

# Symmetry Classification of KdV-Type Nonlinear Evolution Equations

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This paper presents a short summary of results on classification of a class of third-order nonlinear evolution equations with respect to Lie point symmetries. The approach we take is a suitable combination of the concept of equivalence group with the usual infinitesimal technique of Lie based on several results on the structure of low-dimensional Lie algebras.

## 1 Introduction

We classify third-order KdV-type nonlinear evolution equations containing an arbitrary element  $F$  of the form

$$u_t = u_{xxx} + F(x, t, u, u_x, u_{xx}), \quad (1)$$

where  $F$  is a smooth real-valued function of its arguments that admits nontrivial symmetry groups.

We adopt the approach applied to the classification of several partial differential equations such as the most general second order quasi-linear heat-conductivity equation and the Schrödinger equation [1–3]. The approach is basically a combination of Lie's infinitesimal method and the use of equivalence transformations and the classification theory of abstract finite-dimensional Lie algebras.

We implement this classification in an algorithmic way in the following steps. The first step of the procedure is to find the most general form of the symmetry group of the equation under study using the standard Lie algorithm [4–6]. This is equivalent to solving a first order system of linear PDEs. In effect, the coefficients of the infinitesimal generator will satisfy certain relations connecting them with the arbitrary term and its derivatives. We then use the direct method (or, equivalently, infinitesimal one) to construct the equivalence group of the equation.

In the second step, we classify realizations of finite-dimensional Lie algebras by vector fields of the above form up to equivalence transformations. To this end, we use various results on the structure of abstract Lie algebras [7–11]. In the last step, after transforming symmetry generators to canonical forms, we proceed to classifying equations that admit nontrivial symmetries. We do this by inserting these generators into the symmetry condition and solving for  $F$ . With the increasing dimension of the realized algebras, the form of the arbitrary element will be restricted. Typically we continue this procedure until the arbitrary term figuring in the equation is completely specified. At this stage, the obtained equations are expected to depend on arbitrary parameters at most and it is easily checked by Lie algorithm whether the symmetry algebras obtained can be maximal among all those leaving invariant the equation under study.

The full details of the group classification of (1) can be found in [12].

## 2 Determining equations and equivalence transformations

Lie algebra of the symmetry group of equation (1) is realized by vector fields of the form

$$X = \tau(x, t, u)\partial_t + \xi(x, t, u)\partial_x + \phi(x, t, u)\partial_u. \quad (2)$$

Applying the standard Lie algorithm we have

**Proposition 1.** *The symmetry group of the nonlinear equation (1) for an arbitrary (fixed) function  $F$  is generated by the vector field*

$$X = \tau(t)\partial_t + \left(\frac{\dot{\tau}}{3}x + \rho(t)\right)\partial_x + \phi(x, t, u)\partial_u, \quad (3)$$

where the functions  $\tau(t)$ ,  $\rho(t)$  and  $\phi(x, t, u)$  satisfy the determining equation

$$\begin{aligned} & -3u_x\dot{\rho} - xu_x\ddot{\tau} - 9u_xu_{xx}\phi_{uu} - 3u_x^3\phi_{uuu} \\ & + 3\phi_t - 9u_{xx}\phi_{xu} - 9u_x^2\phi_{xuu} - 9u_x\phi_{xxu} - 3\phi_{xxx} + 3(\phi_u - \dot{\tau})F \\ & + (2u_{xx}\dot{\tau} - 3u_{xx}\phi_u - 3u_x^2\phi_{uu} - 6u_x\phi_{xu} - 3\phi_{xx})F_{u_{xx}} \\ & + (u_x\dot{\tau} - 3u_x\phi_u - 3\phi_x)F_{u_x} - 3\phi F_u - 3\tau F_t - (3\rho + x\dot{\tau})F_x = 0. \end{aligned} \quad (4)$$

Here the dot over a symbol stands for time derivative.

