

# Applications of Symmetry to General Relativity

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The equations of general relativity are highly nonlinear partial differential equations and require special techniques to solve exactly. Symmetry considerations have sometimes helped to find solutions. Three examples will be considered here. The first one considers a scaling symmetry in the equations for perfect fluid with a spherical symmetry, which is of historical importance in understanding neutron stars and black holes. The second involves symmetries in the Ernst equation for stationary axially symmetric fields, which are related to the methods worked out in the 1970s for finding solutions to that equation. The third example is an ongoing investigation into critical gravitational collapse, a topic of current interest, using a symmetry of the equations and other analytic techniques.

## 1 Introduction

The Einstein equations of general relativity are, in general, second order highly nonlinear partial differential equations. It has been necessary to employ many different techniques in order to obtain exact solutions of these equations, and many are now known. I will discuss here a few examples of the use of symmetry in such investigations. First I will present some cases of historical interest, and then I will treat, more extensively, a current ongoing research problem.

## 2 Spherically symmetric configurations

My first example has to do with spherically symmetric gravitational fields for a perfect fluid. The first solution of the vacuum Einstein's equations to be found was the Schwarzschild solution,

$$ds^2 = -(1 - 2m/r)dt^2 + (1 - 2m/r)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$$

which is the spherically symmetric solution for a point mass  $m$  and which later proved the foundation for the discovery of black holes. In this equation  $G = c = 1$ . For a perfect fluid, the equations for pressure  $P$ , density  $\rho$ , and mass  $m$  out to radius  $r$  take the Oppenheimer–Volkoff form [1],

$$dP/dr = -(P + \rho)(m + 4\pi r^3 P) / r(r - 2m), \quad (1)$$

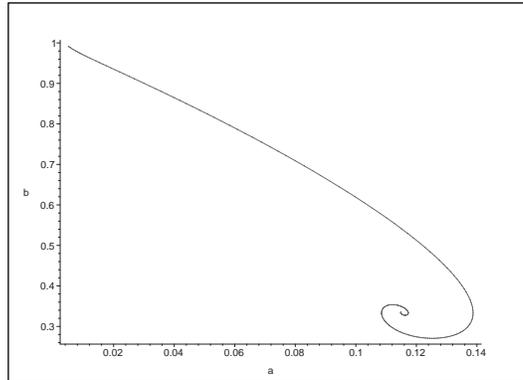
$$dm/dr = 4\pi r^2 \rho \quad (2)$$

in which there is assumed an equation of state  $P = P(\rho)$ . For high densities we expect the equation of state to take the form  $P = k\rho$ , where  $k$  is a constant, which takes the value  $1/3$  for a Fermi gas of neutrons, protons, and electrons. In this case, one can see that equations (1) and (2) admit a scaling symmetry. If we write  $a = m/r$  and  $b = 4\pi r^2 \rho / 3m$  and take  $k = 1/3$ , these equations become [2]

$$rda/dr = a(3b - 1),$$

$$rdb/dr = b[3 - 3b - 4a(3 + b)/(1 - 2a)].$$

Division of the second of these by the first now gives a first order equation for  $b(a)$ . If  $x = 4\pi\rho_0 r^2/3$ , where  $\rho_0$  is the density at  $r = 0$ , then the boundary condition near  $r = 0$  ( $x = 0$ ) is  $a = x + \dots$  and  $b = 1 - 8x/5 + \dots = 1 - 8a/5 + \dots$ . Phase plane analysis of this equation shows that there is a stable focus and saddle points. It is clear that  $2a < 1$  always, ensuring that there is not a black hole event horizon outside the star. Numerical solution (done here with MAPLE) gives Fig. 1, where the focus is obvious.



**Figure 1.**  $b$  vs.  $a$  for large density.

Density decreases as one moves out through the star and one must use a different equation of state, which means that the  $a$ - $b$  trajectory must leave the focus and move down close to the  $a$ -axis, ending at the point where the pressure equals zero at the outer edge of the star. At that point one can determine the radius and mass of the star. But the oscillation at the focus is reflected in the final values. The total mass of the star is an oscillatory function of the central density and there is a maximum mass in the neighborhood of 1.4 solar masses, the Chandrasekhar limit. Thus for masses greater than that there will be collapse. There are two families of stable equilibria, white dwarfs and neutron stars, and this points the way toward the existence of collapsed configurations or black holes [2].

