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Geometrical Interpretation of Constrained Systems

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Quadratic singular Lagrangians are studied using the Hamilton–Jacobi theory of singular systems. An example is studied.

1 Introduction

The standard approach to classical dynamics is to form a Lagrangian which is a function of n generalized coordinates q_i, n generalized velocities $\dot{q}_i$ and parameter τ. The $2n$ variables q_i, $\dot{q}_i$ form the tangent bundle TQ. The passage from TQ to the cotangent bundle T^*Q is achieved by introducing generalized momenta and a Hamiltonian. However, this procedure requires that the rank of Hessian matrix

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

is n. The investigation of this type of Lagrangians (REGULAR) is a usual procedure. If the rank is less than n, system is called SINGULAR.

The study of singular systems started with DIRAC. He obtained the equations of motion of a singular system using consistency conditions and Poisson brackets. He classified the constraints. To quantize a singular system he introduced the Dirac bracket. Bergmann and his collaborators stressed on the relation between invariance principles and constraints in field theories. In fact, their efforts were to construct a Hamiltonian approach of general relativity to quantize the theory since the Einstein's theory of gravitation is a singular theory to its general covariance.

Singular field theories became the center of interest for physicists after the pioneering work of Faddeev, who introduced the Feynman path integral quantization. Nowadays, singular systems find a very wide range of applications in theoretical physics. An invariance under a global gauge transformation implies a singular theory. Hence, starting from the electromagnetic theory, all gauge theories have singular nature.

2 Hamilton–Jacobi theory of singular systems

The aim is to obtain a valid and consistent Hamilton–Jacobi theory of singular systems. Mathematical method which is used is the Caratheodory's equivalent Lagrangians method. The main point of the method is to define the equivalent Lagrangian (variational principle) and then pass to the phase space. This formulation leads us to a set of Hamilton–Jacobi partial differential equations [1–4].

2.1 Construction of phase space

Let us consider a system which is described by the Lagrangian $L\left(q_i, \dot{q}_i, \tau\right)$ such that the rank of the Hessian is $n-p$, $p \leq n$.

Generalized momenta are defined as

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = 1, 2, \ldots, p,$$
$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, \ldots, n - p. \tag{1}$$

Since the rank of the Hessian is $n - p$, one may solve (1) for $\dot{q}_a$ as

$$\dot{q}_a = w_a\left(\tau, q_i, \dot{q}_\mu, p_a\right).$$

So

$$p_\mu = \left.\frac{\partial L}{\partial \dot{q}_\mu}\right|_{\dot{q}_a = w_a} = -H_\mu\left(\tau, q_i, p_a, \dot{q}_\nu\right).$$

Generalized momenta p_μ are NOT independent. Although, it seems that H_μ are functions of $\dot{q}_\nu$, it is a task to show that they do not depend on $\dot{q}_\nu$ explicitly.

Definition 1.

$$H_0 = -L\left(\tau, q_i, \dot{q}_\nu, \dot{q}_a = w_a\right) + p_a w_a + \left.\dot{q}_\mu p_\mu\right|_{p_\mu = -H_\mu}.$$

Like H_μ, H_0 is not an explicit function of $\dot{q}_\mu$.
In fact,

$$\frac{\partial H_0}{\partial \dot{q}_\nu} = -\frac{\partial L}{\partial \dot{q}_\nu} - \frac{\partial L}{\partial \dot{q}_a}\frac{\partial w_a}{\partial \dot{q}_\nu} + p_a \frac{\partial w_a}{\partial \dot{q}_\nu} p_\nu = 0.$$

Therefore, the Hamilton–Jacobi function $S\left(\tau, q_i\right)$ should satisfy the following set of Hamilton–Jacobi Partial Differential Equations (HJPDE) simultaneously for an extremum of the action:

$$p_0 + H_0\left(\tau, q_\nu; q_a, p_a = \frac{\partial S}{\partial q_a}, p_0 = \frac{\partial S}{\partial \tau}\right) = 0,$$
$$p_\mu + H_\mu\left(\tau, q_\nu; q_a, p_a = \frac{\partial S}{\partial q_a}, p_0 = \frac{\partial S}{\partial \tau}\right) = 0$$

or in a compact form

$$H'_\alpha\left(t_p, p_i = \frac{\partial S}{\partial q_i}, p_0 = \frac{\partial S}{\partial t}\right) = 0,$$

where

$$H'_\alpha = p_\alpha + H_\alpha, \qquad \alpha = 0, 1, \ldots, p.$$

2.2 The canonical equations

This method leads us to the following equations:

$$dq_r = \frac{\partial H'_\alpha}{\partial p_r} dt_\alpha, \qquad r = 0, 1, \ldots, n, \quad \alpha = 0, 1, \ldots, p,$$
$$dp_r = -\frac{\partial H'_\alpha}{\partial q_r} dt_\alpha,$$
$$dS = \left(-H'_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dt_\alpha.$$

Simultaneous solutions of these equations determine $S\left(t_\alpha, q_a\right)$ uniquely if the initial conditions are given.

