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Dynamical Space-Time Symmetry for Ageing Far from Equilibrium

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The dynamical scaling of ageing ferromagnetic systems can be generalized to a local scale invariance. This yields a prediction for the causal two-time response function, which has been numerically confirmed in the Glauber–Ising model quenched into the ordered phase. For a dynamical exponent $z = 2$, a new embedding of the Schrödinger group into the conformal group and the resulting conditions for the validity of local scale invariance are discussed.

1 Phenomenology of ageing in simple ferromagnets

Ageing phenomena provide a paradigmatic example of collective behaviour far from equilibrium and have received a lot of attention in recent years [5, 8, 11]. Ageing has been observed first in glassy systems, but for an improved conceptual understanding, it might be more useful to study first ageing phenomena in the simpler ferromagnetic systems, as we shall do here. We consider a ferromagnet with a critical temperature $T_c > 0$ and prepare the system in some initial state (which typically may be disordered). Then the system is suddenly brought into contact (quenched) with a heat bath of temperature $T < T_c$ (or $T \le T_c$). With T fixed, the system's evolution towards its equilibrium state at temperature T is observed. For definiteness, consider an Ising-like system with a microscopic degree of freedom $\sigma_{\boldsymbol{i}} = \pm 1$ which can only take two possible values, and where $\boldsymbol{i}$ denotes the sites of a (hypercubic) lattice.

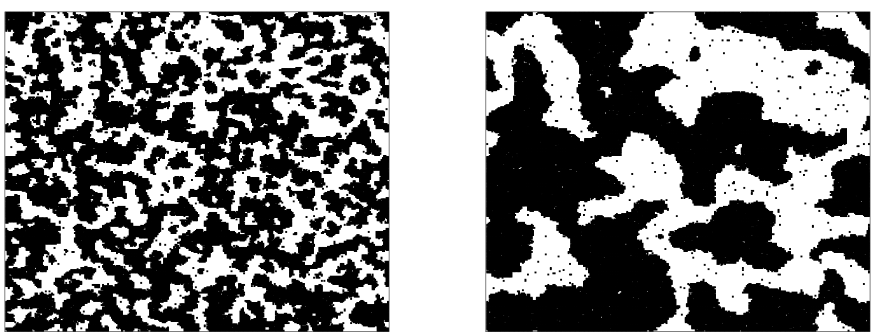

Figure 1. Snapshot of the coarsening of ordered domains in the $2D$ Glauber–Ising model, after a quench to $T = 1.5 < T_c$ from a totally disordered state and at times $t = 25$ (left) and $t = 275$ (right) after the quench. The black and white colors indicate the value of the Ising spins.

A typical example of the microscopic evolution of such a system is shown in Fig. 1. Quite rapidly, ordered domains are formed which slowly move and grow. Empirically, it is found that the typical size of the domains scales with the time t as $L(t) \sim t^{1/z}$, where z is the dynamical exponent. It is known that for dynamical rules chosen such that there is no macroscopic conservation law, $z = 2$ for quenches to $T < T_c$. The slow temporal evolution of macroscopic observables we are interested in results from the slow motion of the domain walls. It is common to consider the coarse-grained order parameter (e.g. mean magnetization for magnetic systems)

$\phi(t,\boldsymbol{r})$ and one tries to capture its time-evolution through a stochastic Langevin equation

$$\frac{\partial\phi}{\partial t} = -\frac{\delta\mathcal{H}[\phi]}{\delta\phi} + \eta, \tag{1}$$

where the Gaussian white noise $\eta = \eta(t,\boldsymbol{r})$ is characterized by its first two moments $\langle\eta(t,\boldsymbol{r})\rangle = 0$ and $\langle\eta(t,\boldsymbol{r})\eta(t',\boldsymbol{r}')\rangle = 2T\delta(t-t')\delta(\boldsymbol{r}-\boldsymbol{r}')$ and $\mathcal{H}$ is the Ginzburg–Landau functional. For systems undergoing a conventional second-order phase-transition, one expects that qualitatively