### 3 Axially symmetric configurations

For my second example, I consider stationary (rotating) axially symmetric fields, which include the Schwarzschild solution and the Kerr solution for a spinning point mass. The standard approach uses the metric

$$ds^2 = -f(dt + \omega d\phi)^2 + \rho^2 f^{-1} d\phi^2 + h f^{-1} (d\rho^2 + dz^2),$$

where  $f$ ,  $\omega$ , and  $h$  are functions of  $\rho$  and  $z$ . We can define a new, conjugate variable  $\psi$ , often called the twist potential, by (subscripts denote derivatives):

$$\omega_\rho = \rho f^{-2} \phi_z, \quad \omega_z = -\rho f^{-2} \psi_\rho.$$

The field equations now are

$$\begin{aligned} f(f_{\rho\rho} + f_\rho/\rho + f_{zz}) &= f_\rho^2 + f_z^2 - \psi_\rho^2 - \psi_z^2, \\ f(\psi_{\rho\rho} + \psi_\rho/\rho + \psi_{zz}) &= 2f_\rho\psi_\rho + 2f_z\psi_z \end{aligned} \tag{3}$$

or, where  $E = f + i\psi$  (the Ernst potential),

$$\Re(E)\nabla^2 E = (\nabla E)^2,$$

the Ernst equation. One can easily calculate the symmetries of equations (3). (Reference [3] has a similar calculation for the Einstein–Maxwell equations.) There are five symmetries: two on the independent variables alone,

$$\partial_z, \quad \rho\partial_\rho + z\partial_z \quad (4)$$

and three on the dependent variables alone,

$$\partial_\psi, \quad f\partial_f + \psi\partial_\psi, \quad f\psi\partial_f + (1/2)(\psi^2 - f^2)\partial_\psi. \quad (5)$$

The symmetries (4), in particular the second (scale) symmetry, can be used to look for special solutions of the Ernst equation (which probably have little physical significance.) The symmetries (5), combined and exponentiated, yield the expression

$$E' = (aE + ib)/(icE + d)$$

the Ehlers transformation [4, 5], where  $a$ ,  $b$ ,  $c$ , and  $d$  are real constants. If  $E$  is a solution of the Ernst equation, then  $E'$  is also. Certain combinations of these variables and their derivatives are potentials for the Ernst equation. W. Kinnersley and his colleagues, through extensive research in the 1970s, pursued relations involving these potentials and higher symmetries of the Ernst equation in the jet space, eventually leading to a method of generating new solutions of the Ernst equation [6]. Alternate methods include the use of Bäcklund transformations, found by the present author and by G. Neugebauer [7, 8], and several other techniques found in the late 1970s.

## 4 Critical gravitational collapse

The third, and main, example is that of critical gravitational collapse. This work is still unfinished, but several interesting results have been obtained. About ten years ago M. Choptuik published a paper considering a massless scalar field in spherically symmetric gravity [9]. The aim was to investigate possible collapse to a black hole. He assumed a metric as follows:

$$ds^2 = -\alpha^2(r, t)dt^2 + a^2(r, t)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

with a time variable suitable near the origin  $T_0 = \int_0^t \alpha(0, t)dt$ . If  $\phi(r, t)$  is the scalar field and where

$$\Phi = \phi_r, \quad \Pi = a\phi_t/\alpha,$$

the Einstein equations are

$$\begin{aligned} \Phi_t &= (\alpha\Pi/a)_r, & \Pi_t &= r^{-2}(r^2\alpha\Phi/a)_r, \\ \alpha_r/\alpha - a_r/a + (1 - a^2)/r &= 0, \\ a_r/a + (a^2 - 1)/(2r) - 2\pi r(\Phi^2 + \Pi^2) &= 0, \\ a_t/\alpha &= 4\pi r\Pi\Phi, \end{aligned} \quad (6)$$

where subscripts denote derivatives. Choptuik does not use the last equation, since it is consistent with the others in the sense that calculation of the cross derivative of  $a$  from the last two equations, with use of the other equations, gives the same value. By analogy to the Schwarzschild case, he takes  $a \rightarrow \infty$  to be the criterion for a black hole.