Next we construct the local group of point transformations preserving the form of the equation but possibly changing the function  $F$ . We require that equation (1) be preserved by local point transformations

$$\tilde{t} = T(x, t, u), \quad \tilde{x} = Y(x, t, u), \quad \tilde{u} = U(x, t, u), \quad (5)$$

where

$$\frac{D(T, Y, U)}{D(t, x, u)} \neq 0.$$

This requirement constrains the form of the transformation and we obtain the following assertion

**Proposition 2.** *The maximal equivalence group  $\mathcal{E}$  has the form*

$$\tilde{t} = T(t), \quad \tilde{x} = \dot{T}^{1/3}x + Y(t), \quad \tilde{u} = U(x, t, u), \quad (6)$$

where  $\dot{T} \neq 0$ ,  $U_u \neq 0$ .

**Proposition 3.** *Vector field (3) is equivalent within a point transformation of the form (5) to one of the following vector fields:*

$$X = \partial_t, \quad X = \partial_x, \quad X = \partial_u. \quad (7)$$

**Proof.** Transformation (6) transform vector field (3) into

$$\begin{aligned} X \rightarrow \tilde{X} = & \tau(t)\dot{T}(t)\partial_{\tilde{t}} + \left[\frac{1}{3}(\tau\dot{T}^{-1}\ddot{T} + \dot{\tau})(\tilde{x} - Y) + \tau\dot{Y} + \rho\dot{T}^{1/3}\right]\partial_{\tilde{x}} \\ & + \left[\tau U_t + \left(\frac{1}{3}\dot{\tau}x + \rho\right)U_x + \phi U_u\right]\partial_{\tilde{u}}. \end{aligned} \quad (8)$$

There are two cases to consider:

i)  $\phi = 0$ . Choose  $U = U(u)$  so that we have

$$\tilde{X} = \tau(t)\dot{T}(t)\partial_{\tilde{t}} + \left[ \frac{1}{3}(\tau\dot{T}^{-1}\ddot{T} + \dot{\tau})(\tilde{x} - Y) + \tau\dot{Y} + \rho\dot{T}^{1/3} \right] \partial_{\tilde{x}} + \phi U_u \partial_{\tilde{u}}. \quad (9)$$

Now if  $\tau = 0$ , then  $\rho \neq 0$  (otherwise  $X$  would be zero) we choose  $T(t)$  to satisfy

$$\dot{T} = \rho^{-3}.$$

In this case  $\tilde{X}$  is transformed into  $\partial_{\tilde{x}}$ .

If  $\tau \neq 0$ , then we choose  $T$  and  $Y$  to satisfy

$$\dot{T} = \tau^{-1}, \quad \tau\dot{Y} + \rho\dot{T}^{1/3} = 0.$$

With this choice of  $T$  and  $Y$  vector field  $\tilde{X}$  is transformed into  $\partial_{\tilde{t}}$ .

ii)  $\phi \neq 0$ . If  $\tau = \rho = 0$  then we can choose  $U$  to satisfy  $\phi U_u = 1$  so that we have  $\tilde{X} = \partial_{\tilde{u}}$ . Otherwise,  $U$  can be chosen to satisfy

$$\tau U_t + \left( \frac{1}{3}\dot{\tau}x + \rho \right) U_x + \phi U_u = 0.$$

Hence we recover Case i.

Summing up, the vector field (3) is equivalent, up to equivalence under  $\mathcal{E}$ , to one of the three standard vector fields  $\partial_x, \partial_t, \partial_u$ . This completes the proof.  $\blacksquare$

### 3 Classification of equations invariant under semi-simple algebras and algebras having nontrivial Levi decompositions

We start out by classifying equations (1) that admit Lie algebras isomorphic to the Lie algebras having nontrivial Levi decomposition. To achieve this, we construct equations admitting semi-simple Lie algebras as invariance algebra.

