orthogonal and pseudo-orthogonal algebras and their representations, *Lett. Math. Phys.*, 1991, V.21, 215–220.

[4] Macdonald I.G., A new class of symmetric functions, *Actes* 20^e *Séminaire Lotharingien*, Publ. I.R.M.A. Strasbourg, 1988, 372/S-20, 131–171.

[5] Olshanetsky M. and Perelomov A., Quantum integrable systems related to Lie algebras, *Phys. Rep.*, 1983, V.94, 313–304.

[6] Gerasimov A., Kharchev S. and Lebedev D., Representation theory and the quantum inverse scattering method: the open Toda chain and the hyperbolic Sutherland model, math.QA/0204206.

[7] Klimyk A. and Schmüdgen K., Quantum groups and their representations, Berlin, Springer, 1997.