Example 1.

$$2L = a_{ij}(\tau, q_k)\,\dot{q}_i\dot{q}_j + 2b_i(\tau, q_k)\,\dot{q}_i - 2c(\tau, q_k), \qquad i, j = 1, 2, 3,$$

where

$$a_{ij} = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & -a_2 & a_2 \\ 0 & a_2 & -a_2 \end{vmatrix}, \qquad a_1 \neq 0, \quad a_2 \neq 0,$$

$$p_1 = a_1\dot{q}_1, \qquad p_2 = a_2(\dot{q}_3 - \dot{q}_2) + b, \qquad p_3 = a_2(\dot{q}_3 - \dot{q}_2),$$

$$w_1 = \frac{p_1}{a_1}, \qquad w_3 = \dot{q}_2 - \frac{p_3}{a_2}, \qquad p_2 = -p_3 + b = -H_2,$$

$$H_0 = -L + p_1 w_1 + p_3 w_3 + (-p_3 + b)\,\dot{q}_2 \qquad \text{or}$$

$$H_0 = \frac{1}{2}\left(\frac{p_1^2}{a_1} - \frac{p_3^2}{a_2}\right) + c.$$

The set of (HJPDE) is

$$\frac{\partial S}{\partial \tau} + H_0\left(q_i, \tau, p_1 = \frac{\partial S}{\partial q_1}, p_3 = \frac{\partial S}{\partial q_3}\right) = 0,$$

$$\frac{\partial S}{\partial q_2} + H_2\left(q_i, \tau, p_1 = \frac{\partial S}{\partial q_1}, p_3 = \frac{\partial S}{\partial q_3}\right) = 0.$$

Differential equations for the characteristics are:

$$dq_1 = \frac{p_1}{a_1}d\tau, \qquad dq_3 = -\frac{p_3}{a_2}d\tau + dq_2,$$

$$dp_i = \left[\frac{1}{2}\left(\frac{p_1^2}{a_1^2}\frac{\partial a_1}{\partial q_i} - \frac{p_3^2}{a_2^2}\frac{\partial a_2}{\partial q_i}\right)\frac{\partial c}{\partial q_i}\right]d\tau + \frac{\partial b}{\partial q_i}dq_2,$$

$$dp_0 = \left[\frac{1}{2}\left(\frac{p_1^2}{a_1^2}\frac{\partial a_1}{\partial q_i} - \frac{p_3^2}{a_2^2}\frac{\partial a_2}{\partial q_i}\right)\frac{\partial c}{\partial \tau}\right]d\tau + \frac{\partial b}{\partial \tau}dq_2.$$

3 Free particle

A particle of mass m is described by generalized coordinates x_i $(i = 1, \ldots, n)$ as

$$2L(x^i, \dot{x}^i) = -mcg_{ij}\dot{x}^i\dot{x}^j, \tag{2}$$

where the metric matrix elements are functions of x_i. One can express the Hamiltonian in the usual way if the rank of the Hessian matrix (metric matrix) is n. In fact, generalized momenta are defined as

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = -mcg_{ij}\dot{x}^j.$$

This definition leads us to the expression

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = -mcg_{ij}\dot{x}^j.$$

Thus,

$$2H = -2L + 2p_i\dot{x}^i = -\frac{1}{mc}g^{km}p_k p_m.$$

Canonical equations

$$\dot{x}^i = \frac{dx^i}{d\tau} = \frac{\partial H}{\partial p_i},$$
$$\dot{p}^i = \frac{dp^i}{d\tau} = -\frac{\partial H}{\partial x_i}$$

lead us to the geodesic equations on the Riemannian manifold identified with the metric g_{ij}. Following calculations give the desired equations

$$\dot{x}_i = -\frac{1}{2mc} g_{ij} p^j,$$
$$\dot{p}_k = -\frac{\partial H}{\partial x^k} = \frac{1}{2mc} \frac{\partial g_{ij}}{\partial x^k} p^i p^j.$$

Thus,

$$\ddot{x}^i = \frac{\partial g^{ij}}{\partial x^k} g_{jm} \dot{x}^k \dot{x}^m - \frac{1}{2} \frac{\partial g_{lk}}{\partial x^j} g^{ji} g^{km} g^{ln} \dot{x}_m \dot{x}_n.$$

After some manupulation we get

$$\ddot{x}^i + \Gamma^i_{km} \dot{x}^k \dot{x}^m = 0, \qquad i = 1, \ldots, n,$$

where

$$\Gamma^i_{km} = -\frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right).$$

Description of the Lagrangian (2) in terms of the metric g_{ij} gives the geodesic equations.