$$\mathcal{H}[\phi] = \phi\Delta\phi + \mathcal{V}[\phi], \qquad \mathcal{V}[\phi] \sim \begin{cases} \phi^2, & \text{if } T > T_{\rm c}, \\ \left(\phi^2 - \phi_0^2\right)^2, & \text{if } T < T_{\rm c}, \end{cases} \tag{2}$$

where $\phi_0 = \phi_0(T)$ are the two equilibrium values of ϕ and Δ is the spatial Laplacian. Physically, systems with $T > T_{\rm c}$ and $T < T_{\rm c}$ are very different, since in the first case, there is a single ground-state (where $\mathcal{H}[\phi] = \min!$) while there are two distinct ground states in the second case. Therefore, for $T > T_{\rm c}$ the system will rapidly relax towards its single ground-state and no ageing occurs. On the other hand, if $T < T_{\rm c}$ it will depend on the microscopic environment of each spin variable to which of the two possible local ground-states $\pm\phi_0$ the system will evolve locally. The competition between these distinct states then leads to ageing phenomena.

The noisy Langevin equation can be turned into an equivalent field theory through the Martin–Siggia–Rose formalism, see e.g. [8]. Schematically, it may be represented through the effective action

$$S[\phi,\widetilde{\phi}] = \int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\left[\widetilde{\phi}\left(\frac{\partial\phi}{\partial t} + \frac{\delta\mathcal{H}[\phi]}{\delta\phi}\right) - T\widetilde{\phi}^2\right], \tag{3}$$

where $\widetilde{\phi}$ is the so-called response field conjugate to ϕ. From this, the classical equations of motion take the form

$$\frac{\partial\phi}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\phi} + 2T\widetilde{\phi}, \qquad \frac{\partial\widetilde{\phi}}{\partial t} = \frac{\delta^2\mathcal{H}}{\delta\phi^2}\,\widetilde{\phi} \tag{4}$$

but initial conditions must still be specified. For interacting fields, fluctuation effects are not taken into account by the equations (4), although they are present in the action S and the associated path integral. It turns out, see [11] for a review, that ageing is more fully revealed through the study of two-time correlators $C(t,s)$ and response functions $R(t,s)$ defined by

$$C(t,s) = \langle\phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r})\rangle, \qquad R(t,s) = \left.\frac{\delta\langle\phi(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{r})}\right|_{h=0} = \langle\phi(t,\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{r})\rangle, \tag{5}$$

where $h(s,\boldsymbol{r})$ is the local magnetic field at time s and the position $\boldsymbol{r}$. The last equation comes from Martin–Siggia–Rose theory. Furthermore, $\langle\widetilde{\phi}\widetilde{\phi}\rangle = 0$.

Definition 1. A statistical system described by a Langevin equation (1) or an effective action (3) is said to undergo *ageing*, if $C = C(t,s)$ or $R = R(t,s)$ depend on both the *observation time* t and the *waiting time* s and not merely on the difference $\tau = t - s$.

If the times t, s and $t-s$ become large simultaneously (as compared to some microscopic time scale $t_{\rm micro}$), one usually finds, for $T < T_{\rm c}$, the following scaling behaviour, see e.g. [11]

$$\begin{aligned} C(t,s) &\simeq M_{\rm eq}^2 f_C(t/s), & f_C(x) &\sim x^{-\lambda_C/z}, & x &\to \infty, \\ R(t,s) &\simeq s^{-1-a} f_R(t/s), & f_R(x) &\sim x^{-\lambda_R/z}, & x &\to \infty, \end{aligned} \tag{6}$$

where $M_{\rm eq}$ is the equilibrium magnetization and $\lambda_{C,R}$ are the autocorrelation [9] and autoresponse [23] exponents, respectively. If long-range initial correlations of the form $C_{\rm ini}(\boldsymbol{r}) \sim$