The lowest order semi-simple Lie algebras are isomorphic to one of the following three-dimensional algebras:

$$\begin{aligned} \mathfrak{sl}(2, \mathbb{R}) : \quad & [X_1, X_3] = -2X_2, \quad [X_1, X_2] = X_1, \quad [X_2, X_3] = X_3; \\ \mathfrak{so}(3) : \quad & [X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2. \end{aligned}$$

Using these commutation relations one can show that there is no realization of  $\mathfrak{so}(3)$  in terms of vector fields (3). We conclude that we cannot have  $\mathfrak{so}(3)$ -invariant equation of the form (1).

**Theorem 1.** *There exist no realizations of the algebra  $\mathfrak{so}(3)$  in terms of vector fields (3). Hence no equation of the form (1) is invariant under  $\mathfrak{so}(3)$  algebra.*

There are three inequivalent realizations of the algebra  $\mathfrak{sl}(2, \mathbb{R})$  by operators of the form (2)

$$\begin{aligned} & \left\{ \partial_t, t\partial_t + \frac{1}{3}x\partial_x, -t^2\partial_t - \frac{2}{3}tx\partial_x \right\}, \\ & \left\{ \partial_t, t\partial_t + \frac{1}{3}x\partial_x, -t^2\partial_t - \frac{2}{3}tx\partial_x - x^3\partial_u \right\}, \\ & \left\{ \partial_u, u\partial_u, -u^2\partial_u \right\}. \end{aligned}$$

Inserting the coefficients of basis operators of the above realizations of the algebra  $\mathfrak{sl}(2, \mathbb{R})$  into invariance relation (4) and solving and analyzing for their maximality we arrive at the following two theorems.

**Theorem 2.** *The class of PDEs (1) contains two inequivalent equations whose invariance algebra are semi-simple ( $\mathfrak{sl}(2, \mathbb{R})$ )*

$$u_t = u_{xxx} - \frac{3}{2}u_x^{-1}u_{xx}^2 + u_x G(x, t);$$

$$u_t = u_{xxx} - x^{-3} \left[ 2xu_x + \frac{1}{9}x^2u_x^2 - G(\omega_1, \omega_2) \right], \quad \omega_1 = 3u - xu_x, \quad \omega_2 = 6u - x^2u_{xx}.$$

The maximal invariance algebras of the above equations for arbitrary  $G$  are

$$\mathfrak{sl}^1(2, \mathbb{R}) = \{ \partial_u, u\partial_u, -u^2\partial_u \};$$

$$\mathfrak{sl}^2(2, \mathbb{R}) = \left\{ \partial_t, t\partial_t + \frac{1}{3}x\partial_x, -t^2\partial_t - \frac{2}{3}tx\partial_x - x^3\partial_u \right\}.$$

**Theorem 3.** *Nonlinear equation (1) whose invariance algebra is isomorphic to a Lie algebra having non-trivial Levi decomposition is represented by one of the following equations:*

$$u_t = u_{xxx} - \frac{3}{2}u_x^{-1}u_{xx}^2 + u_x \tilde{G}(x), \quad \mathfrak{sl}^1(2, \mathbb{R}) \oplus \{ \partial_t \}; \quad (10)$$

$$u_t = u_{xxx} - \frac{3}{2}u_x^{-1}u_{xx}^2 + \lambda x^{-2}u_x, \quad \lambda \neq 0, \quad \mathfrak{sl}^1(2, \mathbb{R}) \oplus \left\{ \partial_t, t\partial_t + \frac{1}{3}x\partial_x \right\}; \quad (11)$$

$$u_t = u_{xxx} - \frac{3}{2}u_x^{-1}u_{xx}^2, \quad \mathfrak{sl}^1(2, \mathbb{R}) \oplus \left\{ \partial_t, \partial_x, t\partial_t + \frac{1}{3}x\partial_x \right\}; \quad (12)$$

$$u_t = u_{xxx} - 2x^{-2}u_x - \frac{1}{9}x^{-1}u_x^2 + x^{-3}\tilde{G}(\sigma), \quad \sigma = x^2u_{xx} - 2xu_x, \quad \mathfrak{sl}^2(2, \mathbb{R}) \oplus \{ \partial_u \}, \quad (13)$$

where  $\tilde{G}$  is an arbitrary function of  $x$  or  $\sigma$ . Moreover, the associated symmetry algebras are maximal.