If the rank of the Hessian matrix is less than n, we have a singular system [5]. Now, the question is whether the geometric approach is applicable or not. To achieve this aim we are going to apply the Canonical formulation to a singular Lagrangian. Let us consider the Lagrangian

$$L = \frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + \dot{\lambda}_1 \left(y\dot{z} - z\dot{y} \right) + \dot{\lambda}_2 \left(z\dot{x} - x\dot{z} \right) + \dot{\lambda}_3 \left(x\dot{y} - y\dot{x} \right). \tag{3}$$

Physically speaking, this is the Lagrangian which describes a particle such that all three components of angular momentum are conserved. Here $\dot{\lambda}_i$ $(i = 1, 2, 3)$ are Lagrange multipliers.

One can express this Lagrangian as

$$L = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j, \qquad i = 1, \ldots, 6,$$

where

$$\dot{q}_1 \equiv \dot{x}, \qquad \dot{q}_2 \equiv \dot{y}, \qquad \dot{q}_3 \equiv \dot{z}, \qquad \dot{q}_4 \equiv \dot{\lambda}_1, \qquad \dot{q}_5 \equiv \dot{\lambda}_2, \qquad \dot{q}_6 \equiv \dot{\lambda}_3$$

and the 6×6 symmetric matrix g_{ij} is

$$g_{ij} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & x \\ 0 & 0 & 1 & y & -x & 0 \\ 0 & -z & y & 0 & 0 & 0 \\ z & 0 & -x & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 & 0 \end{pmatrix} \equiv g_{ji}.$$

One should notice that the rank of g_{ij} is five. Thus, one cannot apply the standard procedure. To find a remedy, we express (3) as

$$L = \frac{1}{2}\left(g_{\nu\mu}\dot{q}_\nu\dot{q}_\mu + 2g_{6\mu}\dot{\lambda}_3\dot{q}_\mu\right), \qquad \mu,\nu = 1,\ldots,5.$$

In order to determine the Hamiltonian, let us define the generalized momenta as

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial \dot{q}_\mu} = g_{\nu\mu}\dot{q}_\nu + g_{6\mu}\dot{\lambda}_3, \qquad (4)\\ p_{\lambda_3} &= \frac{\partial L}{\partial \dot{\lambda}_3} = g_{6\sigma}\dot{q}_\sigma \\ &\qquad = g_{6\sigma}\left[g^{-1}_{\sigma\mu}p_\mu - g^{-1}_{\sigma\mu}a_{6\mu}\dot{\lambda}_3\right] \\ &\qquad = g_{66}p_\mu - g_{66} \equiv 0. \end{aligned}$$

Since the rank of g_{ij} is five, the submatrix $g_{\mu\nu}$ is invertible. Thus, using (4) we get

$$\dot{q}_\sigma = g^{-1}_{\sigma\mu}p_\mu - g^{-1}_{\sigma\mu}a_{6\mu}\dot{\lambda}_3, \qquad \sigma = 1,\ldots,5. \tag{5}$$

Now, we can define the Hamiltonian H_0 as

$$H_0 = -L + p_\mu\dot{q}_\mu + p_{\lambda_3}\dot{\lambda}_3. \tag{6}$$

Substituting (5) in (6), one obtains

$$H_0 = \frac{1}{2}g^{-1}_{\mu\rho}p_\mu p_\rho, \qquad \mu,\rho = 1,\ldots,5$$

which is independent of $\dot{\lambda}_3$. We expect this result due to the singular nature of the Lagrangian. We have two Hamiltonians to describe the system

$$H'_0 = p_0 + \frac{1}{2}g^{-1}_{\mu\rho}p_\mu p_\rho \equiv 0, \tag{7}$$
$$H' = p_{\lambda_3} \equiv 0. \tag{8}$$

Canonical equations read as

$$dq_\mu = \frac{\partial H'_\alpha}{\partial p_\mu}dt_\alpha, \qquad dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu}dt_\alpha, \qquad \alpha = 0,6, \qquad dt_0 \equiv dt, \qquad dt_\alpha \equiv d\lambda_3.$$