$|\boldsymbol{r}|^{-d-\alpha}$ are used, where d is the number of space dimensions and α a free parameter, one has for *ferro*magnets the rigorous bound $\lambda_C \geq (d+\alpha)/2$ [27] (it need not hold for disordered systems [26]). Furthermore, if $\alpha < 0$, the relationship $\lambda_C = \lambda_R + \alpha$ has been conjectured [23], while $\lambda_C = \lambda_R$ for a fully disordered initial state is generally accepted. Finally, the value of the exponent a has recently been shown [16] by scaling arguments to depend on the equilibrium spin-spin correlator $C_{\rm eq}$ as follows. Systems of *class S* have short-ranged correlators $C_{\rm eq}(\boldsymbol{r}) \sim e^{-|\boldsymbol{r}|/\xi}$ and systems of *class L* have long-ranged correlators $C_{\rm eq}(\boldsymbol{r}) \sim |\boldsymbol{r}|^{-(d-2+\eta)}$. Then [16]

$$a = \begin{cases} 1/z, & \text{class S,} \\ (d-2+\eta)/z, & \text{class L.} \end{cases} \tag{7}$$

This concludes our review of those main properties of ageing systems which we shall need below.

2 Beyond scale invariance

Our central question about ageing is the following: Is there a general, model-independent way to predict the form of the scaling functions $f_C(x)$ and $f_R(x)$ as defined in (6)?

The possibility of an affirmative answer might be suggested by the following known facts: (i) Ageing phenomena show a dynamical scaling, i.e. they are scale-invariant. (ii) In equilibrium critical phenomena, the invariance of the theory under dilatations $\boldsymbol{r} \mapsto b\boldsymbol{r}$ may be extended to local scale or conformal transformations $\boldsymbol{r} \mapsto b(\boldsymbol{r})\boldsymbol{r}$ (such that angles are kept fixed). It is well-known that conformal invariance is a very powerful principle in two-dimensional equilibrium critical phenomena, see e.g. [13]. More precisely, we inquire whether the global dynamical scale transformations $t \mapsto b^z t$ and $\boldsymbol{r} \mapsto b\boldsymbol{r}$ can be extended, analogously to conformal invariance where $z = 1$, to space-time dependent dilatation factors $b = b(t, \boldsymbol{r})$ in a meaningful way.

Example 1. Let $z = 2$ and consider d space dimensions. The *Schrödinger group Sch*(d) is defined by [20]

$$t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad \boldsymbol{r} \mapsto \frac{\mathcal{R}\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}}{\gamma t + \delta}, \qquad \alpha\delta - \beta\gamma = 1, \tag{8}$$

where $\mathcal{R} \in SO(d)$, $\boldsymbol{a}, \boldsymbol{v} \in \mathbb{R}^d$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. It is well-known that *Sch*(d) is the maximal kinematic group of the free Schrödinger equation $\mathcal{S}\psi = 0$ with $\mathcal{S} = 2mi\partial_t - \partial_{\boldsymbol{r}}^2$ [20] (that is, it maps any solution of $\mathcal{S}\psi = 0$ to another solution). Additional examples of *Sch*(d) as a kinematic group include certain non-linear Schrödinger equations [21], systems of Schrödinger equations [6] and the Euler equations of fluid dynamics [22]. We denote the Lie algebra of *Sch*(d) by $\mathfrak{sch}_d$. Specifically, $\mathfrak{sch}_1 = \overline{\{X_{\pm 1,0}, Y_{\pm 1/2}, M_0\}}$ with the non-vanishing commutation relations

$$[X_n, X_{n'}] = (n-n')X_{n+n'}, \qquad [X_n, Y_m] = \left(\frac{n}{2} - m\right) Y_{n+m}, \qquad \left[Y_{1/2}, Y_{-1/2}\right] = M_0, \tag{9}$$

where $n, n' \in \{\pm 1, 0\}$ and $m \in \{\pm 1/2\}$.

Example 2. For a more general dynamical exponent $z \neq 2$, we construct infinitesimal generators of local scale transformations from the following requirements [15] (for simplicity, set $d = 1$): (a) Transformations in time are $t \mapsto (\alpha t + \beta)/(\gamma t + \delta)$ with $\alpha\delta - \beta\gamma = 1$. (b) The generator for dilatations is $X_0 = -t\partial_t - z^{-1}r\partial_r - x/z$, where x is the scaling dimension of the fields ϕ, $\widetilde{\phi}$ on which the generators act. (c) Space-translation invariance is required, with generator $-\partial_r$. Starting from these conditions, we can show by explicit construction that there exist generators X_n, $n \in \{\pm 1, 0\}$ and Y_m, $m = -1/z, 1 - 1/z, \ldots$ such that

$$[X_n, X_{n'}] = (n-n')X_{n+n'}, \qquad [X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}. \tag{10}$$