## 4.1 Results from numerical integration

Choptuik integrated the equations numerically, scaling the integration steps carefully, and found several interesting features. (1) For some values of a parameter  $p$  in the initial conditions for the scalar field black holes form, but for other values they do not. There is a critical parameter,  $p^*$ , for which black holes first form. (2) The precisely critical behavior, in which  $p = p^*$ , is *universal*, holding for all types of initial data. He uses variables  $X = \sqrt{(2\pi)r\Phi/a}$ ,  $Y = \sqrt{(2\pi)r\Pi/a}$ ,  $\rho = \ln(kr)$ ,  $\tau = \ln[k(T_0^* - T_0)]$ , and finds the existence of the universal critical solution  $X^*$ ,  $Y^*$ , which occurs in the neighborhood of  $a = \infty$  and  $r = 0$ . (3) There is a scaling behavior. If  $Z$  = either  $X$  or  $Y$ , then this is  $Z^*(\rho - \Delta, \tau - \Delta) = Z^*(\rho, \tau)$ , or  $Z^*(\rho - \tau, \tau - \Delta) = Z^*(\rho - \tau, \tau)$ . Thus there is a periodicity in  $\tau$  with period  $\Delta$ , where  $\Delta$  is a universal constant, approximately equal to 3.43, independent of initial conditions. This indicates that the critical solution is somehow dependent only on the original equations, in which there is no natural length or time parameter;  $\Delta$  thus should naturally come out of the equations. Furthermore, the mass  $M$  of the black hole is given by a power law  $M = c|p - p^*|^\gamma$ , where  $\gamma$  is another universal constant, approximately equal to 0.37, similar to critical phenomena in thermodynamics.

## 4.2 Symmetry treatment of equations

The scaling behavior suggests that an analytical approach, using scaled variables, might be useful. Choptuik already made the observation that the original equations (6) are invariant under the scaling  $r \rightarrow kr$ ,  $t \rightarrow kt$ . We first define

$$A = \sqrt{(2\pi)\Phi}, \quad B = \sqrt{(2\pi)\Pi}, \quad C = \alpha/a, \quad \sigma = \sqrt{(2\pi)\phi}, \quad U = a^2.$$

Then the basic equations become

$$\begin{aligned} A &= \sigma_r, & B &= \sigma_t/C, \\ A_t &= (BC)_r, & B_t &= (AC)_r + 2AC/r, & C_r/C &= (U - 1)/r, \\ U_r/U &+ (U - 1)/r = 2r(A^2 + B^2), & U_t/U &= 4rABC. \end{aligned} \tag{7}$$

We calculate the symmetries of equations (7) and find that the following are invariant variables:  $\xi = rh(t)$ ,  $U$ ,  $F = rA$ ,  $G = rB$ , and  $P = Ch^2/h'$ , where  $h(t)$  is an arbitrary function of  $t$ . In terms of Choptuik's variables, we have  $F = aX$  and  $G = aY$ . We note that  $\alpha^2$ , which is  $-g_{00}$ , is  $(aC)^2$ , which has a factor  $(h'/h^2)^2$ . So if we define a new time coordinate  $t_0$  with  $dt_0 = -h'dt/h^2$ , we see that  $h = 1/t_0$ . We could make this definition of  $h$ , but it is more convenient to define it in terms of Choptuik's function  $T_0(t)$  as  $h^{-1}(t) = T_0^* - T_0(t)$ , where the star indicates critical value. (This then gives the convenient boundary condition  $aP = 1$  at  $r$ , or  $\xi = 0$ .) We now express the equations in terms of the variables  $\xi = rh(t)$  and  $v = t$ . (We introduce a new name for  $t$  to avoid complications with partial derivative notation.) For any function  $H$ , we have  $H_r = hH_\xi$  and  $H_t = H_v + \xi(h'/h)H_\xi$ . We also find it useful to define  $P = -\xi Q$ ,  $z = \ln \xi$ ,  $\tau = \text{const} - \ln h$ . We now note that  $\tau$  is the same as Choptuik's  $\tau$  and  $z = \rho - \tau$ , so we can look for the periodicity with period 3.43. In terms of derivatives with respect to  $z$  and  $\tau$  (which do not appear explicitly), the equations take the following nice form

$$\begin{aligned} F_\tau &= (F + QG)_z, & G_\tau &= (G + QF)_z + 2QF, & Q_z &= Q(U - 2), \\ U_z/U &= 2F^2 + 2G^2 + 1 - U, & U_\tau/U &= 4QFG + 2F^2 + 2G^2 + 1 - U. \end{aligned} \tag{8}$$

## 4.3 Approaches to solution

We now comment on several avenues of approach. (1) If we assume no  $\tau$  dependence, then the last equation of (8) gives

$$U = 1 + 2F^2 + 2G^2 + 4QFG,$$

and the first equation gives

$$F = k - QG,$$

where  $k$  is a constant. The fourth equation is satisfied automatically. We get

$$Q_z = Q(U - 2)$$

and

$$G_z = Q(1 - Q^2)^{-1} [U(2QG - k) - 2QG]$$

so we can write an equation for  $G(Q)$ :

$$dG/dQ = (1 - Q^2)^{-1} (U - 2)^{-1} [U(2QG - k) - 2QG], \quad (9)$$

where  $U$  is given above in terms of  $G$  and  $Q$ . Equation (9) for  $G(Q)$  can be put in various forms. However, I will not pursue those here, since we really want to search for the periodicity in  $\tau$ .