## 4 Equations invariant under solvable algebras

### 4.1 Equations with one-dimensional symmetry algebras

Given an  $F$  we assume that equation (1) is invariant under one-dimensional symmetry groups whose canonical forms are generated by translational vector fields  $\partial_t$ ,  $\partial_x$  and  $\partial_u$  (Proposition 3):

$$A_{1,1} : X_1 = \partial_t, \quad A_{1,2} : X_1 = \partial_x, \quad A_{1,3} : X_1 = \partial_u. \quad (14)$$

The corresponding invariant equations will have the form

$$A_{1,1} : u_t = u_{xxx} + F(x, u, u_x, u_{xx}), \quad (15a)$$

$$A_{1,2} : u_t = u_{xxx} + F(t, u, u_x, u_{xx}), \quad (15b)$$

$$A_{1,3} : u_t = u_{xxx} + F(x, t, u_x, u_{xx}). \quad (15c)$$

### 4.2 Equations with two-dimensional symmetry algebras

There are two isomorphy classes of two-dimensional Lie algebras, Abelian and non-Abelian satisfying the commutation relations  $[X_1, X_2] = \kappa X_2$ ,  $\kappa = 0, 1$ . We denote them by  $A_{2,1}$  and  $A_{2,2}$ :

$$A_{2,1} : [X_1, X_2] = 0, \quad A_{2,2} : [X_1, X_2] = X_2. \quad (16)$$

We construct all inequivalent realizations of the algebras  $A_{2,1}$ ,  $A_{2,2}$  in terms of infinitesimal operators (3) and obtain nine classes of two-dimensional algebras altogether. The representatives of equivalence classes of invariant equations are again found by substituting the coefficients in the symmetry condition (4). By doing this, the form of  $F$  in (15) is further restricted and the number of variables of  $F$  is reduced from four to three.

**Theorem 4.** *There exist nine classes of two-dimensional symmetry algebras admitted by equation (1). Those algebras and functions  $F$  are represented by*

$$\begin{aligned}
A_{2,1}^1 &: X_1 = \partial_t, \quad X_2 = \partial_x, \quad F = F(u, u_x, u_{xx}), \\
A_{2,1}^2 &: X_1 = \partial_t, \quad X_2 = \partial_u, \quad F = F(x, u_x, u_{xx}), \\
A_{2,1}^3 &: X_1 = \partial_x, \quad X_2 = \alpha(t)\partial_x + \partial_u, \quad F = -\dot{\alpha}uu_x + \tilde{F}(t, u_x, u_{xx}), \\
A_{2,1}^4 &: X_1 = \partial_u, \quad X_2 = g(x, t)\partial_u, \quad g_x \neq \text{const}, \\
&\quad F = (g_t - g_{xx})g_x^{-1}u_x + \tilde{F}(x, t, \omega), \quad \omega = g_{xx}u_x - g_x u_{xx}, \\
A_{2,2}^1 &: X_1 = \partial_t, \quad X_2 = -t\partial_t - \frac{x}{3}\partial_x, \\
&\quad F = x^{-3}\tilde{F}(u, \omega_1, \omega_2), \quad \omega_1 = xu_x, \quad \omega_2 = x^2u_{xx}, \\
A_{2,2}^2 &: X_1 = -3t\partial_t - x\partial_x, \quad X_2 = \partial_x, \\
&\quad F = t^{-1}\tilde{F}(u, \omega_1, \omega_2), \quad \omega_1 = t^{1/3}u_x, \quad \omega_2 = t^{2/3}u_{xx}, \\
A_{2,2}^3 &: X_1 = -u\partial_u, \quad X_2 = \partial_u, \quad F = u_x\tilde{F}(x, t, \omega), \quad \omega = u_x^{-1}u_{xx}, \\
A_{2,2}^4 &: X_1 = \partial_x - u\partial_u, \quad X_2 = \partial_u, \\
&\quad F = e^{-x}\tilde{F}(t, \omega_1, \omega_2), \quad \omega_1 = e^x u_x, \quad \omega_2 = e^x u_{xx}, \\
A_{2,2}^5 &: X_1 = \partial_t - u\partial_u, \quad X_2 = \partial_u, \\
&\quad F = u_x\tilde{F}(x, \omega_1, \omega_2), \quad \omega_1 = e^t u_x, \quad \omega_2 = e^t u_{xx}.
\end{aligned}$$