For generic values of z, it is sufficient to specify the 'special' generator

$$X_1 = -t^2\partial_t - Ntr\partial_r - Nxt - \widetilde{\alpha} r^2 \partial_t^{N-1} - \widetilde{\beta} r^2 \partial_r^{2(N-1)/N} - \widetilde{\gamma} \partial_r^{2(N-1)/N} r^2 \tag{11}$$

explicitly, where we wrote $z = 2/N$ and $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ are free constants. Further non-generic solutions exist for $N = 1$ and $N = 2$ [15]. In particular, the generator (11) reproduces for $z = 2$ those of *Sch*(1). The condition $[X_1, Y_{N/2}] = 0$ is only satisfied if either (I) $\widetilde{\beta} = \widetilde{\gamma} = 0$ which we call *type I* or else (II) $\widetilde{\alpha} = 0$ which we call *type II*.

Definition 2. If a statistical system is invariant under the infinitesimal generators of either type I or type II it is said to be *locally scale-invariant* of type I or type II, respectively.

Only the generators of type II are suitable for applications to ageing phenomena.

Theorem 1. *The generators X_n, Y_m of type II form a kinematic symmetry of the differential equation $\mathcal{S}\psi = 0$, where*

$$\mathcal{S} = -(\widetilde{\beta} + \widetilde{\gamma})\partial_t + \frac{1}{z^2}\partial_r^z. \tag{12}$$

For $z = 2$, we recover the $d = 1$ free Schrödinger equation and its maximal kinematic Lie algebra $\mathfrak{sch}_1$. See [15] for the precise definition of the commuting fractional derivatives ∂_r^z.

In order to be able to apply this kinematic symmetry to ageing phenomena, we must consider the subset of $\{X_n, Y_m\}$ where time translations (generated by $X_{-1} = -\partial_t$) are left out. It can be checked that the initial line $t = 0$ is kept invariant. Ageing systems are not time-translation invariant and local scale invariance for them is meant to exclude the generator X_{-1}.

Theorem 2. *Consider an ageing statistical system which is locally scale-invariant of type II. Then the autoresponse function is*

$$R(t,s) = r_0 \left(\frac{t}{s}\right)^{1+a-\lambda_R/z} (t-s)^{-1-a}, \qquad t > s, \tag{13}$$

where r_0 is a normalization constant. Furthermore, consider the following scaling form of the spatio-temporal response

$$R(t,s;\boldsymbol{r}) = \left.\frac{\delta\langle\phi(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{0})}\right|_{h=0} = R(t,s)\Phi\left(\frac{r}{(t-s)^{1/z}}\right). \tag{14}$$

Then $\Phi(u)$ is a solution of the equation

$$\left(\partial_u + z\left(\widetilde{\beta} + \widetilde{\gamma}\right) u\partial_u^{2-z} + 2z(2-z)\widetilde{\gamma}\partial_u^{1-z}\right)\Phi(u) = 0. \tag{15}$$

In the special case $z = 2$, we have

$$R(t,s;\boldsymbol{r}) = R(t,s)\exp\left(-\frac{\mathcal{M}}{2}\frac{\boldsymbol{r}^2}{t-s}\right), \tag{16}$$

where $\mathcal{M} = \widetilde{\beta} + \widetilde{\gamma}$ is a constant.

Proof. [14,15] Consider first the autoresponse $R = R(t,s) = \langle\phi\widetilde{\phi}\rangle$, where ϕ, $\widetilde{\phi}$ have the scaling dimensions x_ϕ, $x_{\widetilde{\phi}}$, respectively. Local scale invariance means that $X_n R = Y_m R = 0$, with $n \geq 0$. Because of spatial translation invariance and the commutators (10), it is sufficient to check that $X_0 R = X_1 R = 0$. The explicit form (11) produces two linear differential equations for $R(t,s)$ which are readily solved. Comparison with the scaling forms (6) then establishes (13). Equation (15) is proven similarly, see [15]. Equation (16) had been found earlier [12]. ■

In these explicit forms of $R(t,s;\boldsymbol{r})$ it is also assumed that the system is rotation-invariant. If that is not the case, $\mathcal{M}$ is no longer uniform, but becomes direction-dependent [17].