In our search for periodic behavior in  $\tau$ , we note that it can be exhibited either in linear equations resembling simple harmonic oscillator equations – which would have variable amplitude – or in nonlinear cases with a limit cycle. The nonlinear character of the equations suggests that the latter is the case; indeed, C. Gundlach in a review paper [10], sketches qualitative limit cycles – which, however, may actually change to nonperiodic behavior.

(2) In the Schwarzschild solution,  $\alpha^2 = 1/a^2 = 1 - 2m/r$ , for constant geometrized mass  $m$ . That suggests that near that solution, we might suspect that  $\alpha \propto 1/a$ , or that  $C \propto \alpha/a \propto 1/a^2 = 1/U$ . For  $U$  large, then,  $C$  (or  $P$  or  $Q$ ) should be small. So let us consider an expansion of equations (8) in powers of  $Q$ . We first define  $V = UQ$ , which satisfies

$$V_z = V(2F^2 + 2G^2 - 1),$$

with

$$Q_z = V - 2Q.$$

We ignore the  $U_\tau$  equation. Expand  $G, F, V$ :

$$G = a + bQ + \dots, \quad F = c + dQ + \dots, \quad V = f + gQ + \dots,$$

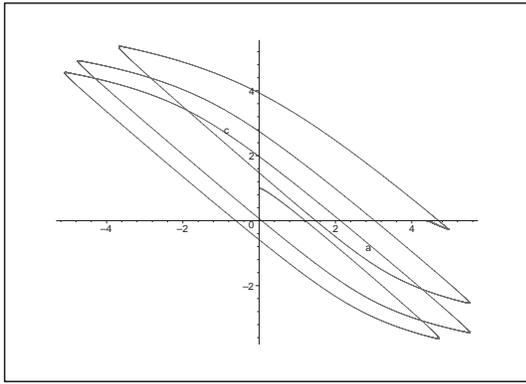
where  $a, b, \dots, g$  are functions of  $\tau$ . We get, from the expansion,

$$\begin{aligned} \dot{c} &= f(a + d), & \dot{d} &= (a + d)(g - 2), & \dot{a} &= f(b + c), \\ \dot{b} &= (b + c)(g - 2) + 2c, & g &= 2a^2 + 2c^2 - 1, \end{aligned}$$

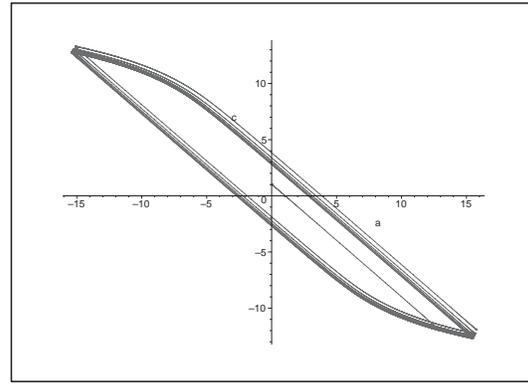
where the dot represents differentiation with respect to  $\tau$ . There is another equation for  $g$  which we ignore. We will also assume  $f$  to be a constant. We solve the first and third equations for  $d$  and  $b$  and substitute them and  $g$  into the second and fourth equations. We get

$$\begin{aligned} \ddot{a} &= \dot{a}(2a^2 + 2c^2 - 3) + f(\dot{c} + 2c), \\ \ddot{c} &= \dot{c}(2a^2 + 2c^2 - 3) + f\dot{a}. \end{aligned}$$

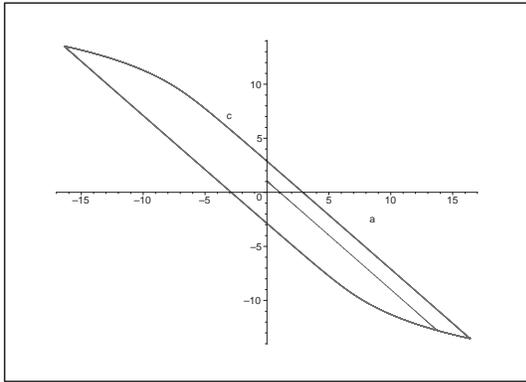
We integrate these with MAPLE. For  $f = 20$ , we get oscillatory behavior with a period of about 2.5, but not a perfect limit cycle in  $a$  and  $c$  (Fig. 2). For  $f = 200$ , we are closer (Fig. 3). For  $f = 225$ , we get what appears to be a perfect limit cycle, with period about 1.8 (Figs. 4 and 5).