### 4.3 Equations with three-dimensional symmetry algebras

Two classes of decomposable three-dimensional Lie algebras exist over  $\mathbb{R}$ . Six classes of indecomposable three-dimensional Lie algebras exist over  $\mathbb{R}$ .

The decomposable three-dimensional algebras are represented by

$$A_{3,1} = 3A_1 = A_1 \oplus A_2 \oplus A_3$$

with  $[X_i, X_j] = 0$  for  $i, j = 1, 2, 3$  and

$$A_{3,2} = A_{2,2} \oplus A_1$$

with  $[X_1, X_2] = X_2$ ,  $[X_1, X_3] = 0$ ,  $[X_2, X_3] = 0$ .

A solvable three-dimensional Lie algebra has a two-dimensional Abelian ideal. We choose  $X_1$  and  $X_2$  as basis elements of the ideal. The commutation relations can be represented in the matrix notation

$$\begin{pmatrix} [X_1, X_3] \\ [X_2, X_3] \end{pmatrix} = J \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad [X_1, X_2] = 0,$$

where  $J \in \mathbb{M}^{2 \times 2}$  can be chosen in its Jordan canonical form.

Over  $\mathbb{R}$  we have

$$\begin{aligned}
A_{3,3} &: J = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & A_{3,4} &: J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & A_{3,5} &: J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
A_{3,6} &: J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & A_{3,7} &: J = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, & & 0 < |q| < 1, \\
A_{3,8} &: J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & A_{3,9} &: J = \begin{pmatrix} q & -1 \\ 1 & q \end{pmatrix}, & & q > 0.
\end{aligned}$$

Two of them,  $A_{3,7}$  and  $A_{3,9}$  depend on a continuous parameter  $q$ . Once algebras in the above list have been realized we can proceed to construct invariant equations (see [12] for details). The final result on three-dimensional symmetry algebras can be summed up as a theorem.

**Theorem 5.** *There are thirty-eight inequivalent three-dimensional solvable symmetry algebras admitted by equation (1).*

The explicit forms of those algebras as well as the associated invariant equations are given in [12].

## 5 Equations with four-dimensional solvable algebras

Ten isomorphism classes of decomposable four-dimensional Lie algebras and twelve decomposable ones exist. Omitting the details we present the result as a theorem. For the forms of inequivalent realizations of four-dimensional algebras and the corresponding invariant equations the reader is referred to Ref. [12].

**Theorem 6.** *There exist fifty-two inequivalent four-dimensional symmetry algebras admitted by equation (1).*

Again, the realizations of these algebras and the corresponding functions  $F$  are too long to present here. The interested reader should consult Ref. [12].

Finally we mention that we performed group classification of the most general third-order linear PDE. To our surprise we established that the nontrivial symmetry group is four-dimensional at most. However, as confirmed by the results of this work, there are nonlinear equations with larger symmetry groups than that of the linear equation. This situation is not the case for second-order PDEs.

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