Equation (13) has been tested and confirmed in several spin systems undergoing ageing, notably the kinetic Ising model with Glauber dynamics in $d=2,3$ through intensive simulations [14,16] and the exactly solvable kinetic spherical model in $d>2$ dimensions [4,11,23], for $T\leq T_{\rm c}$, the random walk [7] and finally (up to logarithmic correction factors) the $2D$ XY model with $T<T_{\rm c}$ [1] and the $2D$ critical voter model [25]. Furthermore, equation (16) has been numerically confirmed in the Glauber–Ising model, again for $d=2,3$ and $T<T_{\rm c}$ [17]. However, small corrections with respect to (13) were found at $T=T_{\rm c}$ in a two-loop ε-expansion [3] and in a recent self-consistent study [19] of the time-dependent Ginzburg–Landau equation at $T=0$ and which improves on the approximate Ohta–Jasnow–Kawasaki theory. We refer to the literature for details.

3 On the Schrödinger group

Having reviewed the important phenomenological result of local scale invariance as given in Theorem 2, we now discuss in more detail how a dynamical symmetry such as local scale invariance might come about. We shall do this here for the case $z=2$ and therefore discuss the Schrödinger group and Schrödinger-invariant systems [18]. For simplicity, we often set $d=1$.

Under the action of an element $g\in Sch(1)$ of the Schrödinger group, a solution $\phi(t,r)$ of the free equation $(2\mathcal{M}\partial_t-\partial_r^2)\phi=0$ where $\mathcal{M}={\rm i}m$ is fixed, transforms projectively, viz. $\phi(t,r)\mapsto(T_g\phi)(t,r)=f_g(g^{-1}(t,r))\phi(g^{-1}(t,r))$ with a known [20] companion function f_g.

Following [10], we treat $\mathcal{M}$ as an additional variable and ask about the maximal kinematic group in this case [18]. First introduce a new coordinate ζ and a new wave function ψ through

$$\phi(t,r)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}{\rm d}\zeta\, e^{-{\rm i}\mathcal{M}\zeta}\psi(\zeta,t,r). \tag{17}$$

We denote time t as the zeroth coordinate and ζ as coordinate number -1. When working out the action of the generators of $\mathfrak{sch}_1$ on the function ψ, it is easily seen that the projective phase factors can be absorbed into certain translations of the variable ζ [18]. Furthermore, the free Schrödinger equation becomes

$$\left(2{\rm i}\frac{\partial^2}{\partial\zeta\partial t}+\frac{\partial^2}{\partial r^2}\right)\psi(\zeta,t,r)=0. \tag{18}$$

In order to find the maximal kinematic symmetry of this equation, we recall that the three-dimensional Klein–Gordon equation $\sum_{\mu=-1}^{1}\partial_\mu\partial^\mu\Psi(\boldsymbol{\xi})=0$ has the $3D$ conformal algebra $\mathfrak{conf}_3\cong so(4,1)\cong B_2$ as maximal kinematic symmetry. By making the following change of variables

$$\zeta=\frac{1}{2}\left(\xi_0+{\rm i}\xi_{-1}\right),\qquad t=\frac{1}{2}\left(-\xi_0+{\rm i}\xi_{-1}\right),\qquad r=\sqrt{\frac{\rm i}{2}}\,\xi_1 \tag{19}$$

and setting $\psi(\zeta,t,r)=\Psi(\boldsymbol{\xi})$, the $3D$ Klein–Gordon equation reduces to (18). Therefore [18]

Theorem 3. *For variable masses $\mathcal{M}$, the maximal kinematic symmetry algebra of the free Schrödinger equation in d dimensions is isomorphic to the conformal algebra $\mathfrak{conf}_{d+2}$ and one has the inclusion of the complexified Lie algebras $(\mathfrak{sch}_d)_{\mathbb{C}}\subset(\mathfrak{conf}_{d+2})_{\mathbb{C}}$.*

For $d=1$, the Cartan subalgebra is spanned by the generators X_0 and $N:=-t\partial_t+\zeta\partial_\zeta$. With respect to the six generators of the Schrödinger algebra $\mathfrak{sch}_1$, there are the four additional ones N, V_+, V_-, W. They are identified in Fig. 2a with the roots in the root space of B_2.