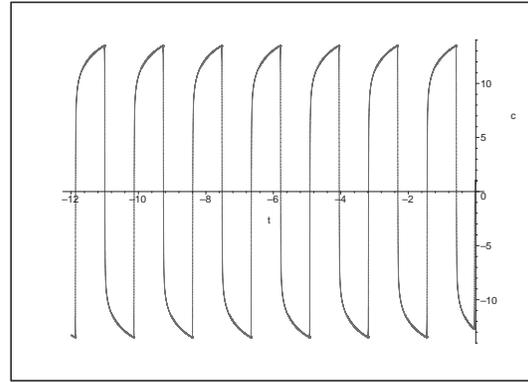
**Figure 2.**  $c$  vs.  $a$  for  $f = 20$ .



**Figure 3.**  $c$  vs.  $a$  for  $f = 200$ .



**Figure 4.**  $c$  vs.  $a$  for  $f = 225$ .



**Figure 5.**  $c$  vs.  $\tau$  (shown as  $t$ ) for  $f = 225$ .

There are several problems associated with this calculation. We ignored the  $U_\tau$  equation. A closely related matter is that we ignored any  $\tau$  dependence in  $Q$ . We also ignored the second equation for  $g$ , which appears to be inconsistent with other equations, and we assumed  $f$  to be constant. There is no particular obvious reason why the value of 225 for  $f$  should be special, and the period obtained is not 3.4 but is roughly half that. Nevertheless, the periodic behavior of  $a$  and  $c$  strongly suggests that we may be on the right track.

(3) In order to clear up some of the problems of the last treatment, we take our functions  $F$ , etc., to be functions of  $Q(z, \tau)$  and  $v (= \tau)$ . Then for any function  $H$ , we have  $H_z = H_Q Q_z = H_Q Q(U - 2)$  and  $H_\tau = H_Q Q_\tau + H_v$ . If we put

$$\beta = 2F^2 + 2G^2 + 1 - U,$$

then from the  $U$  equations we can find

$$U_Q = \beta U / [Q(U - 2)]$$

and

$$Q_\tau = Q(U - 2)(4QFG + \beta - U_v/U)/\beta.$$

We substitute for  $Q_\tau$  in the two equations for  $F$  and  $G$  and get rather messy equations involving  $Q$ ,  $F$ ,  $G$ , and the derivatives of  $F$  and  $G$  with respect to  $v$  and  $Q$ . We expand  $F$  and  $G$  in powers of  $Q$ , with coefficients as functions of  $v$ . We write

$$U = a/Q + b + \dots, \quad F = d + eQ + \dots, \quad G = h + mQ + \dots.$$

We find, where the dot indicates  $d/dv = d/d\tau$ ,

$$b = 2d^2 + 2h^2 - 1, \quad \dot{d} = ha, \quad \dot{h} = da,$$

and equations for  $e$  and  $m$ ,

$$\begin{aligned} a\dot{e} + e\dot{a} &= ha(b - 2) + 2ma^2, \\ a\dot{m} + m\dot{a} &= dab + 2ea^2. \end{aligned}$$

Thus  $d$ , for example, obeys

$$\ddot{d} - \dot{a}\dot{d}/a - a^2d = 0,$$

indicating exponential instead of oscillatory behavior. Inspection of the equations indicates that the change results precisely from the inclusion of  $Q_\tau$ .

(4) We now note that it is more natural to consider an expansion in  $1/U$  than in  $Q$ , since  $U$  is the variable considered to be large near a black hole. Furthermore, it is very convenient to do so since we have expressions for both  $U_z$  and  $U_\tau$  (8). For any  $H(z, \tau)$ , we convert it to  $H(U, v)$ , where  $v = \tau$ . This gives

$$H_z = H_U U_z = H_U U (2F^2 + 2G^2 + 1 - U)$$

and

$$H_\tau = H_v + H_U U_\tau = H_v + H_U U (4QFG + 2F^2 + 2G^2 + 1 - U).$$

The equation for  $Q_z$  now becomes

$$Q_U U (2F^2 + 2G^2 + 1 - U) = Q(U - 2).$$

This expression can be used to replace  $Q_U$  in the  $F$  and  $G$  equations when it occurs. Those equations now become

$$\begin{aligned} F_v + 4QUFGF_U &= QU (2F^2 + 2G^2 + 1 - U) G_U + QG(U - 2), \\ G_v + 4QUFGG_\tau &= QU (2F^2 + 2G^2 + 1 - U) F_U + QFU. \end{aligned}$$