More explicitly

$$\begin{aligned} dq_\mu &= g^{-1}_{\mu\rho}p_\rho dt, \qquad \mu = 1,\ldots,5,\\ dq_6 &\equiv d\lambda_3,\\ dp_\nu &= -\frac{1}{2}\left(\frac{\partial g^{-1}_{\mu\rho}}{\partial q_\nu}\right)p_\mu p_\rho dt,\\ dp_{\lambda_3} &= 0, \qquad dp_0 = 0. \end{aligned} \tag{9}$$

The theory forces us to check the consistency conditions i.e. to check whether the variations of the constraints (7) and (8) are zero or not. Equation (9) guarantees that the variation of H' is zero. Besides,

$$dH'_0 = dp_0 + \frac{1}{2}\left\{d\left(g^{-1}_{\mu\rho}\right)p_\mu p_\rho + g^{-1}_{\mu\rho}dp_\mu p_\rho + g^{-1}_{\mu\rho}p_\mu dp_\rho\right\}.$$

Using the equations of motion we get

$$dH_0' = \frac{1}{2}\left\{\frac{\partial g^{-1}_{\mu\rho}}{\partial q_\sigma}dq_\sigma p_\mu p_\rho + g^{-1}_{\mu\rho}\left(-\frac{1}{2}\frac{\partial g^{-1}_{\alpha\beta}}{\partial q_\mu}p_\alpha p_\beta dt\right)p_\rho + g^{-1}_{\mu\rho}p_\mu\left(-\frac{1}{2}\frac{\partial g^{-1}_{\alpha\beta}}{\partial q_\rho}p_\alpha p_\beta dt\right)\right\} \equiv 0.$$

Thus, the system is integrable.

Up to this point we have followed the standard canonical formulation of a constrained system. Now, we will propose the geodesic equations

$$\ddot{q}_\mu = \frac{1}{2}\left(g^{-1}\right)_{\mu\nu}\left(\frac{\partial g_{\nu\sigma}}{\partial q_\rho} + \frac{\partial g_{\nu\rho}}{\partial q_\sigma} - \frac{\partial g_{\sigma\rho}}{\partial q_\nu}\right)\dot{q}^\sigma\dot{q}^\rho \tag{10}$$

as equivalent to the equations of motion. In other words, we will show that solutions of (10) satisfy the equations of motion. In fact, equations (10) give

$$\begin{aligned}
\ddot{x} &= \frac{x}{x^2+y^2+z^2}\left[2\dot{\lambda}_2\left(z\dot{x}-x\dot{z}\right)+2\dot{\lambda}_1\left(y\dot{z}-z\dot{y}\right)\right],\\
\ddot{y} &= \frac{y}{x^2+y^2+z^2}\left[2\dot{\lambda}_1\left(y\dot{z}-z\dot{y}\right)+2\dot{\lambda}_2\left(z\dot{x}-x\dot{z}\right)\right],\\
\ddot{z} &= \frac{z}{x^2+y^2+z^2}\left[2\dot{\lambda}_2\left(z\dot{x}-x\dot{z}\right)+2\dot{\lambda}_1\left(y\dot{z}-z\dot{y}\right)\right],\\
\ddot{\lambda}_1 &= \frac{1}{x^2+y^2+z^2}\left[-2\dot{\lambda}_1\left(\frac{x^2\dot{z}+z^2\dot{z}}{z}+y\dot{y}\right)+2\dot{\lambda}_2 y\left(\frac{z\dot{x}-x\dot{z}}{z}\right)\right],\\
\ddot{\lambda}_2 &= \frac{1}{x^2+y^2+z^2}\left[2x\dot{\lambda}_1\left(\frac{z\dot{y}-y\dot{z}}{z}\right)-2\dot{\lambda}_2\left(\frac{y^2\dot{z}+z^2\dot{z}}{z}+x\dot{x}\right)\right].
\end{aligned}$$

One should notice that

$$\frac{\ddot{x}}{x} = \frac{\ddot{y}}{y} = \frac{\ddot{z}}{z}.$$

Therefore, equating these ratios to a negative constant k we get the periodic solutions

$$\begin{aligned}
x(t) &= c_1\sin\sqrt{|k|}t + c_2\cos\sqrt{|k|}t,\\
y(t) &= c_3\sin\sqrt{|k|}t + c_4\cos\sqrt{|k|}t,\\
z(t) &= c_5\sin\sqrt{|k|}t + c_6\cos\sqrt{|k|}t.
\end{aligned}$$

These solutions also satisfy the equations of motion.

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