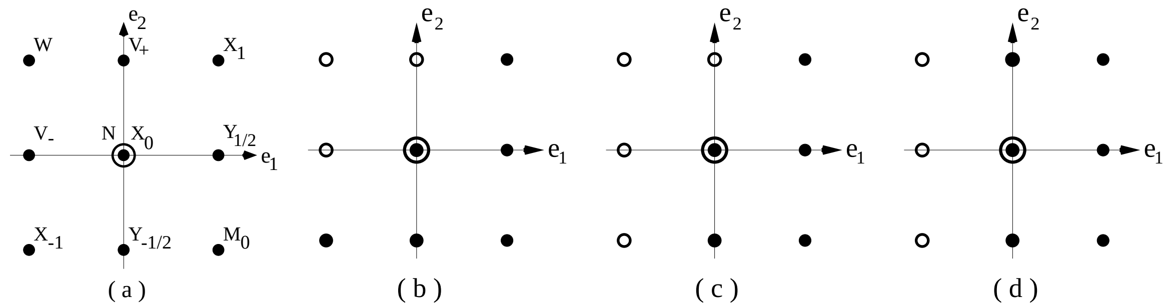

Figure 2. (a) Roots of the complex Lie algebra B_2 and the identification of the generators of the complexified conformal Lie algebra $(\mathfrak{conf}_3)_{\mathbb{C}} \supset (\mathfrak{sch}_1)_{\mathbb{C}}$. The double circle in the center denotes the Cartan subalgebra. The generators belonging to the three non-isomorphic parabolic subalgebras [18] are indicated by the full points, namely (b) $\widetilde{\mathfrak{sch}}_1$, (c) $\widetilde{\mathfrak{age}}_1$ and (d) $\widetilde{\mathfrak{alt}}_1$.

Consider the non-isomorphic parabolic subalgebras of B_2. In Fig. 2, these correspond to convex subsets of roots. There are two maximal parabolic subalgebras, namely (i) $\widetilde{\mathfrak{sch}}_1 := \mathfrak{sch}_1 \oplus \mathbb{C}N$ and (ii) $\widetilde{\mathfrak{alt}}_1 := \mathfrak{alt}_1 \oplus \mathbb{C}N$, where $\mathfrak{alt}_1 := \overline{\{X_0, X_1, Y_{\pm 1/2}, M_0, V_+\}}$ and $V_+ = -2\zeta r\partial_\zeta - 2tr\partial_t - (r^2 + 2\mathrm{i}\zeta t)\partial_r - 2xr$, see also Fig. 2bd. The minimal parabolic subalgebra is $\widetilde{\mathfrak{age}}_1 := \mathfrak{age}_1 \oplus \mathbb{C}N$, where $\mathfrak{age}_1 := \overline{\{X_0, X_1, Y_{\pm 1/2}, M_0\}}$, see Fig. 2c. We see that both $\widetilde{\mathfrak{alt}}_1$ and $\widetilde{\mathfrak{age}}_1$ do not contain the generator X_{-1} of time translations and they may therefore be considered candidates for a dynamical symmetry of ageing systems.

In writing down above the Klein–Gordon equation, we had used units such that the 'speed of light' $c = 1$. It had been claimed in the literature that in a non-relativistic limit $c \to \infty$, there were a group contraction $\mathfrak{conf}_{d+1} \curvearrowright \mathfrak{sch}_d$ provided that an ill-defined "*... transfer of the transformation of mass to the coordinates ...*" is carried out [2]. However, going over first to the variable ζ before letting $c \to \infty$, we do not find a group contraction but rather the map $\mathfrak{conf}_3 \to \widetilde{\mathfrak{alt}}_1 \neq \mathfrak{sch}_1$.