It will be noted that  $Q$  appears as a multiplier in all terms except the  $v$  derivatives, and  $U$  appears in most terms. We define  $R = QU$  and rewrite the equations as

$$\begin{aligned} R_U/R &= (2F^2 + 2G^2 - 1) / [U (2F^2 + 2G^2 + 1 - U)], \\ F_v/R &= (2F^2 + 2G^2 + 1 - U) G_U - 4FGF_U + G(1 - 2/U), \\ G_v/R &= (2F^2 + 2G^2 + 1 - U) F_U - 4FGG_U + F. \end{aligned}$$

We see that the integration of the first equation gives an additive function  $\ln b(v)$  in  $\ln R$ , or a multiplicative function  $b(v)$  in  $R$ . This can be removed in the second and third equations simply by redefining the time variable as  $w = \int b(v)dv$ . This removes the ambiguity in the  $v$  ( $= \tau$ ) dependence of  $Q$  ( $= R/U$ ). Inspection now shows that it is advantageous to define  $x = F + G$  and  $y = F - G$ . The equations are now simplified to

$$\begin{aligned} R_U/R &= (x^2 + y^2 - 1) / [U (x^2 + y^2 + 1 - U)], \\ x_v/R &= 2y^2 x_U + (1 - U) (x_U - x/U) + y/U, \\ y_v/R &= -2x^2 y_U - (1 - U) (y_U - y/U) - x/U. \end{aligned} \tag{10}$$

These are still exact. If they could be solved, then  $x$ ,  $y$ , and  $R$  could be substituted into the  $U_z$  equation to find the  $z$  dependence of  $U$ , then the  $\tau$  ( $= v$ ) dependence could be found from the  $U_\tau$  equation. The obvious approach now is to expand in powers of  $1/U$ . Unfortunately, this does not give oscillatory behavior for  $x$  and  $y$ , but exponential behavior instead, just as in the expansion for small  $Q$ . Variations of approach do not solve the problem. It may be that we have an essential singularity, or to say it another way, that we should assume  $x$  and  $y$  involve, say,  $\exp(-U)$ . It is not clear how to investigate this possibility.

We try to solve the  $x$  and  $y$  equations (10) by a brute force approach. We assume

$$x = \phi(w)S(U), \quad y = \psi(w)Y(U),$$

substitute into (10), and assume that  $\phi$  and  $\psi$  satisfy equations that are purely functions of  $w$ , as follows (prime indicates  $d/dw$ ):

$$\begin{aligned} \phi' &= a\psi^2\phi + b\phi + c\psi, \\ \psi' &= d\phi^2\psi + f\psi + g\phi, \end{aligned}$$

where  $a, b, c, d, f, g$  are constants. We substitute these equations into (10) and equate coefficients, thus obtaining six equations for the two functions  $S$  and  $Y$ . If we assume  $x^2 + y^2$  large, we get  $R = U$ , and then we find some consistency in these six equations, provided that  $Y = cS$ ,  $gc = -1$ ,  $a = -dc^2$ , and  $f = -b$ . If we solve for  $d, f$ , and  $g$  and substitute in the second equation, we find

$$\psi' = -(a/c^2)\phi^2\psi - b\psi - \phi/c.$$

If we let  $\phi = c/\sqrt{a}\phi_0$  and  $\psi = 1/\sqrt{a}\psi_0$ , we see that we may scale  $a$  and  $c$  away. So just put  $a = c = 1$ , and we get

$$\begin{aligned} \phi' &= \psi^2\phi + b\phi + \psi, \\ \psi' &= -\phi^2\psi - b\psi - \phi. \end{aligned} \tag{11}$$

We get inconsistent equations for  $S(U)$ . One of these is

$$S_U/S = (1 - U + b)/[U(1 - U)],$$

yielding

$$S = kU[U/(U - 1)]^b$$

for some constant  $k$ . The other is

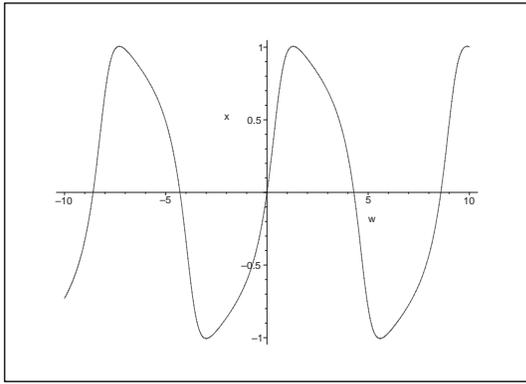
$$2SS_U = 1/U,$$

yielding  $S^2 = \ln U + \text{const}$ . We ignore the second equation and note that for large  $U$ , the first gives approximately  $S = kU$ .