We now discuss some physical consequences of Theorem 3. Consider the effective action $S = S[\psi(\zeta, t, r)]$. For a free field ψ one recovers from the action

$$S = \int \mathrm{d}\zeta\,\mathrm{d}t\,\mathrm{d}r\,\left[2\mathrm{i}\frac{\partial\psi}{\partial\zeta}\frac{\partial\psi}{\partial t} + \left(\frac{\partial\psi}{\partial r}\right)^2\right] + S_{\rm ini}, \tag{20}$$

where $S_{\rm ini}$ describes the initial conditions, the Schrödinger equation (18) as equation of motion. For an infinitesimal coordinate transformation parameterized by ε_ν, $\nu = -1, 0, 1$, a theory is said to be *local* if its action S transforms as (the second integral is restricted to the line $t = 0$)

$$\delta S = \int \mathrm{d}\zeta\,\mathrm{d}t\,\mathrm{d}r\,{T_\mu}^\nu \partial^\mu \varepsilon_\nu + \int_{(t=0)} \mathrm{d}\zeta\,\mathrm{d}r\,U^\nu \varepsilon_\nu, \tag{21}$$

where ${T_\mu}^\nu$ is the energy-momentum tensor.

Theorem 4. *If the action S of a local theory is invariant under translations in ζ and r, scale-invariant with $z = 2$ and Galilei-invariant, then $\delta_{X_1} S = 0$ and S in invariant under the action of $\mathfrak{age}_1$.*

Proof. [18] The line $t = 0$ is invariant under the action of $\mathfrak{age}_1$. Furthermore, locality (21) yields several Ward identities. First, translation-invariance in ζ and r implies $U^{-1} = U^1 = 0$. Dilatation invariance gives ${T_0}^0 + \frac{1}{2}{T_1}^1 = 0$ and Galilei-invariance implies ${T_0}^1 - \mathrm{i}{T_1}^{-1} = 0$. Now, for an infinitesimal special Schrödinger transformation,

$$\delta_{X_1} S = -\varepsilon \int \mathrm{d}\zeta\,\mathrm{d}t\,\mathrm{d}r\,\left[\left(2{T_0}^0 + {T_1}^1\right)t + \left({T_0}^1 - \mathrm{i}{T_1}^{-1}\right)r\right] + \frac{\mathrm{i}\varepsilon}{2}\int_{(t=0)} \mathrm{d}\zeta\,\mathrm{d}r\,r^2 U^{-1} = 0$$

because of the Ward identities derived above. ■

For the free-field action (20), an energy-momentum tensor which satisfies these Ward identities can be explicitly constructed [18]. We stress that Galilei-invariance must be included as a hypothesis and one cannot expect to be able to derive it from weaker assumptions. On the other hand, time-translation invariance is not required.

Finally, we reconsider the derivation of the spatio-temporal response for $z=2$ [18]. A standard result of conformal invariance (see [13]) gives together with Theorem 3

$$\langle\psi_1(\zeta_1,t_1,r_1)\psi_2(\zeta_2,t_2,r_2)\rangle = \langle\Psi(\boldsymbol{\xi}_1)\Psi(\boldsymbol{\xi}_2)\rangle = \psi_0\,\delta_{x_1,x_2}\,t^{-x_1}\left(\zeta+\frac{\mathrm{i}}{2}\frac{r^2}{t}\right)^{-x_1}, \tag{22}$$

where $x_{1,2}$ are the scaling dimensions of the fields $\psi_{1,2}$, ψ_0 is a normalization constant and $\zeta=\zeta_1-\zeta_2$ and so on. Using the convention $\mathcal{M}_{1,2}\geq 0$, it turns out that equations (17), (22) together imply that $\langle\phi_1\phi_2\rangle=0$[1]. However, if we define a *conjugate field* ϕ^* by formal complex conjugation in (17) with $\mathcal{M}\geq 0$, then for $x_1>0$ we find