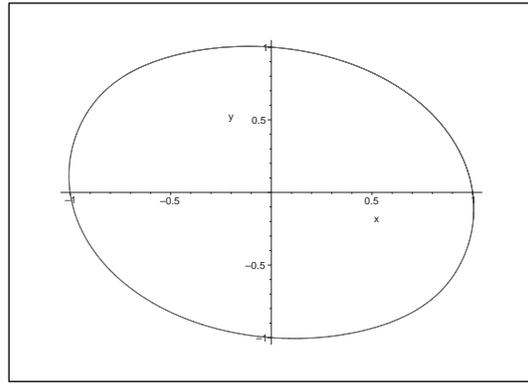
We explore the  $\phi, \psi$  equations (11). In the phase plane of  $\phi$  and  $\psi$  the origin is a critical point, a center. We can actually solve (11) in the phase plane exactly, finding

$$\phi^2 + \psi^2 + 2b \ln(1 + \phi\psi) = \lambda^2,$$

where  $\lambda$  is a constant. If  $b = 0$  this is exactly a circle. Integration of the time equations (Figs. 6, 7) shows that we get oscillations in  $w$  and a limit cycle, which is nearly circular for small  $b$ .  $\lambda$  measures the amplitude. For small  $b$  and  $\lambda = 1$  (fixed by the initial conditions), the period is about 7.2. Calculation from the exact solution indicates that the period should



**Figure 6.**  $\phi$  (shown as  $x$ ) vs.  $t$  for  $b = 0.1$ .



**Figure 7.**  $\psi$  (shown as  $y$ ) vs.  $\phi$  (shown as  $x$ ) for  $b = 0.1$ .

be about  $2\pi$ . The cubic terms in (10) appear to be proportional to both  $U$  and  $\lambda$ , suggesting that  $\lambda$  scales with  $1/U$ . These oscillations are for constant  $U$ . We really want constant  $z$ , the original scaled variable. Attempts to include the time dependence of  $U$  produce the exponential behavior seen before.

Another approach proceeds as follows. Let  $x$  and  $y$  in equation (10) be functions of  $w$  only and write, temporarily,  $\alpha = x^2 + y^2$ . Treat  $\alpha$  as a constant and integrate the  $R$  equation. We get, exactly,

$$R = (1 - (\alpha + 1)/U)^\beta,$$

where

$$\beta = (1 - \alpha)/(1 + \alpha).$$

Expansion for large  $U$  gives approximately

$$R = 1 + (\alpha - 1)/U.$$

Put  $1/U = b$ , which is now to be treated as a parameter, and put this into the  $x$  and  $y$  equations in (10) along with the expression for  $\alpha$ , keep terms of order  $b$ , and get:

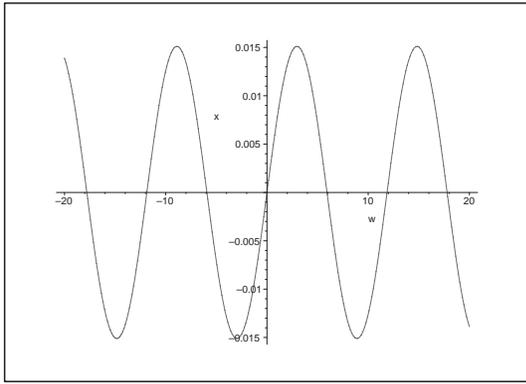
$$\begin{aligned} x_w &= x + b[(x^2 + y^2 - 2) + y], \\ y_w &= -y - b[(x^2 + y^2 - 2) + x]. \end{aligned}$$

These equations have interesting behavior. If  $x^2 + y^2$  is small, the equations may be treated as linear. We find then that the origin in  $(x, y)$  phase space is a center if  $1 > b > 1/3$ . In that region, if we start with small  $x$  and  $y$  they remain small and we get linear equations, with period depending on  $b$ , and arbitrary amplitude. For  $b > 1$ ,  $x$  and  $y$  grow and we get a nonlinear oscillation. For large enough  $b$ , the period is approximately proportional to  $15/b$  and the maximum amplitude of  $x$  and  $y$  is unity (Figs. 8, 9).