$$\begin{aligned}\langle\phi_1(t_1,r_1)\phi_2^*(t_2,r_2)\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}\zeta_1\mathrm{d}\zeta_2\, e^{-\mathrm{i}\mathcal{M}_1\zeta_1+\mathrm{i}\mathcal{M}_2\zeta_2}\langle\psi_1(\zeta_1,t_1,r_1)\psi_2(\zeta_2,t_2,r_2)\rangle\\ &= \phi_0\,\delta_{x_1,x_2}\,\delta(\mathcal{M}_1-\mathcal{M}_2)\,\Theta(t)t^{-x_1}\exp\left(-\frac{\mathcal{M}_1}{2}\frac{r^2}{t}\right)\end{aligned} \tag{23}$$

and we stress that the Θ-function expresses the causality condition $t_1>t_2$ which is required for an interpretation of $\langle\phi_1\phi_2^*\rangle$ as response function. If we recall from Martin–Siggia–Rose theory that $R(t,s;\boldsymbol{r})=\langle\phi(t,\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{0})\rangle=\langle\phi(t,\boldsymbol{r})\phi^*(s,\boldsymbol{0})\rangle$, where effectively ϕ has a 'mass' $\mathcal{M}\geq 0$ and ϕ^* has a 'mass' $\mathcal{M}^*\leq 0$, we have the identification

$$\phi^*(s,\boldsymbol{r})=\widetilde{\phi}(s,\boldsymbol{r}). \tag{24}$$

This result has also been confirmed for the three-point response functions [18].

We now apply this to the parabolic subalgebras. For a system invariant under $\mathfrak{age}_1$, we find

$$\langle\psi_1(\zeta_1,t_1,r_1)\psi_2(\zeta_2,t_2,r_2)\rangle = \psi_0\left(\frac{t_1}{t_2}\right)^{(x_2-x_1)/2} t^{-(x_1+x_2)/2}\left(\zeta+\frac{\mathrm{i}}{2}\frac{r^2}{t}\right)^{-(x_1+x_2)/2}. \tag{25}$$

If we would consider the extension $\widetilde{\mathfrak{age}}_1\to\widetilde{\mathfrak{sch}}_1$, then invariance under time translations generated by $X_{-1}=-\partial_t$ fixes $x_1=x_2$ and we simply recover the result (22) coming form the invariance under $\mathfrak{conf}_3$. On the other hand, the extension $\widetilde{\mathfrak{age}}_1\to\widetilde{\mathfrak{alt}}_1$ requires that the condition $V_+\langle\psi_1\psi_2\rangle=0$ holds. It is easy to see from (25) that this is the case. Therefore, for both $\widetilde{\mathfrak{age}}_1$ and $\widetilde{\mathfrak{alt}}_1$ we find, provided $x_1+x_2>0$

$$\langle\phi_1\phi_2^*\rangle=\phi_0\delta(\mathcal{M}_1-\mathcal{M}_2)\Theta(t)\left(\frac{t_1}{t_2}\right)^{(x_2-x_1)/2} t^{-(x_1+x_2)/2}\exp\left(-\frac{\mathcal{M}_1}{2}\frac{r^2}{t}\right) \tag{26}$$

and we have indeed rederived (16) for ageing systems with $z=2$ in a model-independent way, including the causality condition. There is not yet a criterion which would allow to decide if $\widetilde{\mathfrak{age}}_1$ or $\widetilde{\mathfrak{alt}}_1$, if any, is the dynamic symmetry of ageing systems with $z=2$.

In conclusion, the explicit form of the spatio-temporal two-time response function $R(t,s;\boldsymbol{r})$ as given in (16) is a consequence of the assumed Galilei-invariance of the ageing system. The high-precision numerical confirmation of this form in the kinetic Ising model with Glauber dynamics quenched to $T<T_c$ for $d=2$ and $d=3$ dimensions [17] is a strong indication in favour of Galilei-invariance in that model. However, even for $T=0$ the Langevin equation (1) or the system (4) are for general $\mathcal{H}[\phi]$ not Galilei/Schrödinger-invariant, see [6, 21]. In view of small corrections to (13) suggested by two-loop field-theory calculations [3, 19] and of Theorem 4 on the other hand, it remains to be understood in what precise sense ageing systems might be said to be Galilei-invariant.

[1] This holds true for $T=0$. For the case $0<T<T_c$, see [24].

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