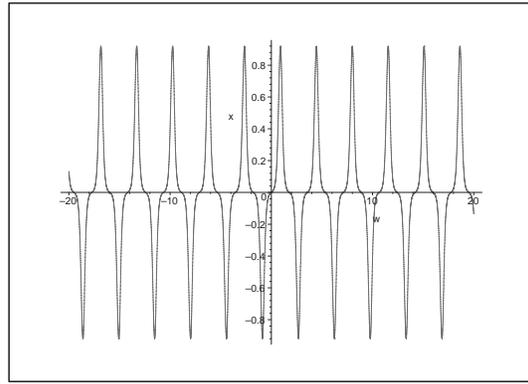
Instead of using the approximate value of  $R$ , we can use the exact value (being careful to take the absolute value of the quantity in parentheses before raising to the power. Then

$$\begin{aligned} x_w &= R[x + b(y - x)], \\ y_w &= -R[y + b(x - y)]. \end{aligned}$$

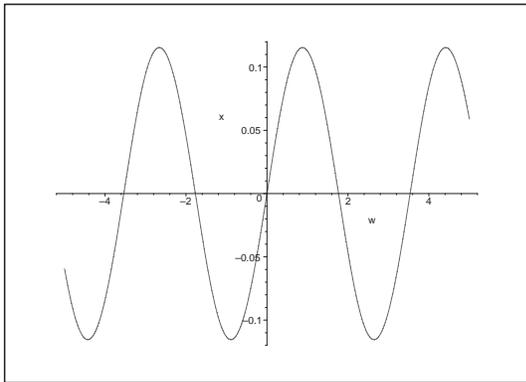
Integration of these equations gives some very nice limit cycles, but the period varies with the choice of  $b$  and of the initial conditions (Figs. 10–12).



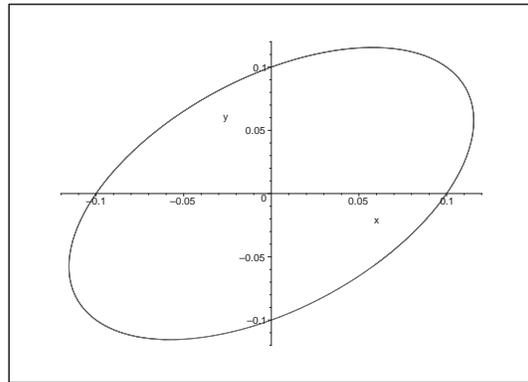
**Figure 8.**  $x$  vs.  $w$ ,  $b = 0.8$ , approximate  $R$ .



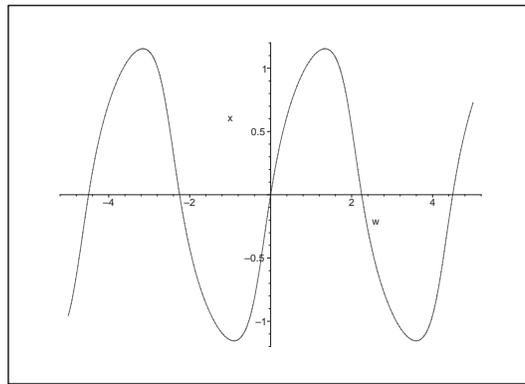
**Figure 9.**  $x$  vs.  $w$ ,  $b = 5$ , approximate  $R$ .



**Figure 10.**  $x$  vs.  $w$ ,  $b = 2$ ,  $y_0 = 0.1$ , exact  $R$ .



**Figure 11.**  $y$  vs.  $x$ ,  $b = 2$ ,  $y_0 = 0.1$ , exact  $R$ .



**Figure 12.**  $x$  vs.  $w$ ,  $b = 2$ ,  $y_0 = 1$ , exact  $R$ .

#### 4.4 Summary of results

In summary, we note several features in this treatment. First, the original equations have been expressed with  $\tau$  and the scaled variable  $z$  as independent variables. Second, in the  $\tau$ -independent case, the equations may be reduced to a single first order equation. The significance of this case is still unclear. Third, changing the independent variables to  $w (= \tau)$  and  $U$  separates the equations into three equations for  $R$ ,  $x$ , and  $y$  and two for the derivatives of  $U$  – a convenient form. Fourth, oscillatory behavior is suggested in four different ways. The amplitude in these may scale as  $1/U$ . Each way has unacceptable assumptions. Trying to remove the assumptions produces exponential behavior. It may be that, as indicated by Gundlach [10] and as mentioned

earlier, there is an epoch of oscillation followed by an epoch of final nonoscillatory behavior, which might be exponential. One would think, however, that there still should be a way of finding the period  $\Delta = 3.43$  analytically. This has not yet been done successfully.

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